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Weak and strong field approximations and circular orbits of Kehagias-Sfetsos space-time

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The Kehagias-Sfetsos asymptotically flat black hole and naked singularity solutions of Hořava-Lifshitz gravity were investigated both in the weak and strong-field regimes. In the weak field limit the Kehagias-Sfetsos spherically symmetric solution generates a weaker gravity compared to the Schwarzschild black hole of general relativity. However in the strong field regime the behavior of gravity depends on the Kehagias-Sfetsos parameter ω_0. When $\omega_0 \gg 1$ the Schwarzschild case is mimicked. For $\omega_0 \ll 1$ (which represents a naked singularity) gravity is weakened when approaching the center. Finally we investigated the stability of the circular orbits. While in the black hole case the square of the angular momentum should be larger than a minimal value, in the naked singularity case there are stable circular orbits for any non-zero angular momentum. In this regime the existence of an infimum of the allowed radii of circular orbits was proved.

1 Introduction

Properties of the Hořava-Lifshitz (HL) gravity theory have been extensively analyzed and several versions have been proposed in the literature (Visser 2011). The infrared (IR)-modified Hořava gravity seems to be consistent with the current observational data (Konoplya 2009, Chen & Jing 2009, Chen & Wang 2010).

The vacuum metric of the HL black holes is given by (Radinschi et al.)

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \ , \quad (1)$$

$$f(r) = 1 + (\omega - \Lambda)r^2 - \sqrt{r\left[\omega\left(\omega - 2\Lambda\right)r^3 + \beta\right]} \ . \quad (2)$$

β is an integration constant, while Λ and ω are real parameters. Depending on the values of β , ω and Λ, there are special cases of the metric (2).

With the choice $\beta = -(\alpha^2)/(\Lambda)$ and $\omega = 0$ the given metric is the Lu-Mei-Pope black hole solution (Lu et al. 2009). If $\beta = 4\omega m$ and $\Lambda = 0$ we get the Kehagias-Sfetsos (KS) space-time solution. The paper focuses on the latter case with $f(r)$ given by (Kehagias & Sfetsos 2009)

$$f(r) = 1 + \omega r^2 - \sqrt{\omega^2 r^4 + 4\omega m r} \ . \quad (3)$$

Beyond the mass m the space-time (3) is characterised by the Hořava-Lifshitz parameter ω. This solution is asymptotically flat and for large r or when $\omega \to \infty$ we obtain the Schwarzschild case. The slowly rotating Kehagias-Sfetsos solution was introduced in Lee et al. 2010. We introduce $\omega_0 = \omega m^2$ as a dimensionless parameter. When $\omega_0 > 1/2$ there are two event horizons at

$$r_\pm = m\left(1 \pm \sqrt{1 - \frac{1}{2\omega_0}}\right) \ . \quad (4)$$

The two event horizons coincide for $\omega_0 = 0.5$ and there is a naked singularity when $\omega_0 < 0.5$. General relativity is recovered for $\omega_0 \to \infty$ but the black hole interpretation continues to hold for any $\omega_0 \geq 0.5$. The value of the parameter ω_0 has been constrained by various methods. The radar echo delay in the Solar system, analyzed in Harko et al. (2011) gave the limit $\omega_{0,\min}^{(red)} = 2 \times 10^{-15}$. Perihelion precession of Mercury and deflection of light by the Sun gave the limits $\omega_{0,\min}^{(pp)} = 6.9 \times 10^{-16}$ and $\omega_{0,\min}^{(ld)} = 1.1 \times 10^{-15}$ respectively. Tighter constraints for ω_0 were presented in Iorio & Ruggiero (2010) analyzing the range-residuals of the planet Mercury: $\omega_{0,\min}^{(residual)} = 7.2 \times 10^{-10}$. A slightly stronger constraint $\omega_{0,\min}^{(Sag)} = 8 \times 10^{-10}$ arises from the observation of the S2 star orbiting the Supermassive Black Hole (Sagittarius A*) in the center of the Galaxy. It has been shown in Horvath et al. (2011) that the forthcoming instrument Large Synoptic Survey Telescope will be able to constrain ω_0 up to 10^{-1} from strong gravitational lensing. We remark that neither of these observations could render the value of ω_0 to the regime where the Kehagias-Sfetsos solution describes a black hole.

The accretion disk is a structure formed by charged particles in orbital motion around a compact astrophysical body. The IR limit of the HL has been explored both for spherically symmetric (Harko et al. 2009) and for slowly rotating (Harko et al. 2011) black holes in the Hořava-Lifshitz

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theory. In the spherically symmetric case they found that the energy flux, the temperature distribution of the disk and the spectrum of the emitted black body radiation, significantly differ from the general relativistic case. In addition they have shown that the intensity of the flux emerging from the disk surface is larger for the slowly rotating KS solution than for the general relativistic rotating Kerr black hole.

The simplest accretion disk model is the steady-state thin disk, based on several simplifying assumptions (Chen & Jing 2012). For accretion disks with sub-Eddington luminosities the inner edge of the accretion disk is located at the innermost stable circular orbit (ISCO) (Paczynski 2000). However, for larger accretion disk luminosities, there is no uniquely defined inner edge. Different definitions lead to different edges, the differences increasing with the luminosity. In Abramowicz et al. (2010) six possible definitions of the inner edge have been listed.

Our paper is organized as follows. In Section 2 we investigate the metric in the weak-field limit. In Section 3 we repeat the analysis for the strong-field regime. In Section 4 we analyze the properties of the stable circular orbits in both the black hole and the naked singularity cases. Finally we summarize our results in Section 5.

2 Approximatios in the weak-field regime

In the following we introduce the parameter $\varepsilon = m/r$ which is small in the weak-field regime, far from the black hole. In this section we will investigate the metric for three specific ranges of ω_0.

First we assume $\omega_0 \gg \varepsilon^3$ which implies $\omega_0^{-1}\varepsilon^3 \ll 1$ and leads to two possible scenarios. If $\omega_0 \geq 0.5$ there is a black hole in the post-Newtonian regime, while $\omega_0 < 0.5$ the solution yields to a naked singularity. With this assumption the Kehagias-Sfetsos metric becomes

$$-g_{tt} = 1 + \omega_0 \left(\frac{r}{m}\right)^2 \left[1 - \left(1 + \frac{4m^3}{\omega_0 r^3}\right)^{1/2}\right] \qquad (5)$$
$$= 1 + \omega_0 \varepsilon^{-2} \left[1 - \left(1 + \frac{4}{\omega_0}\varepsilon^3\right)^{1/2}\right] .$$

Expanding the expression in the bracket

$$\left(1 + \frac{4}{\omega_0}\varepsilon^3\right)^{1/2} \approx 1 + \frac{2}{\omega_0}\varepsilon^3 , \qquad (6)$$

and the metric function as

$$-g_{tt} = \frac{1}{g_{rr}} \simeq 1 - 2\varepsilon \qquad (7)$$

it approximates the Schwarzschild case.

If $\omega_0 \approx \varepsilon^3$ then $\omega_0^{-1}\varepsilon^3 = \mathcal{O}(1)$ thus no Taylor expansion can be performed. However we can write $\omega_0\varepsilon^{-2} = \omega_0\varepsilon^{-3}\varepsilon \simeq \varepsilon$ and the metric function becomes

$$-g_{tt} = 1 + \varepsilon \left[\omega_0\varepsilon^{-3} - \left(\left(\omega_0\varepsilon^{-3}\right) + 4\left(\omega_0\varepsilon^{-3}\right)\right)^{1/2}\right] \qquad (8)$$
$$\simeq 1 - 2\varepsilon\mathcal{O}(1) .$$

Finally when $\omega_0 \ll \varepsilon^3$ then $\omega_0^{-1}\varepsilon^3 \gg 1$, and we get

$$-g_{tt} = 1 + \left(\omega_0\varepsilon^{-3}\right)\varepsilon \left[1 - \left(1 + \frac{4}{\omega_0}\varepsilon^3\right)^{1/2}\right] \qquad (9)$$
$$\simeq 1 - 2\varepsilon\left(\omega_0\varepsilon^{-3}\right)^{1/2} .$$

The second and third choices to ω_0 always lead to naked singularity solutions.

To summarize the different cases we can write

$$-g_{tt} = \frac{1}{g_{rr}} = 1 - 2\varepsilon y , \qquad (10)$$

where we introduce the parameter y, with

$$y = 1,\ if\ \omega_0 \gg \varepsilon^3 \qquad (11)$$
$$y \leq 1,\ if\ \omega_0 \approx \varepsilon^3$$
$$y \ll 1,\ if\ \omega_0 \ll \varepsilon^3 .$$

As one can see in the weak-field regime HL gravity always predicts a weaker gravity for every value of ω_0. An "effective" mass of the KS black hole can be defined as

$$m_{eff} = my , \qquad (12)$$

where m is the Schwarzschild black hole mass.

3 Approximations in the strong-field regime

Close to the black hole $\varepsilon = \mathcal{O}(1)$. When $\omega_0 \gg 1$, the solution tends to Schwarzschild $-g_{tt} = 1 - 2\varepsilon$.

When $\omega_0 \ll 1$ (falling into the naked singularity regime) the approximation yields

$$-g_{tt} = \frac{1}{g_{rr}} = 1 + \omega_0\varepsilon^{-2}\left[1 - \left(1 + \frac{4}{\omega_0}\varepsilon^3\right)^{1/2}\right] \qquad (13)$$
$$\simeq 1 - 2\varepsilon^{-1/2}\omega_0^{1/2} = 1 - 2\left(\frac{\omega_0}{m}\right)^{1/2} r^{1/2} .$$

Approaching towards the singularity, gravity decreases. This is in contrast with the prediction of general relativity and is presented in Fig. 1.

The case $\omega_0 \approx 1$ has been numerically studied in Harko et al. (2009) for $\omega_0 > 0.5$.

4 Circular orbits around Kehagias-Sfetsos black holes

In this section we study the particle motion in the Kehagias-Sfetsos space-time and determine the radius of the innermost stable circular orbit for a massive particle. We assume that the inner edge of the accretion disk is located at this radius.

The trajectory of a massive particle is a timelike geodesic. Hence we can choose the affine parameter τ to be the proper time along the path of the particle. For circular motion in the equatorial plane ($\theta = \pi/2$) we set $R := r/m = constant$, we introduced the normalised radial coordinate

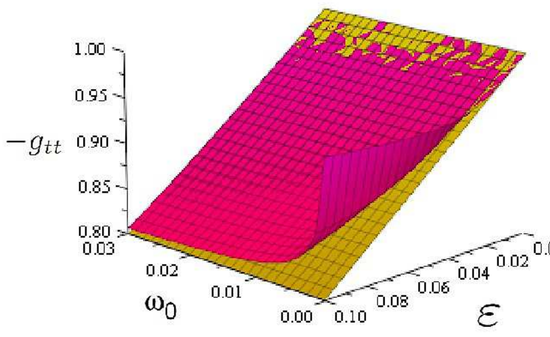

Fig. 1 The behaviour of the metric function $-g_{tt}$ (upper surface) as compared to the respecting metric function of the Schwarzschild black hole (lower surface). For large values of ε but small values of ω_0 the $1/r$ dependence of the Schwarzschild black hole is replaced by $r^{1/2}$ dependence.

R. Then the non-zero t, r and ϕ components of the geodesic equation reduce to

$$\left(\frac{d^2\phi(\tau)}{d\tau^2}\right) = 0, \tag{14}$$

$$\left(\frac{d^2t(\tau)}{d\tau^2}\right) = 0, \tag{15}$$

$$\frac{\omega_0\left(R\Xi - \omega_0 R^3 - 1\right)}{m^2\Xi}\left(\frac{dt(\tau)}{d\tau}\right)^2 = R\left(\frac{d\phi(\tau)}{d\tau}\right)^2, \tag{16}$$

where $\Xi = \sqrt{R\omega_0\left(R^3\omega_0 + 4\right)}$. Eqs. (14) and (15) imply that $\dot{\phi}(\tau) = d\phi(\tau)/d\tau$ and $\dot{t}(\tau) = dt(\tau)/d\tau$ are constants and these are related to the specific angular momentum L and specific energy E of the particle respectively. From Eq. (16) the positivity of ω_0, E and L^2 implies a constraint for the radius R

$$R > \left(\frac{1}{2\omega_0}\right)^{1/3} . \tag{17}$$

We emphasize that this inequality is a strict one, equality is not allowed. Therefore for every circular orbit (either stable or unstable) Eq. (17) must hold both in the black hole and naked singularity regimes.

In the Kehagias-Sfetsos space-time the effective potential studied in Ref. Harko et al. 2009 has the form

$$V_{eff}(L, R, \omega_0) = \left[1 + \omega_0 R^2\left(1 - \sqrt{1 + \frac{4}{\omega_0 R^3}}\right)\right]\left(1 + \frac{L^2}{R^2}\right) . \tag{18}$$

Since this potential depends only on the constants of motion L and R therefore V_{eff} is also a constant of motion ($dV_{eff}/dR = 0$). Stable circular orbits occur at the local minima of the potential, while the local maxima in the potential are the locations of unstable circular orbits.

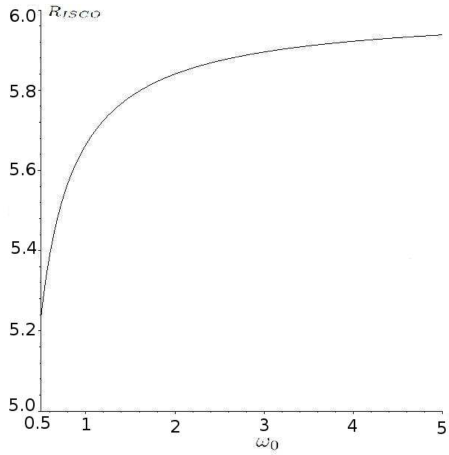

Fig. 2 The ISCO radius as a function of the dimensionless parameter ω_0 in the black hole regime. For large ω_0 values the radius of the ISCO is approaching $R_{ISCO} = 6$ which is the Schwarzschild case.

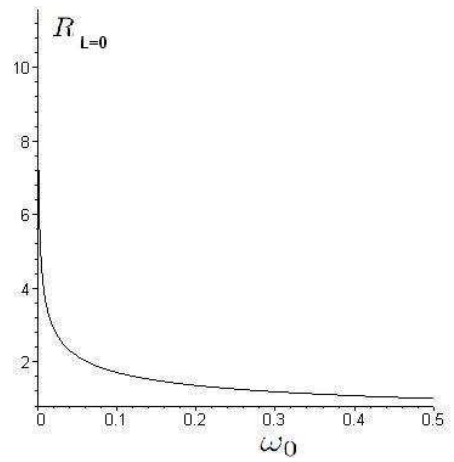

Fig. 3 The $(2\omega_0)^{-1/3}$ curve, where $L^2 = 0$. Stable circular orbits exist for radii lying above this curve.

4.1 Black hole region ($\omega_0 \geq 0.5$)

For every $\omega_0 \geq 0.5$ there exist a given angular momentum L, for which there is one extremum of the effective potential. At this radius the conditions $dV_{eff}/dR = 0$ and $d^2V_{eff}/dR^2 = 0$ must be satisfied. The corresponding marginally stable orbit is the ISCO. The ISCO as function of the parameter ω_0 in the black hole regime is shown on the numerical plot of Figure 2.

4.2 Naked singularity region ($\omega_0 < 0.5$)

Our numerical study has revealed that stable circular orbits exist for any non-zero angular momenta, whenever the inequality (17) holds. This means that the set of the allowed stable circular orbit radii have the infimum $(2\omega_0)^{-1/3}$, but no minimum. This infimum as the function of ω_0 is shown on Figure 3 .

5 Conclusions

We investigated the Kehagias-Sfetsos asymptotically flat black hole solution of Hořava-Lifshitz gravity both in the weak and strong-field regimes. In Section 2 we analyzed the weak-field regime, where the parameter $\varepsilon = m/r$ is small, thus the metric function could be Taylor-expanded in ε. We found that when $\varepsilon \ll 1$, gravity is always weaker in HL theory than predicted by GR regardless of the value of ω_0. In Section 3 the strong-field regime was investigated, close to the singularity, where $\varepsilon = \mathcal{O}(1)$. For the parameter region $\omega_0 \gg 1$ there is a black hole with two event horizons. The related solutions are asymptotically flat and tend to the well known Schwarzschild case. The case $\omega_0 \approx 1$ can only be investigated by numerical methods. When $\omega_0 \ll 1$ there is a naked singularity and approaching towards the center, gravity surprisingly decreases.

In Section 4 we studied the minimum radius for a stable circular orbit in the black hole and naked singularity regimes. In the black hole case the ISCO radius was derived as the function of ω_0, and found a complete agreement with the results of Harko et al. 2009. On the other hand in the naked singularity case, stable circular orbits always exist if the angular momentum is not zero which implies that only an infimum of the stable circular orbit radii can be defined when $\omega_0 < 0.5$. Therefore the curve on Fig. 4 in Abdujabbarov et al. 2011 has no physical meaning, moreover it corresponds to the impossible condition $L^2 < 0$.

In an upcoming article we plan to derive the energy flux of the accretion disk around a naked singularity to restricting the value of the metric parameter ω_0.

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References

Abdujabbarov, A., Ahmedov, B. & Hakimov, A.: 2011, PhRvD 83, 044053
Abramowicz, M. A., Jaroszynski, M., Kato, S., Lasota, J.P., Rozanska, A., Sadowski, A.: 2010, A&A 521, A15
Chen, S. & Jing, J.: 2012, J. PhLB 711, 81-87
Chen S., & Jing, J.: 2009, PhRvD 80, 024036
Chen, J. & Wang, Y.: 2010, IJMPA 25, 1439-1448
Harko, T., Kovacs, Z., Lobo, F.S.N.: 2011, CQGra 28, 165001
Harko, T., Kovacs, Z., Lobo, F.S.N.: 2009, PhRvD 80, 044021
Hořava, P.: 2009a, JHEP 0903, 020
Hořava, P.: 2009b, PhRvD 79, 084008
Horvath, Zs., Gergely, L.A, Keresztes, Z., Harko, T., Lobo, F.S.N.: 2011, PhRvD 84, 083006
Iorio, L. & Ruggiero, M. L.: 2010, OAJ 3, 167
Kehagias, A. & Sfetsos, K.: 2009, PhLB 678, 123
Konoplya, R. A.: 2009, PhLB 679, 499.
Lee, H. W., Kim, Y. W., and Myung, Y. S.: 2010, Eur. Phys. J. C 70, 367.
Lu, H., Mei, J. & Pope, C.N.: 2009, PhRvL 103, 091301
Mosaffa, A.E.: 2011, PhRvD 83, 124006
S. 1973 Black Holes, ed. C. DeWitt and B. DeWitt, New York: Gordon and Breach
Paczynski, B.: 2000, ApJL
ApJ 191, 499.
Radinschi, I., Rahaman, F. & Banerjee, A.: 2011, IJTP 50, 2906-2916
A.: 1973, A&A 24 33.
Visser, M.: 2011, arXiv:1103.5587