ECONOMETRICS

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ECONOMETRICS

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ECONOMETRICS

Week 10.

Univariate time series analysis II.

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Plan

AR, MA, ARMA and ARIMA processes

Box–Jenkins methodology, estimation and goodness of fit test of ARMA models

Forecasting from ARMA models

Material: M 13.4–13.6

AR(1) process

$$X_t = \mu + \alpha X_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim IN(0, \sigma^2)$$

$$X_t = \mu + \alpha(\mu + \alpha X_{t-2} + \varepsilon_{t-1}) + \varepsilon_t = \ldots$$

$$= \mu \frac{1-\alpha^n}{1-\alpha} + \alpha^n X_{t-n} + \sum_{i=0}^{n-1} \alpha^i \varepsilon_{t-i}$$

If $|\alpha| < 1$ then the model is stationary, and:

$$X_t = \mu / (1-\alpha) + \sum_{i=0}^{\infty} \alpha^i \varepsilon_{t-i}$$ (MA(∞) – repr.)

$$E(X_t) = \mu / (1-\alpha)$$

$$\mathrm{Var}(X_t) = \alpha^2 \,\mathrm{Var}(X_{t-1}) + \sigma^2 \Rightarrow \mathrm{Var}(X_t) = \sigma^2 / (1-\alpha^2)$$

ACF, PACF in AR(1) models

$$\gamma_k = \text{cov}(X_t, X_{t-k}) = \text{cov}(\alpha X_{t-1} + \varepsilon_t, X_{t-k}) = \alpha \gamma_{k-1}$$

$$\gamma_k = \alpha^k \sigma^2$$

$$\rho_k = \alpha^k$$

$$\eta_k = \begin{cases} 1, & \text{if } \ \mathrm{k} = 1 \\ 0, & \text{if } \ \mathrm{k} > 1 \end{cases}$$

AR(p) process

$$X_t = c + \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \ldots + \alpha_p X_{t-p} + \varepsilon_t$$

Assuming $c = 0$ and using the $X_{t-k} = L^k X_t$ lag operator,

$$\varepsilon_t = \left(1 - \alpha_1 L - \alpha_2 L^2 - \ldots - \alpha_p L^p\right) X_t .$$

If all π_i roots of $x^p - \alpha_1 x^{p-1} - \ldots - \alpha_p = 0$ satisfy $|\pi_i| < 1$, then X_t is stationary and

$$X_t = \left(1 - \alpha_1 L - \alpha_2 L^2 - \ldots - \alpha_1 L^p\right)^{-1} \varepsilon_t$$

$$= \left\{(1 - \pi_1 L)(1 - \pi_2 L) \ldots (1 - \pi_p L)\right\}^{-1} \varepsilon_t$$

Properties of stationary AR(p) processes

$$\mu = E(X_t) = c/(1-\alpha_1-\alpha_2-\ldots-\alpha_p)$$

ACF: Yule–Walker equations

$$\gamma_k = \operatorname{cov}(X_t, X_{t-k}) = \operatorname{cov}(c+\alpha_1 X_{t-1}+\ldots+\alpha_p X_{t-p}+\varepsilon_t, X_{t-k})$$

$$= \alpha_1\gamma_{k-1}+\alpha_2\gamma_{k-2}+\ldots+\alpha_p\gamma_{k-p}$$

$$\rho_k = \alpha_1\rho_{k-1}+\alpha_2\rho_{k-2}+\ldots+\alpha_p\rho_{k-p}$$

$k \le p$: system of equations $(\rho_{\mathrm{k}} = \rho_{\mathrm{-k}})$,

$k > p$: recursion

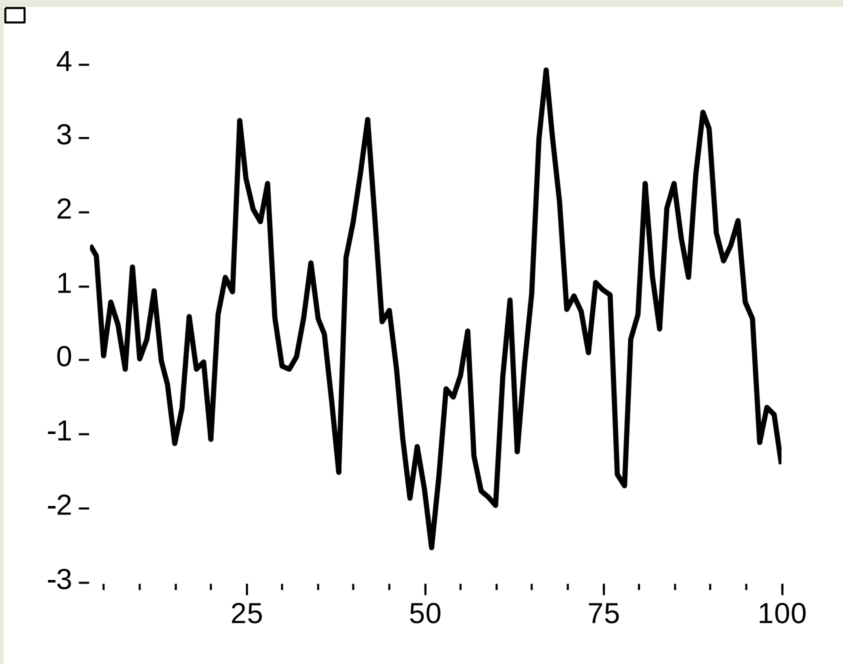
PACF

$$\eta_k = 0 \text{ if } k > p.$$

Example: AR(1) process

$$X_t = 0{,}7X_{t-1} + \varepsilon_t$$

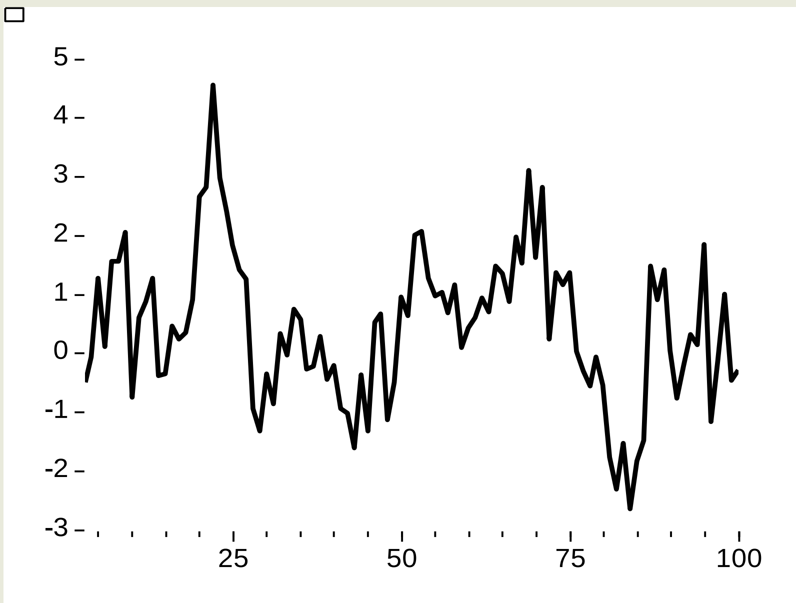
(ACF is easy to calculate)



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.700	0.700	2449.5	0.000
		2	0.494	0.008	3669.4	0.000
		3	0.350	0.003	4281.7	0.000
		4	0.245	-0.006	4580.9	0.000
		5	0.169	-0.004	4723.5	0.000
		6	0.114	-0.006	4788.1	0.000
		7	0.070	-0.013	4812.5	0.000
		8	0.043	0.002	4821.6	0.000
		9	0.030	0.009	4826.2	0.000
		10	0.016	-0.010	4827.5	0.000
		11	0.010	0.005	4828.1	0.000
		12	0.001	-0.011	4828.1	0.000

Example: AR(2) process

$$X_t = 0{,}4X_{t-1} + 0{,}5X_{t-2} + \varepsilon_t$$



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.800	0.800	3197.2	0.000
		2	0.814	0.486	6515.4	0.000
		3	0.722	-0.003	9126.9	0.000
		4	0.685	-0.026	11477.	0.000
		5	0.627	-0.009	13443.	0.000
		6	0.578	-0.022	15118.	0.000
		7	0.536	0.005	16554.	0.000
		8	0.488	-0.017	17744.	0.000
		9	0.453	0.006	18773.	0.000
		10	0.411	-0.009	19619.	0.000
		11	0.386	0.022	20366.	0.000
		12	0.360	0.030	21015.	0.000
		13	0.331	-0.014	21563.	0.000
		14	0.313	0.009	22053.	0.000
		15	0.283	-0.017	22455.	0.000
		16	0.267	0.000	22813.	0.000
		17	0.243	-0.004	23108.	0.000
		18	0.225	-0.006	23363.	0.000
		19	0.206	-0.002	23576.	0.000
		20	0.193	0.012	23763.	0.000
		21	0.181	0.014	23928.	0.000
		22	0.175	0.027	24082.	0.000
		23	0.167	0.013	24223.	0.000
		24	0.158	-0.009	24349.	0.000

MA(1) process

$$X_t = c + \varepsilon_t + \beta\varepsilon_{t-1}$$

$$\mu = E(X_t) = c$$

$$\gamma_0 = \mathrm{Var}(X_t) = \sigma^2(1 + \beta^2)$$

$$\gamma_1 = \sigma^2\beta$$

$$\gamma_k = 0 \text{ if } k > 1$$

$$\rho_1 = \beta/(1 + \beta^2)$$

$$\rho_k = 0 \text{ if } k > 1$$

η_k : decays to 0

X_t : stationary for every β.

MA(q) process

$$X_t = c + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q}$$

$$\mu = E(X_t) = c$$

$$\gamma_0 = \mathrm{Var}(X_t) = \sigma^2 (1 + \beta_1^2 + \ldots + \beta_q^2)$$

$$\gamma_k = \begin{cases} \sigma^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k}, \text{ if } 0 \le k \le q \\ 0, \text{ if } k > q. \end{cases}$$

$$\rho_k = \gamma_k / \gamma_0$$

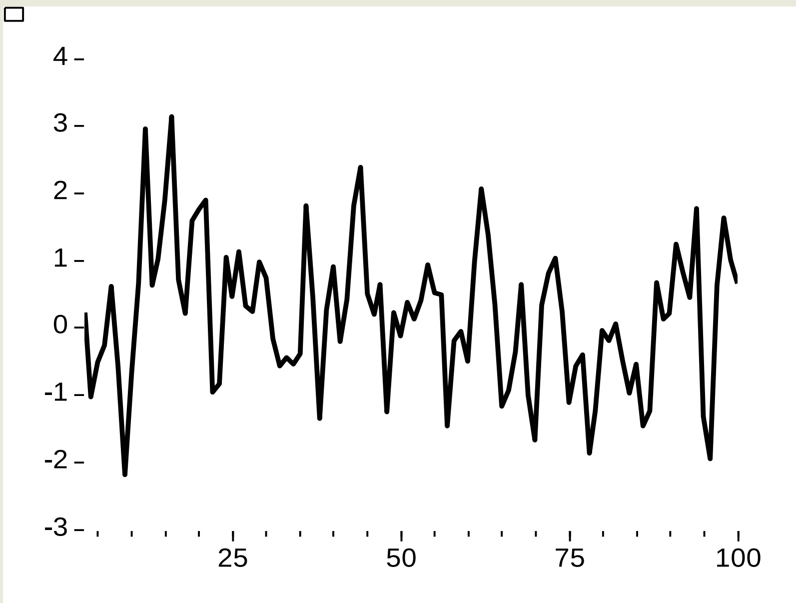
η_k : decays to zero

X_t : stationary for every β_i

Example: MA(1) process

$$X_t = \varepsilon_t + 0{,}7\varepsilon_{t-1}$$

(ACF is easy to calculate)



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.467	0.467	1091.3	0.000
		2	-0.018	-0.302	1092.8	0.000
		3	-0.025	0.178	1095.9	0.000
		4	-0.021	-0.138	1098.2	0.000
		5	-0.021	0.076	1100.4	0.000
		6	-0.013	-0.060	1101.3	0.000
		7	-0.001	0.043	1101.3	0.000
		8	-0.006	-0.041	1101.5	0.000
		9	-0.007	0.024	1101.8	0.000
		10	-0.008	-0.027	1102.1	0.000
		11	-0.008	0.012	1102.4	0.000
		12	-0.002	-0.007	1102.4	0.000
		13	-0.009	-0.010	1102.8	0.000
		14	0.004	0.022	1102.8	0.000
		15	0.024	0.008	1105.6	0.000
		16	0.016	-0.001	1106.9	0.000
		17	0.003	0.002	1106.9	0.000
		18	0.002	0.002	1106.9	0.000
		19	-0.003	-0.008	1107.0	0.000
		20	-0.012	-0.007	1107.7	0.000
		21	-0.011	-0.003	1108.3	0.000
		22	0.002	0.010	1108.4	0.000
		23	0.011	0.003	1109.0	0.000
		24	0.012	0.008	1109.6	0.000

ARMA(p,q) process

$$X_t = c + \alpha_1 X_{t-1} + \ldots + \alpha_p X_{t-p} + \varepsilon_t + \beta_1 \varepsilon_{t-1} + \ldots + \beta_q \varepsilon_{t-q}$$

Stationary if its AR(p) component is stationary (all roots of the characteristic equation…)

Neither ACF nor PACF is 0, but both tend to zero at an exponential rate.

Remark:	ACF	PACF
AR(p)	decays to 0	0 for k>q
MA(q)	0 for k>q	decays to 0

ARIMA(p,d,q) process

X_t is an ARIMA(p,1,q) process if ΔX_t is a stationary ARMA(p,q) process.

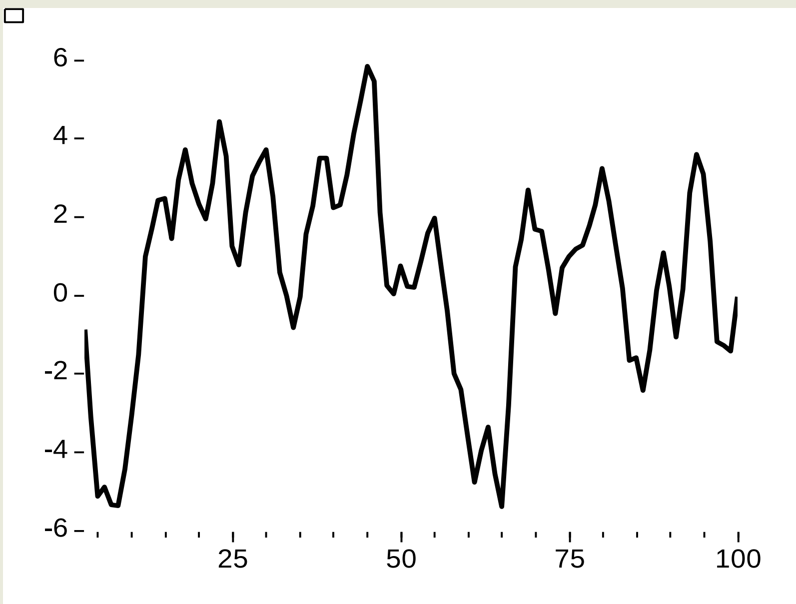
Similarly, X_t is an ARIMA(p,d,q) process if ΔX_t is an ARIMA(p,d–1,q) process.

Order of integration of ARIMA(p,d,q) is I(d).

Examples:

- ARIMA(0,1,0): $X_t - X_{t-1} = \varepsilon_t$ is the random walk.
- ARIMA(1,1,0): $X_t - X_{t-1} = \alpha(X_{t-1} - X_{t-2}) + \varepsilon_t$, where $|\alpha|<1$.
- So $X_t = (1 + \alpha)X_{t-1} - \alpha X_{t-2} + \varepsilon_t$ is a nonstationary AR(2) process.

Example: ARMA(1,1) process



Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.845	0.845	3570.8	0.000
		2	0.586	-0.448	5287.9	0.000
		3	0.396	0.253	6072.8	0.000
		4	0.261	-0.179	6414.2	0.000
		5	0.167	0.107	6553.2	0.000
		6	0.105	-0.062	6608.8	0.000
		7	0.070	0.053	6633.3	0.000
		8	0.047	-0.042	6644.4	0.000
		9	0.039	0.066	6652.0	0.000
		10	0.043	-0.016	6661.2	0.000
		11	0.044	-0.001	6670.7	0.000
		12	0.031	-0.035	6675.5	0.000
		13	0.011	-0.003	6676.1	0.000
		14	-0.003	0.012	6676.2	0.000
		15	-0.008	-0.002	6676.4	0.000
		16	-0.015	-0.030	6677.5	0.000
		17	-0.023	0.013	6680.2	0.000
		18	-0.022	0.010	6682.6	0.000
		19	-0.015	0.001	6683.8	0.000
		20	-0.011	-0.014	6684.4	0.000
		21	-0.011	-0.001	6685.0	0.000
		22	-0.017	-0.031	6686.5	0.000
		23	-0.030	-0.006	6690.9	0.000
		24	-0.042	-0.016	6699.9	0.000

Estimation of ACF

Estimation:

$$\hat{\rho}_k = \frac{\sum_{t=1}^{T-k}\left(X_t - \overline{X}\right)\left(X_{t+k} - \overline{X}\right)}{\sum_{t=1}^{T}\left(X_t - \overline{X}\right)^2}$$

Only makes sense in the stationary case, and in this case it is consistent (i.e. for large T it estimates ρ_k with a small variance).

Box-Jenkins methodology

Taking differences

- The series is differentiated until it becomes stationary

Identification

- Conjecture of the orders p, q of the ARMA model based on the ACF

Estimation

Examining the goodness of fit

Estimation of ARMA models

- Simple in the case of AR models
  - OLS (minimising the sum of squares of the estimated innovations (ε_t))
  - Consistent and asymptotically normal in the stationary case
- In the case of MA or ARMA models
  - Full maximum likelihood or
  - Searching methods
    - Choosing the starting innovations as zero, the subsequent innovations can be calculated as a function of the parameters, and their sum of squares can be minimised

Model selection criteria in ARMA models

These criteria control for the fact that using more parameters in the model may only apparently give a better fit

Minimising a criterion yields to the optimal size of the model.

Examples:

- Akaike information criterion
  - $AIC = n \cdot \log(RSS/(n-s)) + 2s$
- Bayes (Schwartz) information criterion
  - $BIC = n \cdot \log(RSS/(n-s)) + s \cdot \log n$
- Where
  - s: number of estimated parameters
  - RSS: sum of squares of innovations
  - n: sample size

Testing autocorrelation in the residuals

r_k : lag k autocorrelation of the residuals

$$H_0\colon r_1 = r_2 = \ldots = r_m = 0$$

Ljung–Box-test

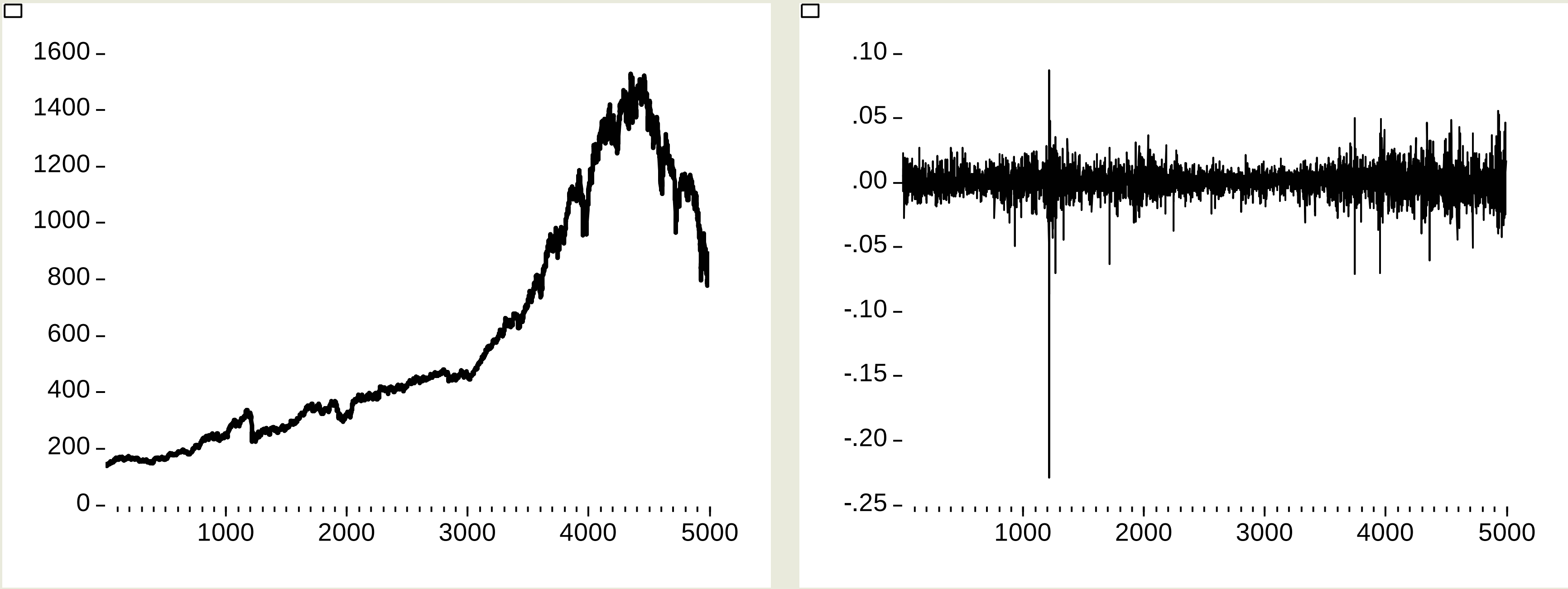
$$Q_{LB} = n(n+2)\sum_{k=1}^{m} \frac{\hat{r}_k^2}{n\text{-}k} \sim \chi^2_{m-s}$$ (under H_0, largesample)

Breusch–Godfrey-test: regression on the innovations

$$\hat{\varepsilon}_t = b_1\hat{\varepsilon}_{t-1} + b_2\hat{\varepsilon}_{t-2} + \ldots + b_m\hat{\varepsilon}_{t-m} + u_t$$

$$NR^2 \sim \chi^2_m$$ (under H_0, largesample)

Example: white noise test for S&P logarithmic returns



White noise test (cont.)

low, perhaps significant autocorrelation
but: one should be careful when drawing conclusions because of heteroscedasticity (changing variance)

Breusch-Godfrey Serial Correlation LM Test:

F-statistic	5.371561	Probability	0.000258
Obs*R-squared	21.41562	Probability	0.000262

Test Equation:
Dependent Variable: RESID
Method: Least Squares
Date: 02/17/09 Time: 13:22
Presample missing value lagged residuals set to zero.

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	-5.10E-07	0.000152	-0.003351	0.9973
RESID(-1)	0.020576	0.014149	1.454221	0.1459
RESID(-2)	-0.047172	0.014146	-3.334611	0.0009
RESID(-3)	-0.030450	0.014152	-2.151600	0.0315
RESID(-4)	-0.026050	0.014163	-1.839263	0.0659

R-squared	0.004284	Mean dependent var	5.78E-18
Adjusted R-squared	0.003486	S.D. dependent var	0.010777
S.E. of regression	0.010758	Akaike info criterion	-6.225313
Sum squared resid	0.577991	Schwarz criterion	-6.218795
Log likelihood	15565.17	F-statistic	5.371561
Durbin-Watson stat	1.998872	Prob(F-statistic)	0.000258

Autocorrelation	Partial Correlation		AC	PAC	Q-Stat	Prob
		1	0.022	0.022	2.3792	0.123
		2	-0.046	-0.047	13.031	0.001
		3	-0.033	-0.031	18.469	0.000
		4	-0.025	-0.026	21.631	0.000
		5	0.012	0.010	22.349	0.000
		6	-0.013	-0.017	23.187	0.001
		7	-0.015	-0.015	24.299	0.001
		8	-0.008	-0.008	24.583	0.002
		9	-0.002	-0.003	24.598	0.003
		10	0.008	0.005	24.892	0.006
		11	-0.014	-0.016	25.920	0.007
		12	0.037	0.038	32.736	0.001

Forecasting from ARIMA models

Estimated innovations

$$\hat{\varepsilon}_0 = 0,\ \hat{\varepsilon}_{-1} = 0$$

$$\hat{\varepsilon}_t = X_t - \hat{\mu} - \hat{\alpha}_1 X_{t-1} - \hat{\alpha}_2 X_{t-2} - \hat{\beta}_1 \hat{\varepsilon}_{t-1} - \hat{\beta}_2 \hat{\varepsilon}_{t-2}$$

Forecasting

$I_t =$ information set in t $(X_t, X_{t-1}, \ldots)$

$$\hat{X}_{t+1} = E\left(X_{t+1} \mid I_t\right) = E\left(\mu + \alpha_1 X_t + \alpha_2 X_{t-1} + \varepsilon_{t+1} + \beta_1 \varepsilon_t + \beta_2 \varepsilon_{t-1} \mid I_t\right)$$

$$= \hat{\mu} + \hat{\alpha}_1 X_t + \hat{\alpha}_2 X_{t-1} + \hat{\beta}_1 \hat{\varepsilon}_t + \hat{\beta}_2 \hat{\varepsilon}_{t-1}$$

$$\hat{X}_{t+1} = E\left(X_{t+2} \mid I_t\right) = E\left(\mu + \alpha_1 X_{t+1} + \alpha_2 X_t + \varepsilon_{t+2} + \beta_1 \varepsilon_{t+1} + \beta_2 \varepsilon_t \mid I_t\right)$$

$$= \hat{\mu} + \hat{\alpha}_1 \hat{X}_{t+1} + \hat{\alpha}_2 X_t + \hat{\beta}_2 \hat{\varepsilon}_t$$

Types of forecasts and evaluation of their performance

Forecasts

- In sample
- Out of sample

Performance evaluation: root mean squared error (RMSE), mean absolute error (MAE)

- Estimation on interval [1,T], evaluation on interval [T+1,T+m]

$$RMSE = \sqrt{\frac{\sum_{t=T+1}^{T+m}\left(X_t - \hat{X}_t\right)^2}{m}}$$

$$MAE = \sqrt{\frac{\sum_{t=T+1}^{T+m}| X_t - \hat{X}_t |}{m}}$$

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Seminar

Univariate time series II.

Exercises I

Simulation of AR(1), MA(1), AR(2) and MA(2) time series

Graphical representation of the ACF and PACF

Determination of the ACF and PACF by the Yule-Walker equations

Evaluation of the stationarity of the AR(2) model by the roots of the characteristic equation

Exercises: series of company bond data I.

- Evaluation of stationarity by visual inspection of the ACF
- Estimation of ARMA models on a subsample
- Goodness of fit test and model selection
  - Significance of parameters
  - Uncorrelatedness of residuals (Ljung–Box and Breusch–Godfrey-tests)
  - Model selection based on AIC and BIC

Exercises: series of company bond data II

Static (multi-period) forecast based on the best performing model

Dynamic (one-period) forecast

- Comparison of its RMSE with that of the naive forecast

Graphical comparison of the forecasts to the observed data