Introduction to Logic and Computer Science
Foundation of Computer Science
Logic in Computer Science

Lecture #3: semantics

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This work was supported by the construction EFOP-3.4.3-16-2016-00021.
The project was supported by the European Union, co-financed by the European Social Fund.

Interpretation

Definition (interpretation)

The function ϱ is an interpretation of the language $L^{(0)}$ if

1. $Dom(\varrho) = Con$
2. If $p \in Con$, then $\varrho(p) \in \{0, 1\}$.

Semantic value

Definition (semantic value)

Let ϱ be an interpretation and $|A|_\varrho$ be the semantic value of the formula A with respect to ϱ.

1. If $p \in Con$, then $|p|_\varrho = \varrho(p)$
2. If $A \in Form$, then $|\neg A|_\varrho = 1 - |A|_\varrho$.
3. If $A, B \in Form$, then
   - $$|(A \supset B)|_\varrho = \begin{cases} 0, & \text{if } |A|_\varrho = 1, \text{ and } |B|_\varrho = 0; \\ 1, & \text{otherwise.} \end{cases}$$
   - $$|(A \wedge B)|_\varrho = \begin{cases} 1, & \text{if } |A|_\varrho = 1, \text{ and } |B|_\varrho = 1; \\ 0, & \text{otherwise.} \end{cases}$$
   - $$|(A \vee B)|_\varrho = \begin{cases} 0, & \text{if } |A|_\varrho = 0, \text{ and } |B|_\varrho = 0; \\ 1, & \text{otherwise.} \end{cases}$$
   - $$|(A \equiv B)|_\varrho = \begin{cases} 1, & \text{if } |A|_\varrho = |B|_\varrho; \\ 0, & \text{otherwise.} \end{cases}$$

Semantic value

Example

$Con = \{p,\ q\}$, let $\varrho(p) = 1$ and $\varrho(q) = 0$.
Then $|q|_\varrho = 0$, $|p|_\varrho = 1$, $|(q \supset p)|_\varrho = 1$, $|\neg(q \supset p)|_\varrho = 0$, $|\neg p|_\varrho = 0$,
$|(\neg(q \supset p) \vee \neg p)|_\varrho = 0$

Models, satisfiable

Definition (model of a set of formulas)

Let $\Gamma \subseteq Form$ be a set of formulas. An interpretation ϱ is a model of the set of formulae Γ, if $|A|_\varrho = 1$ for all $A \in \Gamma$.

Definition (model of a formula)

A model of a formula A is the model of the singleton $\{A\}$.

Definition (satisfiable - set of formulas)

The set of formulae $\Gamma \subseteq Form$ is satisfiable if it has a model.

- If there is an interpretation in which all members of the set Γ are true.

Definition (satisfiable formula)

A formula $A \in Form$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Satisfiability

Remark

- A satisfiable set of formulae does not involve a logical contradiction;
  - its formulae may be true together.
- A satisfiable formula may be true.
- If a set of formulae is satisfiable, then its members are satisfiable.
- But: all members of the set $\{p, \neg p\}$ are satisfiable, and the set is not satisfiable.

Subset property of satisfiability

Theorem

All subsets of a satisfiable set are satisfiable.

Proof

- Let $\Gamma \subseteq \mathit{Form}$ be a set of formulae and $\Delta \subseteq \Gamma$.
- Γ is satisfiable: it has a model. Let ϱ be a model of Γ.
- A property of ϱ:
  - If $A \in \Gamma$, then $|A|_\varrho = 1$
- Since $\Delta \subseteq \Gamma$, if $A \in \Delta$, then $A \in \Gamma$, and so $|A|_\varrho = 1$. That is the interpretation ϱ is a model of Δ, and so Δ is satisfiable.

Unsatisfiable

Definition (unsatisfiable set)

The set $\Gamma \subseteq Form$ is unsatisfiable if it is not satisfiable.

Definition (unsatisfiable formula)

A formula $A \in Form$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

Remark

A unsatisfiable set of formulae involve a logical contradiction.

- *Its members cannot be true together.*

Superset property of unsatisfiability

Theorem

All expansions of an unsatisfiable set of formulae are unsatisfiable.

Indirect proof

- Suppose that $\Gamma \subseteq Form$ is an unsatisfiable set of formulae and $Delta \subseteq Form$ is a set of formulas.
- Indirect condition: Γ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem Γ is satisfiable, and it is a contradiction.

Logical consequence, validity

Definition

A formula A is the logical consequence of the set of formulae Γ if the set $\Gamma \cup \{\neg A\}$ is unsatisfiable. (Notation : $\Gamma \models A$)

Definition

$A \models B$, if $\{A\} \models B$.

Definition

The formula A is valid if $\emptyset \models A$. (Notation: $\models A$)

Definition

The formulae A and B are logically equivalent if $A \models B$ and $B \models A$.

- Notation: $A \Leftrightarrow B$