# Introduction to Logic and Computer Science
# Foundation of Computer Science
# Logic in Computer Science

## Lecture #4: properties of the central concepts

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# Models and logical consequence

## Theorem

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if all models of the set $\Gamma$ are models of formula $A$. (i.e. the singleton $\{A\}$).

## Proof

$\rightarrow$ Indirect condition: $\Gamma \models A$ and there is a model of $\Gamma$ such that it is not a model of the formula $A$.

- Let the interpretation $\varrho$ be this model.
- The properties of $\varrho$:
    1. $|B|_\varrho = 1$ for all $B \in \Gamma$;
    2. $|A|_\varrho = 0$, and so $|\neg A|_\varrho = 1$
- In this case all members of the set $\Gamma \cup \{\neg A\}$ are true wrt $\varrho$, and so $\Gamma \cup \{\neg A\}$ is satisfiable.
- It means that $\Gamma \not\models A$, which is a contradiction.

# Models and logical consequence

## Proof (cont.)

$\leftarrow$ Indirect condition: All models of the set $\Gamma$ are models of formula $A$, but (and) $\Gamma \not\models A$.

- In this case $\Gamma \cup \{\neg A\}$ is satisfiable, i.e. it has a model.
- Let the interpretation $\varrho$ be a model.
- The properties of $\varrho$:
  1. $|B|_\varrho = 1$ for all $B \in \Gamma$;
  2. $|\neg A|_\varrho = 1$, i.e. $|A|_\varrho = 0$
- So the set $\Gamma$ has a model such that it is not a model of formula $A$, which is a contradiction.

## Corollary

Let $\Gamma \subseteq \mathit{Form}$, and $A \in \mathit{Form}$. $\Gamma \models A$ if and only if for all interpretations in which all members of $\Gamma$ are true, the formula $A$ is true.

# Valid formula as a consequence

## Theorem

If A is a valid formula ($\models A$), then $\Gamma \models A$ for all sets of formulae $\Gamma$.
*A valid formula is a consequence of any set of formulas.*

## Proof

- If $A$ is a valid formula, then $\emptyset \models A$ (according to its definition).
- $\emptyset \cup \{\neg A\} = \{\neg A\}$ is unsatisfiable, and so its expansions are unsatisfiable (by theorem).
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

# Unsatisfiable set as hypothesis

## Theorem

If $\Gamma$ is unsatisfiable, then $\Gamma \cup A$ for all $A$.

*All formulae are the consequences of an unsatisfiable set of formulas.*

## Proof

- According to a proved theorem: If $\Gamma$ is unsatisfiable, then all expansions of $\Gamma$ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\Gamma$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

# Deduction theorem

## Theorem

If $\Gamma \cup \{A\} \models B$, then $\Gamma \models (A \supset B)$.

## Proof

- Indirect condition: Suppose, that $\Gamma \cup \{A\} \models B$, and $\Gamma \not\models (A \supset B)$.
- $\Gamma \cup \{\neg(A \supset B)\}$ is satisfiable, and so it has a model.
- Let the interpretation $\varrho$ be this model.
- The properties of $\varrho$:
  1. All members of $\Gamma$ are true wrt $\varrho$.
  2. $|\neg(A \supset B)|_\varrho = 1$
- $|(A \supset B)|_\varrho = 0$, i.e. $|A|_\varrho = 1$ and $|B|_\varrho = 0$. So $|\neg B|_\varrho = 1$.
- All members of $\Gamma \cup \{A\} \cup \{\neg B\}$ are true wrt interpretation $\varrho$, i.e. $\Gamma \cup \{A\} \not\models B$, and it is a contradiction.

# Deduction theorem - opposite direction

## Theorem

If $\Gamma \models (A \supset B)$, then $\Gamma \cup \{A\} \models B$.

## Proof

- Indirect condition: Suppose that $\Gamma \models (A \supset B)$, and $\Gamma \cup \{A\} \not\models B$..
- So $\Gamma \cup \{A\} \cup \{\neg B\}$ is satisfiable, i.e. it has a model.
- Let the interpretation $\varrho$ be this model.
- The properties of $\varrho$:
    1. All members of $\Gamma$ are true wrt the interpretation $\varrho$.
    2. $|A|_\varrho = 1$
    3. $|\neg B|_\varrho = 1$, and so $|B|_\varrho = 0$
- $|(A \supset B)|_\varrho = 0$, $|\neg(A \supset B)|_\varrho = 1$.
- All members of $\Gamma \cup \{\neg(A \supset B)\}$ are true wrt the interpretation $\varrho$, i.e. $\Gamma \not\models (A \supset B)$.

# Corollary of deduction theorem

## Corollary

$A \models B$ if and only if $\models (A \supset B)$

## Proof

Let $\Gamma = \emptyset$ in the former theorems.

# Cut elimination theorem

**Theorem**

If $\Gamma \cup \{A\} \models B$ and $\Delta \models A$, then $\Gamma \cup \Delta \models B$.

**Proof**

Indirect.