Introduction to Logic and Computer Science
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Logic in Computer Science

Lecture #6: Boole functions, Post's theorem

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Boole functions with arity 1

x	a	b	c	d
0	0	0	1	1
1	0	1	0	1

a: $\bot$ (falsity)
b: identity
c: **negation**
d: $\top$ (truth)

Boole functions with arity 2

x	y	e	f	g	h	i	j	k	l	m	n	o	p	q	r	s	t
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

f: conjunction

k: XOR (exclusive or, $\oplus$)

l: disjunction

m: NOR (Pierce's arrow)

n: equivalence

r: implication s: NAND (Sheffer stroke)

Closed classes

Definition (ordering)

Let $\mathbf{x} = \langle x_1, \ldots, x_n \rangle$ and $\mathbf{y} = \langle y_1, \ldots, y_n \rangle$. $\mathbf{x} \leq \mathbf{y}$, by definition, if $x_i \leq y_i$ for all $1 \leq i \leq n$.

Monoton

The function $f : \{0,\ 1\}^n \rightarrow \{0,\ 1\}$ is monoton, if $\mathbf{x}, \mathbf{y} \in \{0,\ 1\}^n$ and $\mathbf{x} \leq \mathbf{y}$ then $f(\mathbf{x}) \leq f(\mathbf{y})$

- Monoton functions: a, b, d, e ($\bot$), f ($\wedge$), h, j, l ($\vee$), t ($\top$)

Linear

The function $f : \{0,\ 1\}^n \rightarrow \{0,\ 1\}$ is linear,
if $f(x_1, \ldots, x_n) = \alpha_0 \oplus \alpha_1 \wedge x_1 \oplus \cdots \oplus \alpha_n \wedge x_n$ for some α where $\alpha_i \in \{0,\ 1\}$.

- The linear functions: a, b, c ($\neg$), d, e ($\bot$), h, j, k (XOR), n ($\equiv$), o, q, t ($\top$)

Function class

Constans-preserving

The function $f : \{0,\ 1\}^n \to \{0,\ 1\}$ 0-preserving, if $f(0,\dots,0) = 0$.

- 0-preserving functions: a, b, e($\bot$), f ($\wedge$), g, h, i, j, k (XOR), l ($\vee$)

The function $f : \{0,\ 1\}^n \to \{0,\ 1\}$ 1-preserving, ha $f(1,\dots,1) = 1$.

- 1-preserving functions: b, d, f ($\wedge$), h, j, l ($\vee$), n ($\equiv$), p, r ($\supset$), t ($\top$),

Self-dual functions

Function $f : \{0,\ 1\}^n \to \{0,\ 1\}$ self-dual,
if $\neg f(\neg x_1,\dots,\neg x_n) = f(x_1,\dots,x_n)$.

- Self-dual functions: b, c, h, j, o, q.

Functional Completeness

Definition

A set of Boole-functions $f_i : \{0, 1\}^{n_i} \to \{0, 1\}$ is functional complete, if any Boole-functions $f : \{0, 1\}^n \to \{0, 1\}$ $(n \geq 1)$ can be expressed in terms of the functions f_i.

Post' functional completeness theorem (1941)

Set of Boolean-functions S is functional complete iff for each of the five classes (linear, monoton, self-dual, 0-preserving, 1-preserving) there is a member of S which does not belong to that class..

Minimal functional complete classes (base)

- signleton: $\{\text{NAND}\}$, $\{\text{NOR}\}$
- two elements: $\{\neg, \wedge\}$, $\{\neg, \vee\}$, $\{\neg, \supset\}$, $\{\bot, \supset\}$
- three elements: $\{\vee, \equiv, \bot\}$, $\{\wedge, \equiv, \bot\}$