Introduction to Logic and Computer Science
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Lecture #7: sequent calculus

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Sequent calculus

Truth tables can be used to determine valid formulas, but if we have too many non-logical constants, it is hard to construct these tables, even for computers. Let's consider a method based on **syntax**.

Definition

If Γ and Δ are two—possibly empty—set of formulae, then $\Gamma \vdash \Delta$ is a sequent.
The axioms of the sequent calculus are $\Gamma \cup \{A\} \vdash \Delta \cup \{A\}$, where A is an atomic formula, Γ and Δ are set of formulae.
Let S be the sequent $\Gamma \vdash \Delta$, where $\Gamma = \{A_1, \ldots, A_n\}$ and $\Delta = \{B_1, \ldots, B_m\}$;
The sequent S is valid, if for every interpretation ϱ where $|A_1|_\varrho = \cdots = |A_n|_\varrho = 1$ then $|B_i|_\varrho = 1$ for some i.

Remark

If a sequent is not valid—i.e. falsifiable—then there exists an interpretation ϱ for which $|A_1|_\varrho = \cdots = |A_n|_\varrho = 1$, but $|B_1|_\varrho \cdots = |B_m|_\varrho = 0$.

Inference rules

For the sake of simplicity we write „Γ, A" in the following instead of $\Gamma \cup \{A\}$. In the following rules the upper sequent(s) and the lower sequent are called the premise(s) and the conclusion of the rule, respectively.

$$\frac{\Gamma \vdash \Delta,\ A \qquad \Gamma \vdash \Delta,\ B}{\Gamma \vdash \Delta,\ A \wedge B} \qquad \frac{\Gamma,\ A,\ B \vdash \Delta}{\Gamma,\ A \wedge B \vdash \Delta}$$

$$\frac{\Gamma \vdash \Delta,\ A,\ B}{\Gamma \vdash \Delta,\ A \vee B} \qquad \frac{\Gamma,\ A \vdash \Delta \qquad \Gamma,\ B \vdash \Delta}{\Gamma,\ A \vee B \vdash \Delta}$$

$$\frac{\Gamma,\ A \vdash \Delta,\ B}{\Gamma \vdash \Delta,\ A \supset B} \qquad \frac{\Gamma \vdash \Delta,\ A \qquad \Gamma,\ B \vdash \Delta}{\Gamma,\ A \supset B \vdash \Delta}$$

$$\frac{\Gamma,\ A \vdash \Delta}{\Gamma \vdash \Delta,\ \neg A} \qquad \frac{\Gamma \vdash \Delta,\ A}{\Gamma,\ \neg A \vdash \Delta}$$

Sound inference rules

Theorem

If the premise(s) of the rule is/are valid, then its conclusion is valid.

Lemma

If some interpretation falsifies the conclusion of the rule, then this interpretation falsifies some of its premise(s).

Theorem

The axioms of the sequent calculus are valid.

Proof

If ϱ is model of the first set of formulae of the sequent, then it is the model of set of these formulae. At least formulae from this set occurs in the second set of formulae, so the sequent is valid.

Inference tree

The inference tree is such a tree, whose nodes contains a sequent, and the edges correspond to the inference rules.

Definition

- Let $\Gamma \vdash \Delta$ be a sequent. This sequent alone is an inference tree, for whose this sequent is its root and its sole leaf.
- Let F be an inference tree, for which one of the leaves contains the sequent $\Gamma \vdash \Delta$.
- Let assume that there exist an inference rule, whose conclusion is $\Gamma \vdash \Delta$, and its premises are $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$
- Let us extend the tree F such that the leaf $\Gamma \vdash \Delta$ is the root of a subtree made up of the sequents $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$!
- Then we get the inference tree F' whose root is the same as the root of F, and its leaves are the leaves of F (minus $\Gamma \vdash \Delta$) and the sequents $\Gamma_1 \vdash \Delta_1, \ldots, \Gamma_n \vdash \Delta_n$.

Proof

Definition

An inference tree is a proof, if all of its leaves are axioms.

Soundness

Theorem (soundness)

The root sequent of a proof is valid.

Proof

Let's assume (indirect) that we have a proof and its root is falsifiable.

- As the root is falsifiable, at least one of its premises is falsifiable.
- By using induction on structure, we get that the tree with falsifiable root has a falsifiable leaf.
- An inference tree with falsifiable leaf is not a proof, and so we have a contradiction.

Completeness

Theorem (completeness)

If sequent S is valid, then there exists a proof with root S.

Lemma

If a sequent contains only atomic formulae, then it is valid iff it is an axiom.

Lemma

Any inference tree can be extended to inference tree with leaves containing only atomic formulae.

Lemma

Any inference tree with valid root has only valid leaves.

Completeness - 2

Proof

- Let S be a valid sequent.
- Extend this root into a maximal inference tree
- All the leaves of the maximal inference tree are valid and contain only atomic formulae.
- Hence all the leaves are axioms, so the tree is a proof.

Existence of falsifying model

Theorem

If the sequent S is not provable, then there exists an interpretation which falsifies it.

Proof

- Let's construct some of the maximals inference tree of S.
- As S is not provable, there exists a sequent in some leaf, which is not an axiom: $A_1, \ldots, A_n \vdash B_1, \ldots, B_m$, where $A_i \neq B_j$ for all i, j.
- Let's define the interpretation ϱ in such way that $|A_i|_\varrho = 1$ and $|B_j|_\varrho = 0$.
- ϱ falsifies the sequent in the leaf, and hence in the root too.