Introduction to Logic and Computer Science
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Lecture #8: First-order logic, semantics

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Syntax

Definition

The language of first-order logic is a

$$L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$$

ordered 5-tuple, where

1. $LC = \{\neg, \supset, \wedge, \vee, \equiv, =, \forall, \exists, (,)\}$
   - the set of logical constants.
2. $Var = \{x_n : n = 0, 1, 2, \dots\}$
   - countable infinite set of variables
3. $Con = \bigcup_{n=0}^{\infty} \left(\mathcal{F}(n) \cup \mathcal{P}(n)\right)$
   - the set of non-logical constants (at best countably infinite)
   - $\mathcal{F}(0)$: the set of name parameters,
   - $\mathcal{F}(n)$: the set of n argument function parameters,
   - $\mathcal{P}(0)$: the set of proposition parameters,
   - $\mathcal{P}(n)$: the set of predicate parameters.

Syntax (cont.)

Definition

4. The sets *LC*, *Var*, $\mathcal{F}(n)$, $\mathcal{P}(n)$ are pairwise disjoint $(n = 0, 1, 2, \dots)$.
5. The set of terms—i.e. the set *Term*—is given by the following inductive definition:
   - $Var \cup \mathcal{F}(0) \subseteq Term$
   - If $f \in \mathcal{F}(n)$, $(n = 1, 2, \dots)$, t_1 ,t_2, $\dots$, $t_n \in Term$, then $f(t_1, t_2, ..., t_n) \in Term$.
6. The set of formulas—i.e. the set *Form*—is given by the following inductive definition:
   - $\mathcal{P}(0) \in Form$
   - If t_1, $t_2 \in Term$, then $(t_1 = t_2) \in Form$
   - If $P \in \mathcal{P}(n)$, $(n = 1, 2, \dots)$, and t_1, t_2,$\dots$, $t_n \in Term$, then $P(t_1, t_2, \dots, t_n) \in Form$.
   - If $A \in Form$, then $\neg A \in Form$.
   - If $A, B \in Form$, then $(A \supset B)$, $(A \wedge B)$, $(A \vee B)$, $(A \equiv B) \in Form$.
   - If $x \in Var$, $A \in Form$, then $\forall x A$, $\exists x A \in Form$.

Atomic formulae

Remark

The formulae constructed by first three points of rule 6 is called an atomic or prime formulae.

Semantics of classical first-order logic

Definition (interpretation)

The ordered pair $\langle U, \varrho \rangle$ is an interpretation of the language $L^{(1)}$ if

- $U \neq \emptyset$ (i.e. U is a nonempty set);
- $Dom(\varrho) = Con$
  - If $a \in \mathcal{F}(0)$, then $\varrho(a) \in U$;
  - If $f \in \mathcal{F}(n)$ $(n \neq 0)$, then $varrho(f) \in U^{U^{(n)}}$
  - If $p \in \mathcal{P}(0)$, then $\varrho(p) \in \{0, 1\}$;
  - If $P \in \mathcal{P}(n)$ $(n \neq 0)$, then $\varrho(P) \subseteq U^{(n)}$ $(\varrho(P) \in \{0, 1\}^{U^{(n)}})$.

Assignment

Definition (assignment)

The function v is an assignment relying on the interpretation $\langle U, \varrho \rangle$ if the followings hold:

- $Dom(v) = Var$;
- If $x \in Var$, then $v(x) \in U$.

Definition (modified assignment)

Let v be an assignment relying on the interpretation $\langle U, \varrho \rangle$, $x \in Var$ and $u \in U$. For all $y \in Var$

$$v[x : u](y) = \begin{cases} u, & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$$

Semantic rules

Definition (value of a term)

Let $\langle U, \varrho \rangle$ be a given interpretation and v be an assignment relying on $\langle U, \varrho \rangle$.

- If $a \in \mathcal{F}(0)$, then $|a|_v^{\langle U,\varrho \rangle} = \varrho(a)$.
- If $x \in Var$, then $|x|_v^{\langle U,\varrho \rangle} = v(x)$.
- If $f \in \mathcal{F}(n)$ $(n = 1, 2, ...)$, and t_1 ,$t_2, \ldots, t_n \in Term$, then $|f(t_1, t_2, ..., t_n)|_v^{\langle U,\varrho \rangle} = \varrho(f) \left(|t_1|_v^{\langle U,\varrho \rangle}, |t_2|_v^{\langle U,\varrho \rangle}, ..., |t_n|_v^{\langle U,\varrho \rangle} \right)$

Semantic rules (cont.)

Definition (value of an atomic formula)

- If $p \in \mathcal{P}(0)$, then $|p|_v^{\langle U,\varrho\rangle} = \varrho(p)$
- If $t_1, t_2 \in \mathit{Term}$, then $|(t_1 = t_2)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } |t_1|_v^{\langle U,\varrho\rangle} = |t_2|_v^{\langle U,\varrho\rangle} \\ 0 & \text{otherwise} \end{cases}$
- If $P \in \mathcal{P}(n)$ $(n = 1, 2, ...)$, $t_1, t_2, \ldots, t_n \in \mathit{Term}$, then $|P(t_1, t_2, \ldots, t_n)|_v^{\langle U,\varrho\rangle} =$ $\begin{cases} 1, & \text{if } \left\langle |t_1|_v^{\langle U,\varrho\rangle}, |t_2|_v^{\langle U,\varrho\rangle}, ..., |t_n|_v^{\langle U,\varrho\rangle} \right\rangle \in \varrho(P) \\ 0 & \text{otherwise} \end{cases}$

Semantic rules (cont.)

Definition (value of a formula)

- If $A \in Form$, then $|\neg A|_v^{\langle U,\varrho\rangle} = 1 - |A|_v^{\langle U,\varrho\rangle}$.
- If A, $B \in Form$, then
  - $|(A \supset B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0, & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$
  - $|(A \wedge B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } |A|_v^{\langle U,\varrho\rangle} = 1, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$
  - $|(A \vee B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 0, & \text{if } |A|_v^{\langle U,\varrho\rangle} = 0, \text{ and } |B|_v^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$
  - $|(A \equiv B)|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if } |A|_v^{\langle U,\varrho\rangle} = |B|_v^{\langle U,\varrho\rangle}; \\ 0, & \text{otherwise.} \end{cases}$

Semantic rules (cont.)

Definition (value of a quantor)

- If $A \in Form$, and $x \in Var$ then
  - $$|\forall x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 0, & \text{if there is an } u \in U, st. |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 0; \\ 1, & \text{otherwise.} \end{cases}$$
  - $$|\exists x A|_v^{\langle U,\varrho\rangle} = \begin{cases} 1, & \text{if there is an } u \in U, st. |A|_{v[x:u]}^{\langle U,\varrho\rangle} = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Models

Definition (model—a set of formulas)

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $\Gamma \subseteq Form$ be a set of formulas. An ordered triple $\langle U, \varrho, v \rangle$ is a model of the set Γ, if

- $\langle U, \varrho \rangle$ is an interpretation of $L^{(1)}$;
- v is an assignment relying on $\langle U, \varrho \rangle$;
- $|A|_v^{\langle U, \varrho \rangle} = 1$ for all $A \in \Gamma$.

Definition (a model of a formula)

A model of a formula A is the model of the singleton $\{A\}$.

Satisfiable

Definition (satisfiable a set of formulas)

The set of formulae $\Gamma \subseteq \mathit{Form}$ is satisfiable if it has a model.

- If there is an interpretation and an assignment in which all members of the set Γ are true.

Definition (satisfiable a formula)

A formula $A \in \mathit{Form}$ is satisfiable, if the singleton $\{A\}$ is satisfiable.

Remark

- A satisfiable set of formulae does not involve a logical contradiction; its formulae may be true together.
- A satisfiable formula may be true.
- If a set of formulae is satisfiable, then its members are satisfiable.
- But: all members of the set $\{P(a), \neg P(a)\}$ are satisfiable, and the set is not satisfiable.

Subset property of satisfiability

Theorem

All subsets of a satisfiable set are satisfiable.

Proof

- Let $\Gamma \subseteq Form$ be a set of formulae and $\Delta \subseteq \Gamma$.
- Γ is satisfiable: it has a model.
- Let $\langle U, \varrho, v \rangle$ be a model of Γ.
- A property of $\langle U, \varrho, v \rangle$:
  - If $A \in \Gamma$, then $|A|_v^{\langle U, \varrho \rangle} = 1$
- Since $\Delta \subseteq \Gamma$, if $A \in \Delta$, then $A \in \Gamma$, and so $|A|_v^{\langle U, \varrho \rangle} = 1$
  - That is the ordered triple $\langle U, \varrho, v \rangle$ is a model of Δ, and so Δ is satisfiable.

Unsatisfiable

Definition (unsatisfiable set)

The set $\Gamma \subseteq Form$ is unsatisfiable if it is not satisfiable.

Definition (unsatisfiable formula)

A formula $A \in Form$ is unsatisfiable if the singleton $\{A\}$ is unsatisfiable.

Remark

- A unsatisfiable set of formulae involve a logical contradiction.
  - Its members cannot be true together.

Superset property of unsatisfiability

Theorem

All expansions of an unsatisfiable set of formulae are unsatisfiable.

Proof (indirect)

- Suppose that $\Gamma \subseteq Form$ is an unsatisfiable set of formulae and $\Delta \subseteq Form$ is a set of formulas.
- Indirect condition: Γ is unsatisfiable, and $\Gamma \cup \Delta$ satisfiable.
- $\Gamma \subseteq \Gamma \cup \Delta$
- According to the former theorem Γ is satisfiable, and it is a contradiction.

Logical consequence

Definition

A formula A is the logical consequence of the set of formulae Γ if the set $\Gamma \cup \{\neg A\}$ is unsatisfiable. (Notation: $\Gamma \models A$)

Definition

$A \models B$, if $\{A\} \models B$.

Definition

The formula A is valid if $\emptyset \models A$. (Notation: $\models A$)

Definition

The formulae A and B are logically equivalent if $A \models B$ and $B \models A$. (Notation: $A \Leftrightarrow B$)

Properties of the logical consequence

Theorem

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if all models of the set Γ are models of formula A (i.e. the singleton $\{A\}$).

Proof

$\rightarrow$ Indirect condition: There is a model of Γ such that it is not a model of the formula A.

- Let the ordered triple $\langle U, \varrho, v\rangle$ be this model.
- The properties of $\langle U, \varrho, v\rangle$:
  1. $|B|_v^{\langle U,\varrho\rangle} = 1$ for all $B \in \Gamma$;
  2. $|A|_v^{\langle U,\varrho\rangle} = 0$, and so $|\neg A|_v^{\langle U,\varrho\rangle} = 1$
- In this case all members of the set $\Gamma \cup \{\neg A\}$ are true wrt the interpretation $\langle U, \varrho\rangle$ and assignment v, so $\Gamma \cup \{\neg A\}$ is satisfiable.
- It means that $\Gamma \not\models A$, and so it is a contradiction.

Properties of the logical consequence - 2

Proof

$\leftarrow$ Indirect condition: All models of the set Γ are models of formula A, but (and) $\Gamma \not\models A$.

- In this case $\Gamma \cup \{\neg A\}$ is satisfiable, i.e. it has a model.
- Let the ordered $\langle U, \varrho, v\rangle$ be a model.
- The properties of $\langle U, \varrho, v\rangle$:
  1. $|B|_v^{\langle U,\varrho\rangle} = 1$ for all $B \in \Gamma$;
  2. $|\neg A|_v^{\langle U,\varrho\rangle} = 1$ i.e. $|A|_v^{\langle U,\varrho\rangle} = 0$
- So the set Γ has a model such that it is not a model of formula A, and it is a contradiction.

Corollary

Let $\Gamma \subseteq Form$, and $A \in Form$. $\Gamma \models A$ if and only if for all interpretations in which all members of Γ are true, the formula A is true.

Properties of the logical consequence - 3

Theorem

If A is a valid formula ($\models A$), then $\Gamma \models A$ for all sets of formulae Γ.

A valid formula is a consequence of any set of formulas.

Proof

- If A is a valid formula, then $\emptyset \models A$ (according to its definition).
- $\emptyset \cup \{\neg A\} = \{\neg A\}$ is unsatisfiable, and so its expansions are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of $\{\neg A\}$, and so it is unsatisfiable, i.e. $\Gamma \models A$.

Properties of the logical consequence - 4

Theorem

If Γ is unsatisfiable, then $\Gamma \models A$ for all A.
All formulae are the consequences of an unsatisfiable set of formulas.

Proof

- According to a proved theorem:
- If Γ is unsatisfiable, the all expansions of Γ are unsatisfiable.
- $\Gamma \cup \{\neg A\}$ is an expansion of Γ, and so it is unsatisfiable, i.e. $\Gamma \models A$.

Deduction theorem

Theorem

If $\Gamma \cup \{A\} \models B$, then $\Gamma \models (A \supset B)$.

Proof

- Indirect condition: Suppose, that $\Gamma \cup \{A\} \models B$, and $\Gamma \not\models (A \supset B)$.
- $\Gamma \cup \{\neg(A \supset B)\}$ is satisfiable, and so it has a model.
- Let the ordered triple $\langle U, \varrho, v\rangle$ be a model.
- The properties of $\langle U, \varrho, v\rangle$:
  1. All members of Γ are true wrt $\langle U, \varrho\rangle$ and v.
  2. $|\neg(A \supset B)|_v^{\langle U,\varrho\rangle} = 1$
- So $|(A \supset B)|_v^{\langle U,\varrho\rangle} = 0$, i.e. $|A|_v^{\langle U,\varrho\rangle} = 1$ and $|B|_v^{\langle U,\varrho\rangle} = 0$.
- Hence $|\neg B|_v^{\langle U,\varrho\rangle} = 1$.
- All members of $\Gamma \cup \{A\} \cup \{\neg B\}$ are true wrt $\langle U, \varrho\rangle$ and v, i.e. $\Gamma \cup A \not\models B$, and so it is a contradiction.

Deduction theorem, opposite direction

Theorem

If $\Gamma \models (A \supset B)$, then $\Gamma \cup \{A\} \models B$.

Proof

- Indirect condition: Suppose that $\Gamma \models (A \supset B)$, and $\Gamma \cup \{A\} \not\models B$.
- So $\Gamma \cup \{A\} \cup \{\neg B\}$ is satisfiable, i.e. it has a model.
- Let the ordered triple $\langle U, \varrho, v\rangle$ be a model.
- The properties of $\langle U, \varrho, v\rangle$:
  1. All members of Γ are true wrt $\langle U, \varrho\rangle$ and v.
  2. $|A|_v^{\langle U,\varrho\rangle} = 1$
  3. $|\neg B|_v^{\langle U,\varrho\rangle} = 1$, and so $|B|_v^{\langle U,\varrho\rangle} = 0$
- Hence $|(A \supset B)|_v^{\langle U,\varrho\rangle} = 0$, so $|\neg(A \supset B)|_v^{\langle U,\varrho\rangle} = 1$.
- All members of $\Gamma \cup \{\neg(A \supset B)\}$ are true wrt $\langle U, \varrho\rangle$ and v, i.e. $\Gamma \not\models (A \supset B)$.

Properties of the logical consequence - 5

Corollary (of deduction theorem)

$A \models B$ if and only if $\models (A \supset B)$

Proof

Let $\Gamma = \emptyset$ in the former theorems.

Cut elimination theorem

If $\Gamma \cup \{A\} \models B$ and $\Delta \models A$, then $\Gamma \cup \Delta \models B$.

Proof

Indirect.