Introduction to Logic and Computer Science
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Lecture #9: First-order logic, syntax

Tamás Mihálydeák, László Aszalós

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Free variables

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and let $A \in Form$ be a formula. The set of free variables of the formula A—in notation: $FreeVar(A)$—is given by the following inductive definition:

- If A is an atomic formula—i.e. $A \in AtForm$—then the members of the set $FreeVar(A)$ are the variables occurring in A.
- If the formula A is $\neg B$, then $FreeVar(A) = FreeVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, then $FreeVar(A) = FreeVar(B) \cup FreeVar(C)$.
- If the formula A is $\forall x B$ or $\exists x B$, then $FreeVar(A) = FreeVar(B) \setminus \{x\}$.

Bound variables

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and let $A \in Form$ be a formula. The set of bound variables of the formula A—in notation: $BoundVar(A)$—is given by the following inductive definition:

- If A is an atomic formula—i.e. $A \in AtForm$—then $BoundVar(A) = \emptyset$.
- If the formula A is $\neg B$, then $BoundVar(A)$ = BoundVar(B)$.
- If the formula A is $(B \supset C)$, $(B \wedge C)$, $(B \vee C)$ or $(B \equiv C)$, $BoundVar(A) = BoundVar(B) \cup BoundVar(C)$.
- If the formula A is $\forall xB$ or $\exists xB$, then $BoundVar(A) = BoundVar(B) \cup x$.

Properties

Remark

- The bases of inductive definitions of sets of free and bound variables are given by the first requirement of the corresponding definitions.
- The sets of free and bound variables of a formula are not necessarily disjoint:
  $FreeVar((P(x) \wedge \exists x R(x))) = \{x\} = BoundVar((P(x) \wedge \exists x R(x)))$

Fixed occurences

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, and $x \in Var$ be a variable.

- A fixed occurrence of the variable x in the formula A is free if it is not in the subformulae $\forall xB$ or $\exists xB$ of the formula A.
- A fixed occurrence of the variable x in the formula A is bound if it is not free.

Properties - 2

Remark

- If x is a free variable of the formula A—i.e. $x \in FreeVar(A)$—then it has at least one free occurrence in A.
- If x is a bound variable of the formula A—i.e. $x \in BoundVar(A)$—then it has at least one bound occurrence in A.
- A fixed occurrence of a variable x in the formula A is free if it does not follow a universal or an existential quantifier, or it is not in a scope of a $\forall x$ or a $\exists x$ quantification.
- A variable x may be a free and a bound variable of the formula A:

$$(P(x) \wedge \exists x R(x))$$

Open and closed formulae

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.

- If $FreeVar(A) \neq \emptyset$, then the formula A is an open formula.
- If $FreeVar(A) = \emptyset$, then the formula A is a closed formula.

Remark

- The formula A is open if there is at least one variable which has at least one free occurrence in A.
- The formula A is closed if there is no variable which has a free occurrence in A.

de Morgan laws

Remark

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x | inVar$ be a variable. Then

- $\neg \exists x A \Leftrightarrow \forall x \neg A$
- $\neg \forall x A \Leftrightarrow \exists x \neg A$

Expressibilty of quantifications

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x | inVar$ be a variable. Then

- $\exists x A \Leftrightarrow \neg \forall x \neg A$
- $\forall x A \Leftrightarrow \neg \exists x \neg A$

Quantifications and operators

Conjunction

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A, B \in Form$ be formulae and $x \in Var$ be a variable. If $x \notin FreeVar(A)$, then

- $A \wedge \forall x B \Leftrightarrow \forall x(A \wedge B)$
- $A \wedge \exists x B \Leftrightarrow \exists x(A \wedge B)$

Remark

According to the commutativity of conjunction the followings hold: If $x \notin FreeVar(A)$, then

- $\forall x B \wedge A \Leftrightarrow \forall x(B \wedge A)$
- $\exists x B \wedge A \Leftrightarrow \exists x(B \wedge A)$

Quantifications and operators - 2

Disjunction

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A, B \in Form$ be formulae and $x \in Var$ be a variable. If $x \notin FreeVar(A)$, then

- $A \vee \forall x B \Leftrightarrow \forall x(A \vee B)$
- $A \vee \exists x B \Leftrightarrow \exists x(A?B)$

Remark

According to the commutativity of disjunction the followings hold: If $x \notin FreeVar(A)$, then

- $\forall x B \vee A \Leftrightarrow \forall x(B \vee A)$
- $\exists x B \vee A \Leftrightarrow \exists x(B \vee A)$

Quantifications and operators - 3

Implication with existential quantification

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A, B \in Form$ be formulae and $x \in Var$ be a variable. If $x \notin FreeVar(A)$, then

- $A \supset \exists x B \Leftrightarrow \exists x (A \supset B)$
- $\exists x B \supset A \Leftrightarrow \forall x (B \supset A)$

Implication with universal quantification

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A, B \in Form$ be formulae and $x \in Var$ be a variable. If $x \notin FreeVar(A)$, then

- $A \supset \forall x B \Leftrightarrow \forall x (A \supset B)$
- $\forall x B \supset A \Leftrightarrow \exists x (B \supset A)$

Substitutabily a variable with an other variable

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula and $x, y \in Var$ be variables. The variable x is substitutable with the variable y in the formula A if there is no free occurrence of x in A which is in the subformulae $\forall yB$ or $\exists yB$ of A.

Example

In the formula $\forall z P(x, z)$ the variable x is substitutable with the variable y, but x is not substitutable with the variable z.

Substitutabily a variable with a term

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, $x \in Var$ be a variable and $t \in Term$ be a term.
The variable x is substitutable with the term t in the formula A if in the formula A the variable x is substitutable with all variables occurring in the term t.

Example

In the formula $\forall z P(x, z)$ the variable x is substitutable with the term $f(y_1, y_2)$, but x is not substitutable with the term $f(y, z)$.

Result of a substitution

Definition

If the variable x is substitutable with the term t in the formula A, then $[A]_x^t$ denotes the formula which appears when all free occurrences of the variable x in A are substituted with the term t.

Renaming

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, $A \in Form$ be a formula, and $x, y \in Var$ be variables. If the variable x is substitutable with the variable y in the formula A and $y \notin FreeVar(A)$, then

- the formula $\forall y[A]_x^y$ is a regular renaming of the formula $\forall xA$;
- the formula $\exists y[A]_x^y$ is a regular renaming of the formula $\exists xA$.

Congruent formulae

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language, and $A \in Form$ be a formula. The set $Cong(A)$—the set of formulae which are congruent with A—is given by the following inductive definition:

- $A \in Cong(A)$;
- if $\neg B \in Cong(A)$ and $B' \in Cong(B)$, then $\neg B' \in Cong(A)$;
- if $(B \circ C) \in Cong(A)$, $B' \in Cong(B)$ and $C' \in Cong(C)$, then $(B' \circ C') \in Cong(A)$ (where $circ \in \{\supset, \wedge, \vee, \equiv\}$);
- if $\forall xB \in Cong(A)$ and $\forall y[B]_x^y$ is a regular renaming of the formula $\forall xB$, then $\forall y[B]_x^y \in Cong(A)$;
- if $\exists xB \in Cong(A)$ and $\exists y[B]_x^y$ is a regular renaming of the formula $\exists xB$, then $\exists y[B]_x^y \in Cong(A)$;

Syntactical synonym

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and let A, $B \in Form$ be formulae.

- If $B \in Cong(A)$, then the formula A is congruent with the formula B.
- If $B \in Cong(A)$, then the formula B is a syntactical synonym of the formula A.

Theorem

Congruent formulae are logically equivalent, i.e. if $B \in Cong(A)$, then $A \Leftrightarrow B$.

Standardised formula

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula. The formula A is standardised if

- $FreeVar(A) \cap BoundVar(A) = \emptyset$;
- all bound variables of the formula A have exactly one occurrence next to a quantifier.

Theorem

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula. Then there is a formula $B \in Form$ such that

- the formula B is standardised;
- the formula B is congruent with the formula A, i.e. $B \in Cong(A)$.

Prenex form

Definition

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula. The formula A is prenex if

- there is no quantifier in A or
- the formula A is in the form $Q_1 x_1 Q_2 x_2 \dots Q_n x_n B$ $(n = 1, 2, ...)$ where
  - there is no quantifier in the formula $B \in Form$;
  - $x_1, x_2, \dots, x_n \in Var$ are different variables;
  - $Q_1, Q_2, \dots, Q_n \in \{\forall, \exists\}$ are quantifiers.

Theorem

Let $L^{(1)} = \langle LC, Var, Con, Term, Form \rangle$ be a first order language and $A \in Form$ be a formula.
Then there exists a formula $B \in Form$ such that

- the formula B is prenex;
- $A \Leftrightarrow B$.