Mathematics for Engineers

Pál Burai

Complex numbers

This work was supported by the construction EFOP-3.4.3-16-2016-00021. The project was supported by the European Union, co-financed by the European Social Fund.

The imaginary unit

Let us solve the following equation:

$$x^2 + 1 = 0$$

The imaginary unit

Let us solve the following equation:

$$x^2 + 1 = 0$$

The number which square is -1 is called **imaginary unit** and it is denoted by i, in other words

$$i^2 = i \cdot i = -1.$$

The imaginary unit

Let us solve the following equation:

$$x^2 + 1 = 0$$

The number which square is -1 is called **imaginary unit** and it is denoted by i, in other words

$$i^2 = i \cdot i = -1.$$

Algebraic form of complex numbers

Let a, b be real numbers, then

$$z = a + ib$$

is said to be a **complex number** in **algebraic form**, where a is the **real part** and b is the **imaginary part**. In notation: $Re(z) = a, Im(z) = b$.

The imaginary unit

Let us solve the following equation:

$$x^2 + 1 = 0$$

The number which square is -1 is called **imaginary unit** and it is denoted by i, in other words

$$i^2 = i \cdot i = -1.$$

Algebraic form of complex numbers

Let a, b be real numbers, then

$$z = a + ib$$

is said to be a **complex number** in **algebraic form**, where a is the **real part** and b is the **imaginary part**. In notation: $Re(z) = a, Im(z) = b$.

The set of complex numbers is denoted by $\mathbb{C}$.

Exercise

Solve the following equations, and plot the solutions on the complex plane!

(a) $x^3 + 7x = 0$,

(b) $x^2 - 2x + 5 = 0$.

Exercise

Solve the following equations, and plot the solutions on the complex plane!

(a) $x^3 + 7x = 0$,

(b) $x^2 - 2x + 5 = 0$.

Calculation with complex numbers which are given in algebraic form

Let $z = a + ib, w = u + iv$ be complex numbers, then

$$z + w = (a + u) + i(b + v), \quad z \cdot w = (au - bv) + i(av + bu).$$

Exercise

Solve the following equations, and plot the solutions on the complex plane!

(a) $x^3 + 7x = 0,$ (b) $x^2 - 2x + 5 = 0.$

Calculation with complex numbers which are given in algebraic form

Let $z = a + ib, w = u + iv$ be complex numbers, then

$$z + w = (a + u) + i(b + v), \quad z \cdot w = (au - bv) + i(av + bu).$$

Exercise

Give the algebraic form of the following complex numbers:

(a) $(2 - i)(2 + i), \quad (-2 - 5i)(5 - 2i), \quad (-1 - i)(1 + i)(7 + 6i),$

(b) $\sqrt{2}i(1 + \sqrt{2}i), \quad (1 + i)^3, \quad i^9 + i^7 - i^4 + i^2 - i - 1, \quad i^{2017}.$

Conjugate and multiplicative inverse of complex numbers

If $z = a + ib$, then the complex number

$$\overline{z} = a - ib$$

is called the **conjugate** of z.

Conjugate and multiplicative inverse of complex numbers

If $z = a + ib$, then the complex number

$$\overline{z} = a - ib$$

is called the **conjugate** of z.If $z \neq 0$, then

$$\frac{1}{z} = z^{-1} = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}$$

is the **multiplicative inverse** of z, in other words, $z \cdot z^{-1} = 1$.

Conjugate and multiplicative inverse of complex numbers

If $z = a + ib$, then the complex number

$$\overline{z} = a - ib$$

is called the **conjugate** of z.If $z \neq 0$, then

$$\frac{1}{z} = z^{-1} = \frac{a}{a^2 + b^2} - i\frac{b}{a^2 + b^2}$$

is the **multiplicative inverse** of z, in other words, $z \cdot z^{-1} = 1$.

Exercise

Give the algebraic form of the following complex numbers:

(a) $\overline{2 - i}$, $\quad \overline{(-3 + i)}(1 + i)$,

(b) $\dfrac{-3 + 3i}{1 - i}$, $\quad \dfrac{1 + i}{-i - 3}$.

Absolute value and argument of a complex number

If $z = a + ib$ is identified with the vector (a, b) on the plane, then according to the Pythagorean theorem the length of this vector is

$$|z| = \sqrt{a^2 + b^2},$$

which is called the **absolute value** of z.

Absolute value and argument of a complex number

If $z = a + ib$ is identified with the vector (a, b) on the plane, then according to the Pythagorean theorem the length of this vector is

$$|z| = \sqrt{a^2 + b^2},$$

which is called the **absolute value** of z. If z is different from zero, then the **argument** of z is the angle is the radius of the vector (a, b) with the positive real axis the angle (φ_z). If a and b are positive, then

$$\tan \varphi_z = \frac{b}{a}.$$

The angle can be calculated similarly if a and/or b is non-negative. The zero complex number has no argument!

Absolute value and argument of a complex number

If $z = a + ib$ is identified with the vector (a, b) on the plane, then according to the Pythagorean theorem the length of this vector is

$$|z| = \sqrt{a^2 + b^2},$$

which is called the **absolute value** of z. If z is different from zero, then the **argument** of z is the angle is the radius of the vector (a, b) with the positive real axis the angle (φ_z). If a and b are positive, then

$$\tan \varphi_z = \frac{b}{a}.$$

The angle can be calculated similarly if a and/or b is non-negative.
The zero complex number has no argument!

If φ_z is the argument of z and $|z|$ its length, then it can be written in the form

$$z = |z|(\cos \varphi_z + i \sin \varphi_z),$$

which is called the **polar form** of z.

Exercise

Give the polar form of the following complex numbers!

(a) 1, (b) i, (c) $1 - i$, (d) $-1 - \sqrt{3}i$,

Exercise

Give the polar form of the following complex numbers!

(a) 1, (b) i, (c) $1-i$, (d) $-1-\sqrt{3}i$,

Calculation with complex numbers given in polar form

If $z=|z|(\cos\varphi_z+i\sin\varphi_z)$ and $w=|w|(\cos\varphi_w+i\sin\varphi_w)$, then

$$z\cdot w=|z||w|(\cos(\varphi_z+\varphi_w)+i\sin(\varphi_z+\varphi_w)).$$

If $w\neq 0$, then

$$\frac{z}{w}=\frac{|z|}{|w|}(\cos(\varphi_z-\varphi_w)+i\sin(\varphi_z-\varphi_w)).$$

Exercise

Let us consider $x = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $y = 11\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$. Determine the value of the following expressions!

(a) xy, (b) xy^{-1}, (c) x^3, (d) y^5, (e) $\frac{1}{x}$,

Exercise

Let us consider $x = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $y = 11\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$. Determine the value of the following expressions!

(a) xy, (b) xy^{-1}, (c) x^3, (d) y^5, (e) $\frac{1}{x}$,

nth roots of a complex number.

If $z = |z|(\cos\varphi_z + i\sin\varphi_z)$ is a given complex number and n is a given natural number, then the equation $\zeta^n = z$ has n different complex solutions, which are called the ***n*th roots of** z z. They can be expressed in the following way:

$$\zeta_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi_z + 2k\pi}{n} + i\sin\frac{\varphi_z + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Exercise

Let us consider $x = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ and $y = 11\left(\cos\frac{\pi}{7} + i\sin\frac{\pi}{7}\right)$. Determine the value of the following expressions!

(a) xy, (b) xy^{-1}, (c) x^3, (d) y^5, (e) $\frac{1}{x}$,

nth roots of a complex number.

If $z = |z|(\cos\varphi_z + i\sin\varphi_z)$ is a given complex number and n is a given natural number, then the equation $\zeta^n = z$ has n different complex solutions, which are called the **nth roots of** z z. They can be expressed in the following way:

$$\zeta_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi_z + 2k\pi}{n} + i\sin\frac{\varphi_z + 2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Exercise

Calculate the second, the third and the fourth roots of the complex number $z = 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$! Plot the roots on the complex plain!

Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

Exponential form of complex numbers

If the polar form of a complex number is $z = |z|(\cos\varphi + i\sin\varphi)$, then the form

$$z = |z|e^{i\varphi}$$

is called the **exponential form of z.**

Euler's formula

$$e^{i\varphi} = \cos\varphi + i\sin\varphi.$$

Exponential form of complex numbers

If the polar form of a complex number is $z = |z|(\cos\varphi + i\sin\varphi)$, then the form

$$z = |z|e^{i\varphi}$$

is called the **exponential form of z.**

Multiplication and division in exponential form

Let $z = |z|e^{i\varphi}$ and $w = |w|e^{i\psi}$, then

$$zw = |z||w|e^{i(\varphi+\psi)}, \qquad \text{and} \qquad \frac{z}{w} = \frac{|z|}{|w|}e^{i(\varphi-\psi)}.$$

Raising to a power in exponential form

$z^n = |z|^n e^{in\varphi}.$

Raising to a power in exponential form

$z^n = |z|^n e^{in\varphi}$.

Periodicity

$|z|e^{i\varphi} = |z|e^{i(\varphi+2\pi)}$

Raising to a power in exponential form

$z^n = |z|^n e^{in\varphi}$.

Periodicity

$|z|e^{i\varphi} = |z|e^{i(\varphi+2\pi)}$

Exercises

Calculate the product zw and the ratio $\frac{z}{w}$!

- $z = 2e^{i\frac{\pi}{2}}$, $w = 4e^{i\frac{\pi}{4}}$.
- $z = -3e^{i10}$, $w = 3e^{-i10}$.
- $z = \sqrt{2}e^{i\frac{\pi}{3}}$, $w = \sqrt{18}e^{i\frac{\pi}{6}}$.

Raising to a power in exponential form

$z^n = |z|^n e^{in\varphi}$.

Periodicity

$|z|e^{i\varphi} = |z|e^{i(\varphi+2\pi)}$

Exercises

Calculate the product zw and the ratio $\frac{z}{w}$!

- $z = 2e^{i\frac{\pi}{2}}$, $w = 4e^{i\frac{\pi}{4}}$.
- $z = -3e^{i10}$, $w = 3e^{-i10}$.
- $z = \sqrt{2}e^{i\frac{\pi}{3}}$, $w = \sqrt{18}e^{i\frac{\pi}{6}}$.

Exercises

Calculate the nth power of z!

- $z = \sqrt[3]{5}e^{i\frac{\pi}{6}}$, $n = 3$.
- $z = 2e^{i\frac{\pi}{5}}$, $n = 5$.
- $z = e^{i\frac{\pi}{5}}$, $n = 10$.

Exercises

- Plot the following complex numbers on the complex plane! Determine the real and imaginary parts of them! $z_1 = 2 + 3i$, $z_2 = -10 + 2i$, $z_3 = 10 + 2i$, $z_4 = 2 - 3i$, $z_5 = -2 - 3i$.
- Using Euler's formula, compute $\cos\varphi$ and $\sin\varphi$ and convert to algebraic form the following complex numbers! $e^{i\pi}$, $e^{i\frac{\pi}{3}}$.
- Transform the complex number $z_1 = 1 - i$ into polar, and exponential form. Calculate its fourth power!
- Determine the real and imaginary parts of $\frac{(1+i)^2}{\sqrt{2}(1-i)}$!
- Evaluate the roots of the following quadratic equations:
  - $x^2 + 4x + 13$.
  - $x^2 + \frac{3}{2}x + \frac{25}{16}$.
- Give the exponential form of the third, fourth and fifth roots of unity!