Mathematics for Engineers

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Linear algebra

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Vectors

The ordered pairs of numbers are called two dimensional **vectors**. We denote them with small Latin (boldfaced) letters, e.g **a**, **b**, ... We make a distinction between row vectors and column vectors. If otherwise not stated all the vectors are column vectors.

Examples

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{a}^T = [1, 2], \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}^T = [x_1, x_2]$$

One can define higher dimensional complex or real vectors in a similar way. We denote by $\mathbb{R}^2$, $\mathbb{R}^n$, $\mathbb{C}^n$ the set of two dimensional real, the set of n dimensional real and the set of n dimensional complex vectors respectively.

Examples

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{x}^T = [x_1, \dots, x_n], \quad \mathbf{w} = \begin{bmatrix} 2 + 4i \\ -1 - i \end{bmatrix} \in \mathbb{C}^2.$$

Addition of vectors

We add the same dimensional column vectors (row vectors) coordinatewise.

Examples

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 8 \\ 1 \end{bmatrix}, \qquad [1+i, -1+2i] + [i, 3+3i] = [1+2i, 2+5i],$$

$$\begin{bmatrix} \pi + \sqrt{2}i \\ \sqrt{3} - i \\ 1 - i \end{bmatrix} + \begin{bmatrix} -\pi - \pi i \\ \sqrt[3]{5} + 4i \\ \sin(1) + \cos(1)i \end{bmatrix} = \begin{bmatrix} (\sqrt{2} + \pi)i \\ \sqrt{3} + \sqrt[3]{5} + 3i \\ 1 + \sin(1) + (\cos(1) - 1)i \end{bmatrix}.$$

In general

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Multiplication by a scalar

We multiply a vector by a scalar coordinatewise.

Examples

$$3\begin{bmatrix}-1\\2\\4\end{bmatrix}=\begin{bmatrix}-3\\6\\12\end{bmatrix},\qquad -2\left[10,0,2,3\right]=\left[-20,0,-4,-6\right].$$

$$i\begin{bmatrix}2i\\0\\4\end{bmatrix}=\begin{bmatrix}-2\\0\\4i\end{bmatrix},\qquad (1-i)\left[1,0,1+i\right]=\left[1-i,0,2\right].$$

In general

$$\mathbf{x}=\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}\text{ is a vector, and }\lambda\text{ is a scalar, then}\quad \lambda\cdot\begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}=\begin{bmatrix}\lambda x_1\\\vdots\\\lambda x_n\end{bmatrix}.$$

Properties of vector addition

- Addition of vectors is a **binary operation**, that is to say, the result of an addition of two vectors of the same type will be a vector of the same type.
- The vector addition is **associative**, in other words, if we add two vectors and we add a third one to the result, we get the same result if we add the second and the third, and then we add the first to the sum.
- There is an **additive unit**, which is the full zero vector. It has the property that there is no effect when we add it to an arbitrary vector.
- There is an **additive inverse** of every vector $\mathbf{a}$ denoted by $-\mathbf{a}$ such that $\mathbf{a} + (-\mathbf{a})$.
- Vector addition is **commutative**, which means that changing of the order of the vectors in the additions does not change their sum.

Properties of multiplication by a scalar

- Multiplying an arbitrary vector by zero the result will be the additive unit vector.
- Multiplying an arbitrary vector by one the result will be the vector itself.
- If we multiply a vector by a scalar and we multiply the product again by an another scalar the result will be the same if we multiply the vector by the product of the scalars.
- **Distributivity**: Multiplying by a scalar of the sum of vectors we have the same result if we multiply both vectors by the scalar and then add them. Similarly, if we multiply a vector by the sum of two scalars, the result will be the same if we multiply the vector by both scalars and then add them.

Vector spaces

If a set V posses the previous properties then it is said to be a **vector space**.

Examples

- The plane as a vector space: $\mathbb{R}^2 = \{\mathbf{x} \mid \mathbf{x}^T = (x_1, x_2),\ x_1, x_2 \in \mathbb{R}\ \}$.
- The three dimensional real space:
  $\mathbb{R}^3 = \{\mathbf{x} \mid \mathbf{x}^T = (x_1, x_2, x_3),\ x_1, x_2, x_3 \in \mathbb{R}\ \}$.
- The n dimensional real space:
  $\mathbb{R}^n = \{\mathbf{x} \mid \mathbf{x}^T = (x_1, \ldots, x_n),\ x_1, \ldots, x_n \in \mathbb{R}\ \}$.
- The n complex space:
  $\mathbb{C}^n = \{\mathbf{x} \mid \mathbf{x}^T = (x_1, \ldots, x_n),\ x_1, \ldots, x_n \in \mathbb{C}\ \}$.
- The space of real polynomials with the degree at most n:
  $P_n = \{p \colon \mathbb{R} \to \mathbb{R} \mid p(t) = a_n t^n + \cdots + a_1 t + a_0,\ a_0, \ldots, a_n \in \mathbb{R}\ \}$.
- The space of real polynomials: $\cup_n P_n$.
- The space of continuous functions defined on the same interval.
- The space of n times continuously differentiable functions defined on the same interval.

Length of vectors and inner product

According to the Pythagorean theorem the **length** of the vector $\mathbf{a}^T = (a_1, a_2)$ is $\sqrt{a_1^2 + a_2^2}$. It is denoted by $\|\mathbf{a}\|$, and it is also called the **norm** of $\mathbf{a}$. The norm can be calculated using the **inner product** of vectors, which is defined in the following way:

$$\mathbf{a}^T \cdot \mathbf{b} = [a_1, a_2] \cdot \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = a_1 b_1 + a_2 b_2 \quad \text{in the real case,}$$

and

$$\mathbf{a}^T \cdot \overline{\mathbf{b}} = [a_1, a_2] \cdot \begin{bmatrix} \bar{b_1} \\ \bar{b_2} \end{bmatrix} = a_1 \overline{b_1} + a_2 \overline{b_2} \quad \text{in the complex case.}$$

Then

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^T \cdot \mathbf{a}} \quad \text{in the real case,}$$

and

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}^T \cdot \overline{\mathbf{a}}} \quad \text{in the complex case.}$$

In general

$$\mathbf{x}^T \cdot \mathbf{y} = [x_1, \ldots, x_n] \cdot \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + \cdots + x_n y_n \quad \text{in the real case}$$

and

$$\mathbf{x}^T \cdot \overline{\mathbf{y}} = [x_1, \ldots, x_n] \cdot \begin{bmatrix} \bar{y}_1 \\ \vdots \\ \bar{y}_n \end{bmatrix} = x_1 \overline{y_1} + \cdots + x_n \overline{y_n} \quad \text{in the complex case.}$$

Properties of the norm

- If the norm of a vector is zero, then it must be the zero vector.
- The norm of a vector multiplied by a scalar is equal to the norm of the vector multiplied by the absolute value of the scalar.
- The norm of the sum of two vectors is less then or equal to the sum of the norms of the vectors.

Geometric interpretation of the inner product

The inner or scalar product of two vectors is equal to the product of their magnitude and the cosine of the angle between their directions:

$$x^T \cdot y = \|x\| \cdot \|y\| \cos \alpha_{x,y}.$$

So, the scalar product of two vectors x and y is equal to the product of the magnitude of vector x with the projection of y on x.

Corollary

Two vectors are **perpendicular** if and only if their inner product is zero.

Exercise

Let $x^T = (1, 2)$. Find a vector y which is perpendicular to x! Can you characterize all such ys?

Properties of the inner product

Real case:

- Symmetry: $x^T \cdot y = y^T \cdot x$.
- Distributivity: $(x + y)^T \cdot z = x^T \cdot z + y^T \cdot z$.
- Homogeneity $(\lambda x)^T \cdot y = \lambda(x^T \cdot y)$.
- Positive definiteness: $x^T \cdot x \geq 0$ for any x, and $x^T \cdot x = 0$ if and only if $x = 0$.

Exercise

Find the properties in the complex case!

Exercises

Let

$$\mathbf{a} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 0 \\ -1 \\ 7 \end{pmatrix},$$

$$\mathbf{u} = \begin{pmatrix} -1+i \\ 2i \end{pmatrix}, \mathbf{v} = \begin{pmatrix} \sqrt{2}+i \\ i^2-3i \end{pmatrix}, \mathbf{w} = \begin{pmatrix} i-1 \\ i-2 \\ 1 \end{pmatrix}, \mathbf{z} = \begin{pmatrix} 0 \\ -1 \\ 7 \end{pmatrix}.$$

Find the following vectors!

$$\mathbf{a}+\mathbf{b}, \quad -5\mathbf{c}, \quad 12\mathbf{a}+4\mathbf{b}, \quad 3\mathbf{c}+\mathbf{d}, \quad \|\mathbf{b}\|, \quad \|\mathbf{c}+\mathbf{d}\|, \quad \mathbf{u}-i\mathbf{v},$$

$$(3+2i)\mathbf{w}-i\mathbf{z}, \quad \mathbf{v}-3i\mathbf{u}, \quad \|\mathbf{w}\|, \quad \|\mathbf{z}\|, \quad \|i\mathbf{z}+w\|.$$

Linear combination of vectors, linear dependency and independency

Let $\mathbf{x},\ \mathbf{y}$ be vectors of the same type, and $\alpha,\ \beta$ be scalars, then the vector

$$\alpha\mathbf{x} + \beta\mathbf{y}$$

is called a **linear combination** of $\mathbf{x},\ \mathbf{y}$ with the coefficients $\alpha,\ \beta$.

Examples

$$2\begin{bmatrix}2\\1\\-1\end{bmatrix} + 4\begin{bmatrix}1\\4\\0\end{bmatrix} = \begin{bmatrix}8\\18\\-2\end{bmatrix}, \quad i\begin{bmatrix}2-i\\1+i\end{bmatrix} + (1+i)\begin{bmatrix}1\\2+2i\end{bmatrix} = \begin{bmatrix}2+3i\\-1+5i\end{bmatrix}$$

In general, if $\mathbf{x}_1, \dots, \mathbf{x}_n$ is of the same type, and $\alpha_1, \dots \alpha_n$ are scalars, then the expression

$$\alpha_1\mathbf{x}_1 + \dots + \alpha_n\mathbf{x}_n$$

is said to be the **linear combination** of the vectors $\mathbf{x}_1, \dots, \mathbf{x}_n$ with the coefficients $\alpha_1, \dots \alpha_n$.

Linear combination of vectors, linear dependency and independency

Linear combination is not necessarily unique in every cases, e.g.:

$$2\begin{bmatrix}2\\1\\-1\end{bmatrix}+4\begin{bmatrix}1\\4\\0\end{bmatrix}+0\begin{bmatrix}3\\5\\-1\end{bmatrix}=\begin{bmatrix}8\\18\\-2\end{bmatrix},\quad 0\begin{bmatrix}2\\1\\-1\end{bmatrix}+2\begin{bmatrix}1\\4\\0\end{bmatrix}+2\begin{bmatrix}3\\5\\-1\end{bmatrix}=\begin{bmatrix}8\\18\\-2\end{bmatrix}$$

In this case, the zero vector can be written as a linear combinations of the vectors with not only zero coefficients.

$$\begin{bmatrix}2\\1\\-1\end{bmatrix}+\begin{bmatrix}1\\4\\0\end{bmatrix}-\begin{bmatrix}3\\5\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

In this case we say that the zero vector is a **non trivial linear combination** of the vectors.

Linear dependency and independency

If the zero vector can be written as a non trivial linear combination of a system of vector, then this system is called a **linearly dependent system**. Otherwise, it is called a **linearly independent system**.

Generating set, basis, dimension

A system of vectors is called a **generating set** of a vector space, if all the vectors of the space can be written as a linear combination of vectors from the generating set. In this case we say that this system **generate**s the space or the space **is generated** by the system.

Example

The vectors $a^T = [2, 0, 0],\ b^T = [1, 2, 0]$ do not generate the space $\mathbb{R}^3$.

A vector space is said to be **finitely generated**, if it has a finite generating set.

Example

The vectors $a^T = [2, 0, 0],\ b^T = [1, 2, 0],\ c^T = [2, 1, -1],\ d^T = [1, 1, 1]$ generate $\mathbb{R}^3$, however this set is not linearly independent.

If a generating set is linearly independent then it is called a **basis** of the space. If a vector space is finitely generated, then all its bases contains the same number of vectors, this common number is called the **dimension** of the space.

Orthogonal basis

In $\mathbb{R}^n$ and in $\mathbb{C}^n$ besides the vector space structure there is an inner product. Such spaces are called **Euclidean space**s.

If $b_1, \ldots, b_n$ is a basis in an n dimensional Euclidean space, then it is called **orthogonal basis** if b_i is perpendicular to b_j for all $i, j = 1, \ldots, n,\ i \neq j$.

An orthogonal basis is called **orthonormal basis** if $\|b_i\| = 1,\ i = 1, \ldots, n$.

Theorem

In any n-dimensional Euclidean space, there exists an orthonormal basis.

Gramm-Schmidt orthogonalization in $\mathbb{R}^3$

Let b_1, b_2, b_3 be a basis in $\mathbb{R}^3$, then the vectors

$$q_1 = \frac{b_1}{\|b_1\|}, \qquad \bar{q}_2 = b_2 - (b_2^T \cdot q_1)q_1, \qquad q_2 = \frac{\bar{q}_2}{\|\bar{q}_2\|},$$

$$\bar{q}_3 = b_3 - (b_3^T \cdot q_1)q_1 - (b_3^T \cdot q_2)q_2, \qquad q_3 = \frac{\bar{q}_3}{\|\bar{q}_3\|}$$

constitute an orthonormal basis in $\mathbb{R}^3$.

Exercises

- Calculate the products $q_1^T \cdot q_2$, $q_1^T \cdot q_3$, $q_2^T \cdot q_3$!
- Apply the Gramm-Schmidt orthogonalization for the following vectors! Check the orthogonality of the resulting systems!
  - $b_1^T = (1,1)$, $b_2^T = (1,2)$.
  - $b_1^T = (-1,0)$, $b_2^T = (-1,1)$.
  - $b_1^T = (0,0,1)$, $b_2^T = (0,1,1)$, $b_3^T = (1,0,1)$.
  - $b_1^T = (1,2,2)$, $b_2^T = (-1,0,2)$, $b_3^T = (0,0,1)$.
  - $b_1^T = (0,1,2)$, $b_2^T = (1,1,2)$, $b_3^T = (1,0,1)$.

Matrices, addition of matrices

The object

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix},$$

where $a_{ij} \in \mathbb{R}$ or $a_{ij} \in \mathbb{C}$, $i = 1, \ldots, n$, $j = 1, \ldots, m$ is called an $n \times m$ real or complex **matrix**.The set of $n \times m$ real matrices is denoted by $\mathcal{M}_{nm}(\mathbb{R})$, and similarly, the set of $n \times m$ complex matrices is denoted by $\mathcal{M}_{nm}(\mathbb{C})$.

If A and B are of the same type matrices, the their sum is defined in the following way

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & \cdots & a_{1,m} + b_{1m} \\ \vdots & & \vdots \\ a_{n1} + b_{n1} & \cdots & a_{nm} + b_{nm} \end{bmatrix}.$$

Properties of matrix addition

- Matrix addition is a **binary operation** on the set of matrices, in other words, the sum of two $n \times m$ real/complex matrices will be an $n \times m$ real/complex matrix.
- Matrix addition is **associative**:

$$(A + B) + C = A + (B + C).$$

- There exists an **additive unit**: the full zero matrix denoted by 0, with the property:

$$A + 0 = 0 + A = A.$$

- For every matrix A has an **additive inverse** denoted by $-A$ such that

$$A + (-A) = (-A) + A = 0.$$

- Matrix addition is a **commutative** operation, that is to say for two arbitrary A, B $n \times m$ real/complex matrices we have

$$A + B = A + B.$$

Multiplication of matrices by a scalar

Let λ be a scalar and A be a matrix, then their product is defined in the following way:

$$\lambda A = \lambda \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \cdots & \lambda a_{1m} \\ \vdots & & \vdots \\ \lambda a_{n1} & \cdots & \lambda a_{nm} \end{bmatrix}.$$

Example

$$3 \begin{bmatrix} 1 & 2 & -1 \\ -2 & 4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & -3 \\ -6 & 12 & 12 \end{bmatrix}.$$

Properties of by a scalar

- Arbitrary matrix multiplied by zero results the full zero matrix.
- Multiplication by one has no effect.
- If we multiply a matrix by a scalar and we multiply the result by an another scalar, the product will be the same if we multiply the matrix by the product of the scalars.
- **Distributivity**: To multiply a sum of matrices by a factor gives the same result if each summand is multiplied by the factor and the resulting products are added. Similarly, to multiply a matrix by a sum of factors results the same if the matrix multiplied by each factors and the resulting products are added.

Matrix multiplication

Let us consider two matrices. Assume that the first one has the same number of columns as the number of rows of the second, that is to say $A \in \mathcal{M}n \times m$ and $B \in \mathcal{M}m \times k$. In this situation we can define the product $C = AB$, where $C \in \mathcal{M}n \times k$ and the jth element of the i row of C is the inner product of the ith row of A and the jth column of B, in more detail

$$c_{ij} = \sum_{t=1}^{m} a_{it} b_{tj}.$$

Exercise

Calculate the products $A^T B,\ AB^T,\ CD,\ CB,\ A^T D$, where

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ i \\ 1-2i \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 5 \\ -1 & i & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & -i \\ 3 & 2 \\ i & i+2 \end{bmatrix}$$

Properties of matrix multiplication

- Matrix multiplication is not commutative.
- Matrix multiplication is **associative**.
- Transpose of a product is equal to the product of the transposes in the reverse order, that is$(AB)^T = B^T A^T$.
- **Distributivity**: $(A+B)C = AC + BC$ and $A(B+C) = AB + AC$.

Exercise

Do the following calculations!
$AB,\ BA,\ (AB)^T,\ A^T B^T,\ (A+B)C,\ AC+BC$, where

$$A = \begin{bmatrix} -1 & i & 4 \\ 5 & 1 & 0 \\ -i & 0 & 1+i \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 2 \\ -3 & 3 & 1 \\ 5 & 2 & 2 \end{bmatrix}$$

Inverse of matrices

The $n \times n$ matrix

$$I_n = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

is said to be the n dimensional **unit matrix**. If there is no ambiguity, we write I instead of I_n. Let $A \in \mathcal{M}_{n \times n}$. We call A **invertible**, if there is such a matrix $B \in \mathcal{M}_{n \times n}$ for which $AB = BA = I$.

Exercise

Calculate the inverse of the following matrices!

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Determinant of 2×2 matrices

Let's consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. We assign a number to A in the following manner:

$$\det A = a_{11}a_{22} - a_{12}a_{21}.$$

This number is called the **determinant** of A.

Exercise

Calculate the determinant of the matrices below!

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},$$

$$\begin{bmatrix} 1+i & i \\ -i & 2 \end{bmatrix}, \quad \begin{bmatrix} 2+i & 3i \\ -i & 2i \end{bmatrix}, \quad \begin{bmatrix} 1+i & 2+2i \\ -i & -2i \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

Geometric meaning of the determinant.

Determinant of 3×3 matrices

Let us consider the matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. We assign a number to A in the following way:

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

This number is said to be the **determinant** of A.

Exercise

Calculate the determinants of the matrices below!

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & \frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} i & 1 & -i \\ 2 & 4 & 1 \\ 3i & -2i & 1 \end{bmatrix}, \quad \begin{bmatrix} 1-i & 0 & 2 \\ 0 & -1-i & 3 \\ \pi & 0 & 1 \end{bmatrix}.$$

Laplace expansion theorem

The ij **minor** of A is denoted by A_{ij} ,that is , the determinant of the $(n-1)\times(n-1)$ matrix that results from deleting the ith row and the jth column of A.

Theorem (Laplace expansion theorem)

If A is an $n\times n$ matrix, then its determinant can be calculated using the cofactor expansion below:

$$\det A=\sum_{t=1}^{n}(-1)^{i+t}a_{it}\det A_{it},\quad \textit{(expansion with respect to the } i\textit{th row)}$$

or

$$\det A=\sum_{t=1}^{n}(-1)^{t+j}a_{tj}\det A_{tj},\ \textit{(expansion with respect to the } j\textit{th column).}$$

Exercise

Let's calculate the determinant of an arbitrary 4×4 matrix!

Linear system of equations, Gaussian elimination

Consider a set of three equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned} \quad \text{or} \quad \underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b1 \\ b_2 \\ b_3 \end{bmatrix}}_{b}$$

The basic idea of the Gaussian elimination method is the transformation of this set of equations into a staggered set:

$$\begin{aligned} a'_{11}x_1 + \quad a'_{12}x_2 + \quad a'_{13}x_3 &= b'_1 \\ a'_{22}x_2 + \quad a'_{23}x_3 &= b'_2 \\ a'_{33}x_3 &= b'_3 \end{aligned}$$

All coefficients a below the diagonal are zero. The last equation is solved for x_3. Now, the second can be solved by inserting the value of x_3. This procedure can be repeated for the uppermost equation. We have to eliminate x_1 in all but the first equation. This can be done by subtracting $\frac{a_{21}}{a_{11}}$ times the first equation from the second equation and $\frac{a_{31}}{a_{11}}$ times the first equation from the third equation. We have to eliminate x_2 from the third equation in a pretty similar way. This procedure is called the **Gaussian method of elimination**.

Linear system of equations, Gaussian elimination

Example

Solve the following linear system of equations using Gaussian method of elimination:

$$\begin{array}{rrrcr} 6x_1 & -12x_2 & +6x_3 & = & 6 \\ 3x_1 & -5x_2 & +5x_3 & = & 13 \\ 2x_1 & -6x_2 & +0x_3 & = & -10 \end{array}$$

If $A \in \mathcal{M}_{n\times m}$ and $b \in \mathbb{R}^n$ or $\mathbb{C}^n$, then the system

$$\text{(LE)} \qquad Ax = b, \qquad \text{in matrix form} \qquad (A|b)$$

is called a **linear system of equations**, where x is an m dimensional unknown vector. If $b = 0$, then the system (LE) is said to be a **homogeneous linear system of equations**, otherwise it is called an **inhomogeneous linear system of equations**.

Linear system of equations, Gauss-Jordan elimination

Let us consider whether a set of n linear equations with n variables can be transformed by successive elimination of the variables into the form

$$\begin{array}{cccccc} x_1 & +0 & +\cdots & +0 & = & c_1 \\ 0 & +x_2 & +\cdots & +0 & = & c_2 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & +0 & +\cdots & +x_n & = & c_n \end{array}$$

The transformed set of equations gives the solution for all variables directly. The transformation is achieved by the following method, which is basically an extension of the Gaussian elimination method. At each step, the elimination of x_j has to be carried out not only for the coefficients below the diagonal, but also for the coefficients above the diagonal. In addition, the equation is divided by the coefficient a_{jj}. The above form is available only case, when $\det A \neq 0$. This method is called **Gauss-Jordan elimination**.

Example

Solve the linear system on the previous slide with Gauss-Jordan elimination!

Determination of the inverse matrix with Gauss-Jordan elimination

For a given matrix A we are looking for a matrix X such that $AX = I$. If A $n \times n$, then this task is equivalent to the solution of n linear system of equations. Namely, we have to combine linearly from the columns of A the natural basis vectors $e_i,\ i = 1, \ldots, n$ of $\mathbb{R}^n$, where $e_i^T = (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)$. We can do this with a simultaneous Gauss-Jordan elimination.

$$(A|I) \longrightarrow (I|A^{-1})$$

Exercise

Calculate the inverses of the following matrices with simultaneous Gauss-Jordan elimination!

$$\begin{bmatrix} 2 & 0 & 0 \\ 2 & 1 & -6 \\ 6 & 0 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Linear system of equations, existence of solution

Assume that n linear equations with n variables are given. This system is called **linearly dependent** if one of the equations can be written as a linear combination of the others, otherwise the system is said to be **linearly independent**. The set of n linear equations with n variables has a unique solution if the system is linearly independent. If the set of equations constitute a linearly dependent set, then there is no unique solution. If the maximal number of linearly independent equations is $m < n$, then $m - n$ variables can be freely chosen. If we have more equations, than variables the system is called **overdetermined**. The zero vector is always a solution of the homogeneous system, this is called the **trivial solution**.If there exist non zero solution v, then for all λ scalar the λv is also a solution. Moreover, arbitrary linear combination of solutions will be again a solution.

Theorem

The solution set of a homogeneous linear system of equations constitutes a subspace.

Inhomogeneous linear system of equations

Let v be a vector and S be a subspace of the vector space V. Then the set

$$v + S = \{\ x \in V \mid x = v + s,\ \text{valmely } s \in S\text{-re}\ \}$$

is called **affine subspace**.

Theorem

The solution set of an inhomogeneous linear system of equations constitutes an affine subset, that is to say, the solution set can be written int the form $v + S$, where S is the solution set of the homogeneous part and v is a particular solution of the inhomogeneous system.

Theorem

An inhomogeneous system is solvable if and only if the vector b can be written as a linear combination of the columns of A.

Rank of matrices

The dimension of the space which is generated by the columns of a matrix is called the **column rank** of the matrix. One can define in a pretty similar way the **row rank** of a matrix.

Theorem

The row rank and the column rank of a matrix is the same.

The number defined by the previous theorem is called the **rank** of a matrix.

Exercise

Calculate the rank of the following matrices!

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 2 & 2 \end{bmatrix}$$

Eigenvalues, eigenvectors

Let $A \in \mathcal{M}_{n\times n}$ be a matrix and x be an n dimensional, non-zero vector. If there exists a real number λ such that

$$Ax = \lambda x,$$

then λ is called an **eigenvalue** of the matrix A, and x is said to be an **eigenvector** belonging to λ.

Example

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

Theorem

Eigenvector belonging to the same eigenvalue constitutes a subspace.

Determination of eigenvalues

The polynomial

$$p_A(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial** of the matrix A.

Theorem

The number λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic polynomial of A.

Exercise

Calculate the eigenvalues and eigenvectors of the following matrices!

$$\begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 5 & 1 \\ 4 & 5 \end{bmatrix}, \quad \begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}.$$