Mathematics for Engineers 1.

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Seminar
Complex numbers

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1. Solve the following equations! Plot the solutions on the complex plane!

(a)$*$ $x^2 + 4 = 0$,

(b)$*$ $x^2 + 2x + 2 = 0$,

(c) $x^2 - x + 1 = 0$,

(d) $x^3 - 6x^2 + 13x = 0$.

$*$**2.** Give the algebraic form of the following expressions!

(a) $(3+i)(2+3i)$, $(-2+3i)(5-2i)$, $i(1+2i)$, $(-1+i)(1-2i)(1+2i)$,

(b) $\overline{5-2i}$, $\overline{(3+4i)}(2+i)$,

(c) $(2-i)^3$, $i^6 + 3i^5 - 2i^3 + i^2 - 1$, i^{2008}, i^{103},

(d) $\dfrac{5+3i}{i}$, $\dfrac{1-i}{2+i}$, $\dfrac{1-2i}{1-3i}$, $\dfrac{2-i}{(3-2i)(2+5i)}$.

3. Solve the following equations!

(a)* $\overline{z} + 2z = 9 + 2i$,

(b) $\overline{z} + |z|^2 = 31 - i$,

(c) $i^3 \cdot \overline{z} = -3 - 2i$,

(d)* $z^2 + |z|^2 = 2 - 6i$,

(e) $\overline{z} \cdot z^2 = 8i$,

(f) $z^2 = i$.

*__4.__ Plot the following sets on the complex plane!

(a) $A = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$,

(b) $B = \{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$,

(c) $C = \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 0\}$,

(d) $D = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 2\}$,

(e) $E = \{z \in \mathbb{C} : |z| \leq 1\}$,

(f) $F = \{z \in \mathbb{C} : \mathrm{Re}(z) = \mathrm{Im}(z)\}$.

5. Give the trigonometric form of the following complex numbers!

(a)∗ 3,
(b) -2,
(c) i,
(d)∗ $-4i$,
(e)∗ $2+2i$,
(f) $1-i$,
(g) $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$,
(h)∗ $-1+\sqrt{3}i$,
(i) $-3-3i$,
(j) $\frac{5}{8}+\frac{5}{8}\sqrt{3}i$,
(k) $\sqrt{3}-i$.

∗**6.** Consider the complex numbers $x=3\left(\cos\frac{\pi}{9}+i\sin\frac{\pi}{9}\right)$ and $y=2\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)$. Calculate the value of the following expressions!

(a) $x\cdot y$,
(b) $\dfrac{x}{y}$,
(c) x^3,
(d) y^5,
(e) $\dfrac{1}{x}$,
(f) x^2y.

7. Calculate the followings, use trigonometric forms!

(a)∗ $\sqrt{i}$,
(b) $\sqrt[3]{i}$,
(c) $(2+2i)^{2008}$,
(d)∗ $(1+\sqrt{3}i)^{301}$.

$*$**8.** Calculate the second, the third and the fourth roots of $z = 81\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$! Plot them on the complex plane!

9. Solve the following equations!

(a)$*$ $z^2 - 3iz + 4 = 0$,
(b)$*$ $z^3 + z^2 + z = 0$,
(c) $z^5 - z = 0$,
(d) $z^2 + (2+4i)z - 3 + 3i = 0$,
(e) $2iz^2 + (4+5i)z + 5 = 0$.

10. Let thae values for $e^{i\varphi}$ and $e^{-i\varphi}$ be given. Compute the values of $\sin\varphi$ and $\cos\varphi$.

(a)$*$ $e^{i\varphi} = 1$, $e^{-i\varphi} = 1$,
(b)$*$ $e^{i\varphi} = -i$, $e^{-i\varphi} = i$,
(c) $e^{i\varphi} = -1$, $e^{-i\varphi} = -1$,
(d) $e^{i\varphi} = \frac{1}{2}\sqrt{3} + \frac{i}{2}$,
$e^{-i\varphi} = \frac{1}{2}\sqrt{3} - \frac{i}{2}$.

11. Given the complex number $z = x + iy$. Put it in the form $w = re^{i\varphi}$.

(a)$*$ $z = 3 + 2i$,

(b) $z = 2 - \frac{i}{2}$.

12. Compute the product $z_1 z_2$:

(a)$*$ $z_1 = 2e^{i\frac{\pi}{2}}$, $z_2 = \frac{1}{2}e^{i\frac{\pi}{2}}$,

(b) $z_1 = \frac{1}{2}e^{i\frac{\pi}{4}}$, $z_2 = \frac{3}{2}e^{i\frac{-3\pi}{4}}$.

13. Give the exponential form of the following complex numbers:

(a)$*$ $z = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$,

(b) $z = 5\left(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}\right)$.

Complex numbers in Matlab

- `1i` imaginary unit
- `z=1.5+2*1i` is the $z = 1.5 + 2i$ complex number
- `real(z)` is the real part of z
- `imag(z)` is the imaginary part of z
- `conj(z)` is the conjugate of z
- `abs(z)` is the absolute value of z

If z and w are complex numbers then

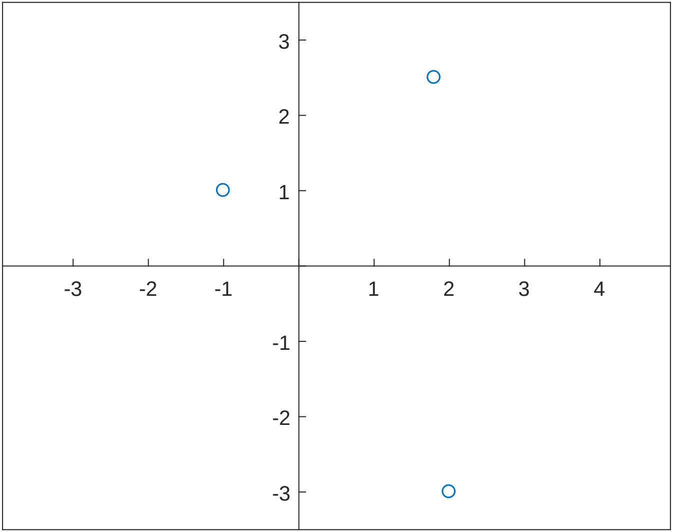
- `z+w` is their sum
- `z-w` is their difference
- `z*w` is their product
- `z/w` is their ratio

Remark: The default value of the variables `i` and `j` is $i = \sqrt{-1}$, but this can be overwritten, so it is better to use `1i` instead of `i` or `j`.

Example

Plot the complex numbers $2-3i$, $-1+i$, $1.8+2.5i$ *on the complex plane!*

```
v=[2-3*1i, -1+1i, 1.8+2.5*1i];
figure; plot(v,'o')
axis([-1.5 2.5 -3.5 3.5])
axis equal
ax=gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```


3
2
1
-3
-2
-1
1
2
3
4
-1
-2
-3

for-loop

The colon operator

- $a = (1, 2, 3, 4, 5)$
  `a = 1:5`
- $b = (5, 4, 3, 2, 1)$
  `b = 5:-1:1`
- $c = (2, 2.2, 2.4, 2.6, 2.8, 3)$
  `c=2:0.2:3`

In general:

```
x=firs element:step length:last element
```

where the step length can be negative, or

```
x=first element:last element
```

in the latter case the step length is 1.

for-loop

```
for loop variable=vector
  commands
end
```

Examples

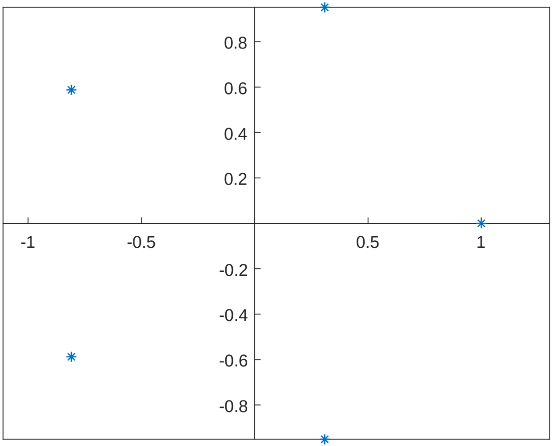
```
s=0;
for i=1:100
  s=s+i;
end
```

```
s=100;
for i=98:-2:2
  s=s+i;
end
```

Example

Plot the fifth roots of unity!

```
z=zeros(1,5);
for k=1:5
  alfa=2*(k-1)*pi/5;
  z(k)=cos(alfa)+1i*sin(alfa);
end
figure; plot(z,'*')
axis equal
ax=gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
```


0.8
0.6
0.4
0.2
-1
-0.5
0.5
1
-0.2
-0.4
-0.6
-0.8

Writing functions

Structure of a Matlab function:

```
function output variable=name of the function(input variables)
  commands
end
```

Important! The above function should be saved as function.m.

Examples

```
function  y=square(x)
   y=2*x.^2-3*x+5;
end
```

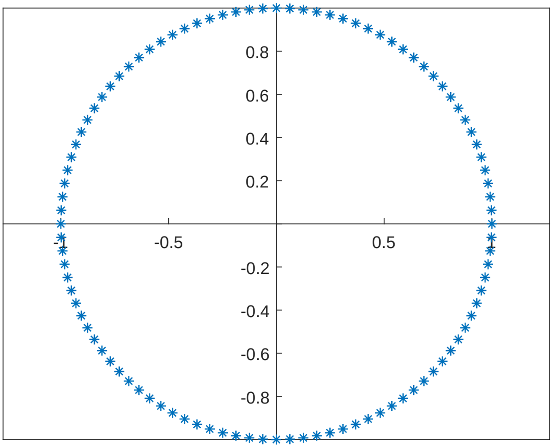
The result of the command `y=masodf(x)` will be the value of the expression $2x^2 - 3x + 5$, where x can be a vector. In this case the calculation is executed componentwise and the result y will be also a vector. This is permitted of by the dotted operation (see earlier slides)

Example

Le us write a function which plots the nth roots of unity.

```
function z=unity(n)
   z=zeros(1,n);
   for k=1:n
     alfa=2*(k-1)*pi/n;
     z(k)=cos(alfa)+1i*sin(alfa);
   end
   figure; plot(z,'*')
   axis equal
   ax=gca;
   ax.XAxisLocation = 'origin';
   ax.YAxisLocation = 'origin';
   end
```

```
>> unity(100)
```



Exercise

Plot the powers of $z = 1 + 1.2i$ *up to fifth order.*

Exercise

Write a Matlab function which calculates the nth roots of a complex number z and plots the powers of z up to nth order.

```
function zroot=roots(z,n)
A = nthroot(abs(z),n);
alfa = angle(z);
for k = 0:n-1
    Nalfa = (alfa+k*2*pi)/n;
    zroot(k+1) =
    A*(cos(Nalfa)+1i*sin(Nalfa));
end
figure; plot(zroot,'*');
axis equal; ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin'
end
```

Exercise

Find the algebraic form of the following complex numbers with Matlab!

$$e^{i\frac{\pi}{2}}, \qquad e^{i\frac{\pi}{4}}, \qquad e^{i\frac{\pi}{6}}, \qquad e^{-i\frac{\pi}{2}}, \qquad e^{-i\pi}.$$

Exercise

Plot the complex numbers $e^{ik\frac{2\pi}{n}}$ with Matlab, where

- $n = 4$ *and* $k = 1, 2, 3$;
- $n = 6$ *and* $k = 1, \ldots, 5$;
- $n = 10$ *and* $k = 1, \ldots, 9$.

Exercise

Plot the first five power of $z = 1 + 1,2$!

Exercise

Write a Matlab function which calculates and plots the firs n power of a given complex number z!