Mathematics for Engineers 2.

Ágnes Baran, Pál Burai, Csaba Noszály

Seminar
Numerics of ODE

This work was supported by the construction EFOP-3.4.3-16-2016-00021. The project was supported by the European Union, co-financed by the European Social Fund.

Successive Approximation, Euler's Method

1. Exercise

Solve the following initial value problems, and approximate the solution with Banach iteration ($2-4$ steps)!

- $x' = t + x, \qquad x(0) = 1;$
- $x' = \cos(x), \qquad x(0) = 1;$
- $x' = \frac{t}{1+x^2}, \qquad x(0) = 1;$
- $x' = \sqrt{1-x^2}, \qquad x(0) = 0.$

Successive Approximation, Euler's Method

1. Exercise

Solve the following initial value problems, and approximate the solution with Banach iteration ($2-4$ steps)!

- $x' = t + x, \qquad x(0) = 1;$
- $x' = \cos(x), \qquad x(0) = 1;$
- $x' = \frac{t}{1+x^2}, \qquad x(0) = 1;$
- $x' = \sqrt{1-x^2}, \qquad x(0) = 0.$

2. Exercise

Approximate with Euler's method the solutions of the previous initial value problems over the interval$[0, 1]$ with step size $h = 0.25$l! Compare these with the exact solution and the successive approximation solution! What is your experience?

Euler's Method

1. Exercise

Write a Matlab code which calculates the approximate solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

with Euler's method over the interval $[x_0, b]$ with step size h.

Euler's Method

1. Exercise

Write a Matlab code which calculates the approximate solution of the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

with Euler's method over the interval $[x_0, b]$ with step size h.

2. Exercise

Using the previous code, solve the initial value problem

$$y' = 2y + x, \quad y(0) = 1$$

over the interval $[0, 1]$ with step sizes $h = 0.25$, $h = 0.2$, $h = 0.1$ and $h = 0.01$! Plot the exact and the numerical solutions on the same figure.

Euler's method

3. Exercise

Using the previous code solve the initial value problem

$$y' = -100(y - \sin(x)) + \cos(x), \qquad y(0) = 0$$

over the interval $[0, 1]$ with different step sizes! What is your experience? Plot the exact, $y(x) = \sin(x)$, and the numerical solutions on the same figure in every cases!

Euler's method

3. Exercise

Using the previous code solve the initial value problem

$$y' = -100(y - \sin(x)) + \cos(x), \qquad y(0) = 0$$

over the interval $[0, 1]$ with different step sizes! What is your experience? Plot the exact, $y(x) = \sin(x)$, and the numerical solutions on the same figure in every cases!

4. Exercise

Solve the previous ODE with Matlab symbolically, and execute the previous task with initial data $y(0) = 1$!

Taylor method

Exercise

Write a code which use the approximation

$$y(x+h) \approx y(x) + hy'(x) + \frac{h^2}{2}y''(x) + \frac{h^3}{6}y'''(x) + \frac{h^4}{24}y^{(4)}(x)$$

for the numerical solution of the initial value problem

$$y' = 2x(x^2+y), \quad y(0) = 0$$

over the interval $[0, 1]$.
Plot the exact solution

$$y = -1 - x^2 + e^{x^2}$$

and the numerical solutions belonging to the step sizes $h = 0.1$ and $h = 0.01$ on the same figure!

Runge-Kutta method

1. Exercise

Write a Matlab code which solves numerically the initial value problem $y'(x) = f(x, y)$, $y(\xi) = \eta$ using the Runge-Kutta I scheme

$$x_0 = \xi, \quad y_0 = \eta$$
$$x_{i+1} = x_i + h$$
$$k_1 = f(x_i, y_i)$$
$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{h}{2}k_1\right)$$
$$y_{i+1} = y_i + h \cdot k_2$$

over the interval $[\xi, b]$, where h is a given step size.

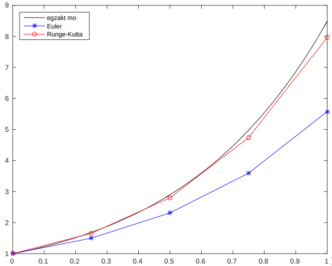
Runge-Kutta method

2. Exercise

Using the previous code, approximate the solution of the initial value problem

$$y' = 2y + x, \quad y(0) = 1$$

over the interval $[0, 1]$. Try different step sizes for the numerical solution. Plot on the same figure the exact and the numerical solution belonging to the step size $h = 0.25$. Plot also on this figure the numerical solution of the problem with Euler's method (use the same step size $h = 0.25$).



Runge-Kutta method

3. Exercise

Using the previous code, approximate the solution of the initial value problem

$$y'(x) = \frac{y}{x} + 10x\cos(10x), \quad y\left(\frac{\pi}{20}\right) = \frac{\pi}{20}$$

over the interval $\left[\frac{\pi}{20}, 2\pi\right]$. Try different step sizes.
Plot on the same figure the exact and the numerical solution.

Numerical solution with Matlab, the ode45 function

Példa

Approximate the solution of the initial value problem

$$y' = 2y + x, \quad y(0) = 1$$

over the interval $[0, 1]$ with Matlab command `ode45`.

Solution: The general scheme is

```
[x,y] = ode45(f,[xmin,xmax],y0)
```

where the initial value problem is

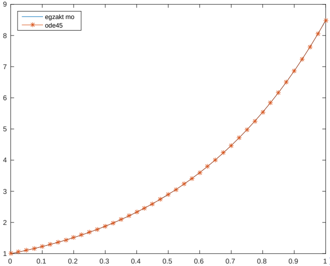
$$y' = f(x, y), \quad y(xmin) = y0$$

and one would like to approximate the solution over the interval $[xmin, xmax]$.
The approximation of the solution at `x(i)` is stored in `y(i)`.

```
>> f=@(x,y) 2*y+x;
>> [xm,ym]=ode45(f,[0,1],1);
```

Plot on the same figure the exact solution $y = \frac{5}{4}e^{2x} - \frac{x}{2} - \frac{1}{4}$, and the numerical solution!

```
>> xx=linspace(0,1);
>> yy=5*exp(2*xx)/4 - xx/2 - 1/4;
>> figure; plot(xx,yy,xm,ym,'-*')
>> legend('egzakt mo', 'ode45', 'Location','NorthWest')
```



Solution of higher order equations with command ode45

Example

Approximate the solution of the initial value problem

$$y'' - 2y' - 3y = 3x^2 + 4x - 5, \quad y(0) = 1, \quad y'(0) = 2$$

over the interval $[0, 2]$ using the Matlab function ode45!

Solution: Rewrite the higher order equation into a system of first order equations.

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= 2y_2 + 3y_1 + 3x^2 + 4x - 5, \quad y_2(0) = 2 \end{aligned}$$

Let's write a function comes back with the right hand sides of the two equations:

```
f=@(x,y) [y(2); 2*y(2)+3*y(1)+3*x^2+4*x-5];
```

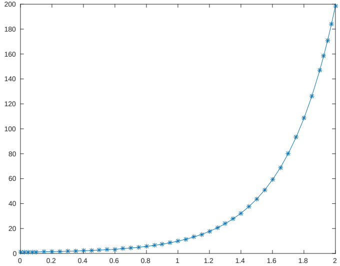
Call the function ode45:

```
>> [x,y] = ode45(f,[0 2],[1; 2]);
```

The values at x of y_1 and y_2 will be in the first and in the second column of y respectively.

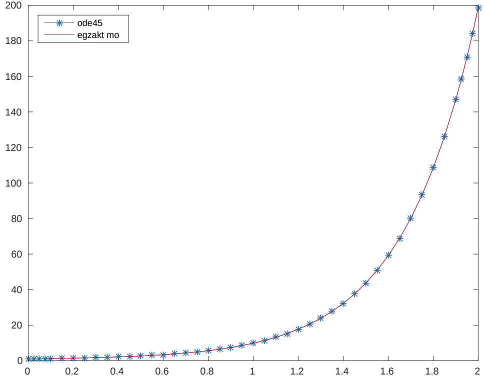
Plot y_1 the approximate solution of the original equation:

```
>> figure; plot(x,y(:,1),'*-')
```



Plot the exact solution ($y = -\frac{1}{2}e^{-x} + \frac{1}{2}e^{3x} - x^2 + 1$) on the same figure.

```
>> hold on; plot(x,-exp(-x)/2+exp(3*x)/2-x.^2+1,'r')
>> legend( 'ode45','exact', 'Location','NorthWest')
```



Numerical solution of higher order equation

1. Exercise

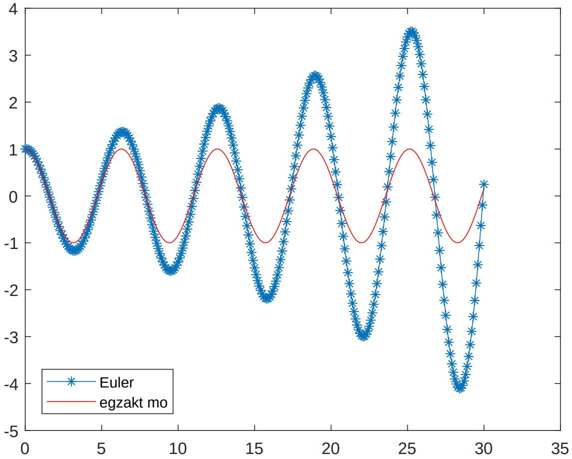
Write a Matlab code which solves the previous initial value problem with Euler method!

2. Exercise

Approximate the solution of the initial value problem

$$m \cdot x'' = -kx, \quad x(0) = x_0, \quad x'(0) = 0$$

using the previous code. Let $x_0 = 1$, $\frac{k}{m} = 1$, and $h = 0.1$. Plot on the same figure the exact $(x = x_0 \cos(t \cdot \frac{k}{m}))$ and the approximate solution over the interval $[0, 30]$.



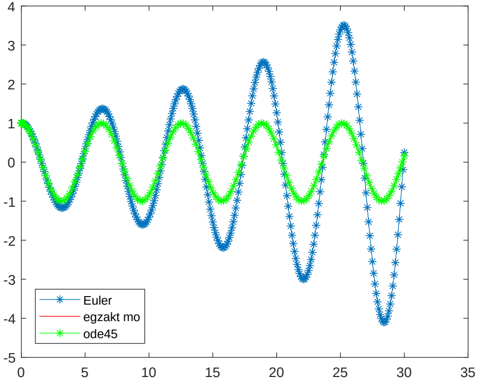
The figure above shows the approximate solution and the exact solution of the initial value problem

$$m \cdot x'' = -kx, \quad x(0) = x_0, \quad x'(0) = 0$$

if $x_0 = 1$, $\frac{k}{m} = 1$, and $h = 0.1$.

3. Exercise

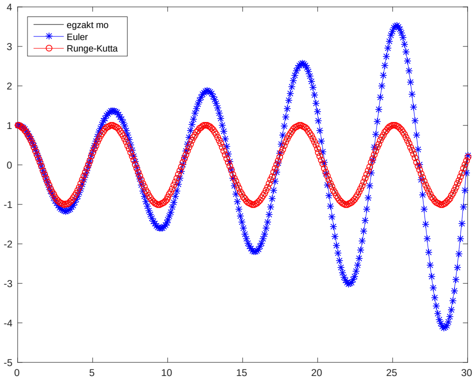
Approximate the solution of the previous initial value problem with `ode45`.



Numerical solution of higher order equations

4. Exercise

Approximate the solution of the previous initial value problem with Runge-Kutta method.



Numerical solution of higher order equations

5. Exercise

Approximate the solution of the initial value problem

$$y^{(4)} + 2y'' + y = 0, \quad y(0) = 1, \quad y'(0) = 2, \quad y''(0) = 1, \quad y'''(0) = -4$$

with Runge-Kutta method II and with Euler's Method over the interval $[0, 30]$. Compare the results with the exact solution $y = \cos(x) + \sin(x) + x\cos(x) + x\sin(x)$!