Mathematics for Engineers 2.

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Seminar
Power series, Fourier series

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Power series

1. Exercise

Give some terms of the Maclauren series of the following functions up to the first three-five terms!

- $f(x) = \sqrt{1-x}$,
- $f(x) = \log((1+x)^5)$,
- $f(x) = \tan(x)$,
- $f(x) = \cosh(x)$.

2. Exercise

Plot with Matlab the previous functions and their approximations on the same figure.

3. Exercise

Obtain the radius of convergence of the following power series!

1. $\sum_{n=0}^{\infty} \frac{5^n}{n^2+1} x^n$
2. $\sum_{n=0}^{\infty} \frac{1}{n3^n} x^n$
3. $\sum_{n=0}^{\infty} n^n x^n$
4. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n} x^n$
5. $\sum_{n=0}^{\infty} \frac{n!}{2^n} x^n$
6. $\sum_{n=0}^{\infty} \frac{n}{2^n} (x-3)^n$
7. $\sum_{n=0}^{\infty} \frac{(3x+4)^n}{(n^3+2)3^n}$,
8. $\sum_{n=0}^{\infty} \frac{n!(5x+3)^n}{(n+1)^n+4n}$,
9. $\sum_{n=0}^{\infty} \frac{(-1)^n n(x+3)^n}{4^n}$,
10. $\sum_{n=0}^{\infty} \frac{2^n(4x-8)^n}{n}$.

Power series

4. Exercise

Expand the following functions at x_0 in a Taylor series!

- $f(x) = \sin(x),\ x_0 = \pi$,
- $f(x) = \cos(x),\ x_0 = \pi$,
- $f(x) = \exp(x),\ x_0 = \log(2)$,
- $f(x) = \sqrt{x},\ x_0 = 1$,
- $f(x) = \log(x),\ x_0 = 1$.

5. Exercise

Obtain the approximate value of $\sqrt{2} = \sqrt{1+1}$ and $\sqrt{42} = \sqrt{36+6}$!

6. Exercise

Obtain the approximate value of $\log(2)$!

Power series

Example

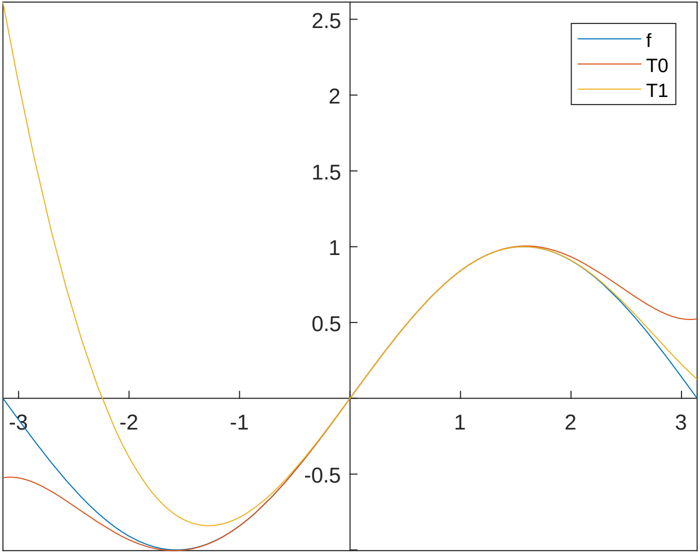
With the Matlab command `taylor` expand the function sin at $x_0 = 0$ and at $x_0 = 1$ in Taylor series up to the first six terms!

```
>> syms x
>> T0 = taylor(sin(x),x,0,'Order',6)
>> T1 = taylor(sin(x),x,1,'Order',6)
```

Plot the function and its approximations on the same figure over the interval $[-\pi, \pi]$!

```
>> f = sin(x);
>> fplot([f,T0,T1],[-pi,pi])
>> legend('f','T0','T1');
>> ax = gca;
>> ax.XAxisLocation = 'origin';
>> ax.YAxisLocation = 'origin';
```

Power series



Fourier-series

1. exercise

Find the periodic functions and their period!

(a) $\tan(x)$

(b) $\sin(2x)$

(c) $\ln(x)$

(d) $\{x\}$ (fractional part function or sawtooth function)

(e) $\sin(x)\cos(x)$

(f) $x^2 - 1$

(g) $\sin(x) + \cos(x)$

2. exercise

Find the even and odd functions!

(a) $\tan(x)$

(b) $x^2 + 1$

(c) $(x+1)^2$

(d) $-2x^5$

(e) $x^5 + 1$

(f) $|x|$

(g) e^{-x^2}

(h) $\frac{e^x - 1}{e^x + 1}$

Fourier-series

Example

Let $f:\mathbb{R}\to\mathbb{R}$ be the periodic extension of $f_0:[-\pi,\pi]\to\mathbb{R}$, $f_0(x)=x^2$. Obtain the Fourier-series of f!

Solution: f is even, so $b_k=0$, $k=1,2,\dots$ and

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x^2dx=\frac{2\pi^2}{3}$$

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{x^2}_{f}\underbrace{\cos(kx)}_{g'}dx$$

$$= \frac{1}{\pi}\underbrace{\left[x^2\frac{1}{k}\sin(kx)\right]_{-\pi}^{\pi}}_{0} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{2}{k}\underbrace{x}_{u}\underbrace{\sin(kx)}_{v'}dx$$

$$= \frac{2}{k^2\pi}\left[x\cos(kx)\right]_{-\pi}^{\pi} - \frac{2}{k^2\pi}\int_{-\pi}^{\pi}\cos(kx)dx$$

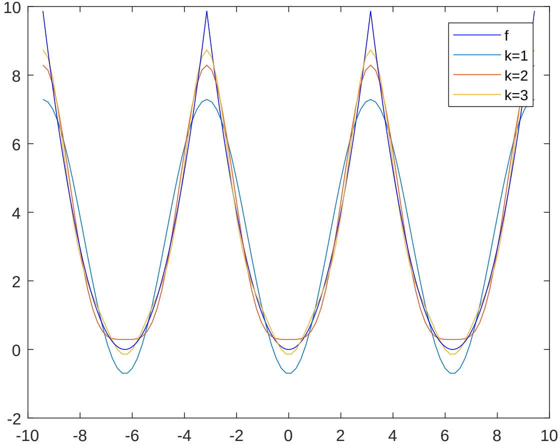
$$= \frac{4}{k^2}\cos(k\pi) - \frac{2}{k^3\pi}\left[\sin(kx)\right]_{-\pi}^{\pi} = \frac{4}{k^2}(-1)^k$$

Hence

$$f(x) = \frac{\pi^2}{3} + \sum_{k=1}^{\infty}\frac{4}{k^2}(-1)^k\cos(kx)$$

exercise

Plot the previous function and the partial sums of its Fourier-series up to the third component over the interval $[-3\pi, 3\pi]$.



Example

Let $f : \mathbb{R} \to \mathbb{R}$ be the periodic extension of $f_0 : [-\pi, \pi) \to \mathbb{R}$, $f_0(x) = x^3$. Obtain the Fourier-series of f!

Solution: f is odd so $a_k = 0$, $k = 0, 1, 2, \ldots$ and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{x^3}_{f} \underbrace{\sin(kx)}_{g'} dx$$

$$= \underbrace{-\frac{1}{k\pi} [x^3 \cos(kx)]_{-\pi}^{\pi}}_{(-1)^{k+1} \frac{2\pi^2}{k}} + \frac{3}{k} \cdot \underbrace{\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(kx) dx}_{A}$$

where $A = \frac{4}{k^2}(-1)^k$, and

$$f(x) = \sum_{k=1}^{\infty} (-1)^k \left(\frac{12}{k^3} - \frac{2\pi^2}{k} \right) \sin(kx)$$

exercise

Plot the previous function and the partial sums of its Fourier-series up to the twelfth component over the interval $[-3\pi, 3\pi]$.