Mathematics for Engineers 2.

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Seminar
Fourier transform

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Fourier transform

Example

Determine the Fourier transform of the following function!

$$f(x) = \begin{cases} 1, & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{otherwise.} \end{cases}$$

Solution: It is known

$$\mathcal{F}[f](\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx \quad \text{és} \quad e^{ix} = \cos(x) + i\sin(x),$$

furthermore, cos is even and sin is odd, so

$$e^{-ix} = \cos(x) - i\sin(x).$$

From this

$$\int_{-\infty}^{\infty} f(t)e^{-i\omega x}dx = \int_{-1/2}^{1/2} \cos(\omega x)dx - i \underbrace{\int_{-1/2}^{1/2} \sin(\omega x)dx}_{=0}$$

$$= 2\int_{0}^{1/2} \cos(\omega x)dx = 2\left[\frac{\sin(\omega x)}{\omega}\right]_{x=0}^{\frac{1}{2}} = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)$$

That is

$$\mathcal{F}[f](\omega) = \frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)$$

Fourier transform

1. exercise

Obtain the Fourier transform of the following functions!

- $f(x) = \begin{cases} x, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$
- $f(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$
- $f(x) = \begin{cases} 1, & \text{if } x \in [2,3] \\ 0, & \text{otherwise.} \end{cases}$
- $f(x) = \begin{cases} -1, & \text{if } x \in [-1,0] \\ 1, & \text{if } x \in]0,1] \\ 0, & \text{otherwise.} \end{cases}$
- $f(x) = e^{-|x|}$.

Fourier transform

Example

Obtain the Fourier transform of the following function!

$$f(x) = \begin{cases} 1, & \text{ha } x \in [2,3] \\ 0, & \text{egyébként.} \end{cases}$$

Solution. We use the translation rule

$$\mathcal{F}[f(x-x_0)](\omega) = e^{-i\omega x_0} F(\omega).$$

If

$$g(x) = \begin{cases} 1, & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right] \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\mathcal{F}[g](\omega) = \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right).$$

Fourier transform

Since $f(x) = g\left(x - \frac{5}{2}\right)$, we have

$$\mathcal{F}[f](\omega) = \mathcal{F}\left[g\left(x - \frac{5}{2}\right)\right](\omega) = e^{-\frac{5}{2}i\omega}\frac{2}{\omega}\sin\left(\frac{\omega}{2}\right)$$
$$= \left(\cos\frac{5\omega}{2} - i\sin\frac{5\omega}{2}\right)\frac{2}{\omega}\sin\left(\frac{\omega}{2}\right).$$

2. exercise

Obtain the Fourier transform of the following function!

$$f(x) = \begin{cases} -1, & \text{if } x \in [-1, 0] \\ 1, & \text{if } x \in]0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Fourier transform

Example

Obtain the Fourier transform of the function $g(x) = e^{-|2x+3|}$, if we know that the Fourier transform of $f(x) = e^{-|x|}$ is $\mathcal{F}[f](\omega) = \frac{2}{1+\omega^2}$.

Solution. According to the dilatation and translation rule of Fourier transform we have

$$\mathcal{F}[f(x - x_0)](\omega) = e^{-i\omega x_0} F(\omega) \quad \text{and} \quad \mathcal{F}\left[f\left(\frac{x}{a}\right)\right](\omega) = |a|\mathcal{F}[f](a\omega).$$

Now $g(x) = h(2x)$, where $h(x) = e^{-|x+3|} = f(x+3)$, so

$$\mathcal{F}[g](\omega) = \mathcal{F}[h(2x)](\omega) = \frac{1}{2}\mathcal{F}[h]\left(\frac{\omega}{2}\right) = \frac{1}{2}\mathcal{F}[f(x+3)]\left(\frac{\omega}{2}\right)$$
$$= \frac{1}{2}e^{\frac{3}{2}i\omega}\mathcal{F}[f]\left(\frac{\omega}{2}\right) = \frac{1}{2}e^{\frac{3}{2}i\omega}\frac{2}{1+\left(\frac{\omega}{2}\right)^2} = e^{\frac{3}{2}i\omega}\frac{4}{4+\omega^2}.$$

Fourier transform

3. Exercise

Obtain the Fourier transform of the following functions!

1. $g(x) = e^{-|1-x|}$
2. $g(x) = 5e^{-\left|\frac{x}{3}-2\right|}$
3. $g(x) = 2e^{-|3x-1|} - e^{-|x+2|}$

4. Exercise

Find the Fourier transform of g if the Fourier transform of f

$$f(x) = \begin{cases} e^{-x}, & \text{if } x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

is $F(\omega) = \frac{1}{1+\omega i}$.

$$g(x) = \begin{cases} e^{4x}, & \text{if } x < 0 \\ 0, & \text{otherwise.} \end{cases}$$

Fourier transform

Example.

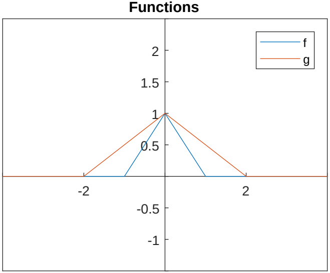
Obtain the Fourier transform of g if we know that the Fourier transform of f is $F(\omega) = \frac{2}{\omega^2}(1 - \cos(\omega))$. Plot the functions and their Fourier transform.

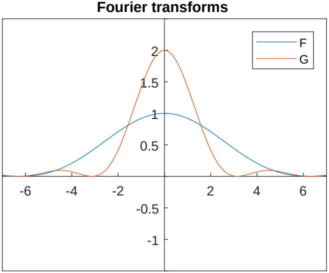
$$f(x) = \begin{cases} 1 + x, & \text{if } x \in [-1, 0] \\ 1 - x, & \text{if } x \in [0, 1] \\ 0, & \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} 1 + \frac{x}{2}, & \text{if } x \in [-2, 0] \\ 1 - \frac{x}{2}, & \text{if } x \in [0, 2] \\ 0, & \text{otherwise.} \end{cases}$$

Solution.

$$\mathcal{F}[g](\omega) = \mathcal{F}\left[f\left(\frac{x}{2}\right)\right](\omega) = 2\mathcal{F}[f](2\omega) = \frac{1}{\omega^2}(1 - \cos(2\omega)).$$





Slowly changing signal → narrow frequency domain
Rapidly changing signal → wide frequency domain

Fourier transform

5. Exercise

Let $a, b > 0,\ x_0 \in \mathbb{R}$ be given and f is the rectangle impulse from $x_0 - \frac{b}{2}$ to az $x_0 - \frac{b}{2}$ with height a. Determine its Fourier transform (depending on the parameters)! What happens if we change the parameters?

6. Exercise

What will be the Fourier transform of the functions below if we know that $\mathcal{F}\left[e^{-\frac{x^2}{2}}\right] = \sqrt{2\pi}e^{-\frac{\omega^2}{2}}$?

- $f(x) = 3e^{-2(x-3)^2}$,
- $g(x) = e^{-x^2+2x}$,

Whose Fourier transform is the function $F(\omega) = 3e^{-2(\omega-3)^2}$?

Fourier transform

Example

Find with Matlab the Fourier transform of $f(x) = e^{-|x|}$, plot f and F on the same figure.

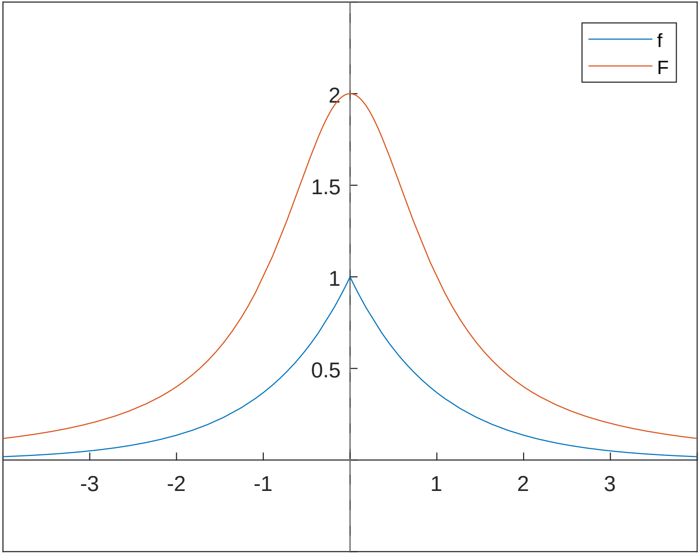
Solution: Calculate the Fourier transform with the command `fourier`!

```
>> syms t x
>> f = exp(-abs(t));
>> F = fourier(f,t,x)
```

Plot f and F on the same figure!

```
>> fplot([f F])
>> legend('f','F');
>> axis([-4,4,-.5,2.5])
>> ax = gca;
>> ax.XAxisLocation = 'origin';
>> ax.YAxisLocation = 'origin';
```

Fourier transform



Fourier transform

7. exercise

Obtain the Fourier transform of the functions from the 1. exercise with Matlab! Plot the function f, and if the transform F is real plot it on the same figure with f!

Fourier transform

Example

Find with Matlab the inverse Fourier transform of $F(x) = e^{-x^2}$ and plot f and F on the same figure!

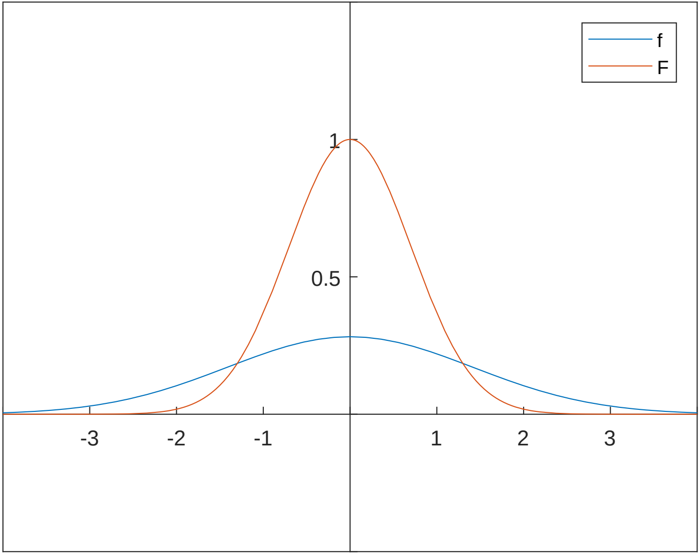
Solution: Calculate the inverse Fourier transform with the command `ifourier`!

```
>> syms t x
>> F = exp(-(x)^2);
>> f = ifourier(F,x,t)
```

Plot f and F on the same figure!

```
>> fplot([f F])
>> legend('f','F');
>> axis([-4,4,-.5,1.5])
>> ax = gca;
>> ax.XAxisLocation = 'origin';
>> ax.YAxisLocation = 'origin';
```

Fourier transform



Discrete Fourier transform, (DFT)

Example

Obtain the DFT of the four-point signal $x = [2, 3, -1, 1]$ 4 by hand!

Solution:

$$F_k = \sum_{n=0}^{N-1} f_n e^{-i\frac{2\pi}{N}kn}, \qquad k = 0, 1, \ldots, N-1,$$

that is

$$F_k = \sum_{n=0}^{3} f_n e^{-i\frac{2\pi}{4}kn}, \qquad k = 0, 1, 2, 3.$$

So

$$F_0 = 2 + 3 - 1 + 1 = 5, \quad F_1 = 2 + 3e^{-i\frac{2\pi}{4}} - e^{-i\frac{2\pi 2}{4}} + e^{-i\frac{2\pi 3}{4}} = 3 - 2i,$$

$$F_2 = 2 + 3e^{-i\frac{2\pi 2}{4}} - e^{-i\frac{2\pi 4}{4}} + e^{-i\frac{2\pi 6}{4}} = -3, \quad F_4 = \cdots = 3 + 2i.$$

DFT

1. exercise

Obtain the *DFT* of the following vectors! Give the solutions in matrix form!

- $x = [20, 5]$;
- $x = [3, 2, 5, 1]$.

2. exercise

- The even coordinates of the *DFT* of a 9-point signal are

$$[3.1,\ 2.5 + 4.6i,\ -1.7 + 5.2i,\ 9.3 + 6.3i,\ 5.5 - 8i]\,.$$

  Obtain the odd coordinates!

- Find the missing data of the *DFT*:

$$[1 - 0i,\ ?,\ 3 + 1i,\ 4 - 1i,\ 1 - 0i,\ ?,\ ?,\ 2 - 2i]$$

DFT

3. exercise

- The *DFT* of a signal x is

$$X = [10, -2+2i, -2, -2-2i]$$

Calculate the DFT of $y(t) = x(t+1)$! Check the solution with Matlab!

Inverse Discrete Fourier transform $(IDFT)$

Example

Obtain the IDFT of the following four-points signal
$x = [5, 3-2i, -3, 3+2i]$ 4 by hand!

Solution:

$$f_k = \frac{1}{N}\sum_{n=0}^{N-1} F_n e^{i\frac{2\pi}{N}kn}, \qquad k = 0, 1, \ldots, N-1,$$

that is

$$f_k = \frac{1}{4}\sum_{n=0}^{3} F_n e^{i\frac{2\pi}{4}kn}, \qquad k = 0, 1, 2, 3.$$

So

$$f_0 = \frac{1}{4}(5+3-2i-3+3+2i) = 2, \quad f_1 = \frac{1}{4}(5+(3-2i)e^{i\frac{2\pi}{4}} - 3e^{i\frac{2\pi 2}{4}} + (3+2i)e^{i\frac{2\pi 3}{4}}) = 3,$$

$$f_2 = \cdots = -1, \qquad f_3 = \cdots = 1.$$

IDFT

1. Exercise

Obtain the IDFT of the following vectors! Write the solution in matrix form!

- $x = [2, 3, -1, 1]$;
- $x = [5, 3 - 2i, -3, 3 + 2i]$.

2. Exercise

Ask the help of the Matlab commands `fft` és az `ifft`! Check our earlier solutions with Matlab!

DFT mixture

Example

- Multiply the following polynomials!

$$p(x) = 3x^4 - 2x^2 - x + 5$$
$$q(x) = x^3 - 2x - 2$$
$$p(x)q(x) = ?$$

DFT mixture

Example

Solution:

- By hand:

$$3x^7 - 8x^5 - 7x^4 + 9x^3 + 6x^2 - 8x - 10$$

- using the command `conv`:

```
>> p=[5, -1, -2, 0, 3]; q=[-2, -2, 0, 1]; conv(p, q)
ans =
-10 -8 6 9 -7 -8 0 3
```

- by the commands `fft, ifft`:

```
>> Q=fft([q, [0, 0, 0, 0]]);
>> P=fft([p, [0, 0, 0]]);
>> ifft(P.* Q)
ans =
-10 -8 6 9 -7 -8 0 3
```

DFT mixture

1. Exercise

Find the square of the polynomial

$$x^{20000} + x^{19999} + \ldots + x^2 + x + 1$$

with Matlab! Compare the solution time of the commands `conv` andz `fft`!

2. Exercise

Find the square of the number

$$10^{20} + 10^{19} + \ldots + 10 + 1 = 111\ldots111.$$

DFT mixture

Example

Take a 7-point sample (equidistant) from the function $f(t) = 2sin(t) + cos(2t)$ on the interval $[0, 2\pi]$. Calculate the *DFT* of the sample vector!

```
>> N=7; t=(0:(N-1))*2*pi/N;
>> f=@(t) 2*sin(t)+cos(2*t);
>> x=f(t); X=fft(x)
X = 0.00000 + 0.00000i
-0.00000 - 7.00000i
3.50000 - 0.00000i
0.00000 - 0.00000i
0.00000 + 0.00000i
3.50000 + 0.00000i
-0.00000 + 7.00000i
```