Verified Localization of Trajectories with Prescribed Behaviour in the Forced Damped Pendulum

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Motivation

- The conjecture on the chaotic behaviour of Hubbard's forced damped pendulum was presented more than 10 years ago.
- Surprisingly, the existence of chaos was proved in Bánhelyi *et al.* only in 2008, but the problem of finding chaotic trajectories remained entirely open.

Definition of chaos

- Consider two areas in the plane (L and R), and a continuous map (φ) from plane to plane.
- If one point is in L (or R), then we say it is "L" (or "R")
- The notation for a bi-infinite L/R sequence is $..., e_{-1}, e_0, e_1, ...$ where $e_i \in \{L, R\}$, for all i.
- If we can give a point (p) in a region for all bi-infinite L/R sequences for which

$$..., \varphi^{-1}(p) \in e_{-1}, \varphi^{0}(p) \in e_0, \varphi^{1}(p) \in e_1, ...,$$

then we say that the system is chaotic in that region.

The forced damped pendulum

- Consider the forced damped pendulum, which is a simple mechanical system of one degree of freedom consisting of a mass point of mass m hung with a weightless solid rod of length l.
- The motions of this system are described by the second order differential equation

  $$mlx''(t) = -mg\sin(x(t)) - \gamma lx'(t) + A\cos(t),$$

  where
  - t is the time,
  - $x(t)$ is the angle of the pendulum,
  - $x'(t)$ is the angle velocity,
  - γ is the friction factor,
  - and A is the degree of force.

The forced damped pendulum

- Suppose that the parameters are chosen so that the equation of motion is

$$x''(t) = \sin(x(t)) - 0.1x'(t) + \cos(t).$$

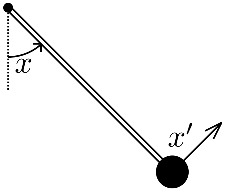


Figure: Illustration of the studied forced damped pendulum.

Chaotic trajectories

- Let an I_k be a time interval: $[2k\pi, 2(k+1)\pi]$. Let us consider those motions, for which one of the following events happens during the I_k time interval:
  - the pendulum goes clockwise through the bottom position exactly once ($\epsilon_k = \ominus$),
  - the pendulum does not go through the bottom position ($\epsilon_k = \otimes$), or
  - the pendulum goes counterclockwise through exactly once ($\epsilon_k = \oplus$).
- We do not consider those motions where the pendulum does something else.
- A trajectory of the forced damped pendulum is a sequence $\ldots, \epsilon_{-2}, \epsilon_{-1} \nabla \epsilon_0, \epsilon_1, \epsilon_2, \ldots$ where $\epsilon_k \in \{\otimes, \oplus, \ominus\}$ and ϵ_k happens during the time interval I_k for all k.

Verified location of trajectories

- We present a fitting verified numerical technique capable to find long trajectory segments with prescribed qualitative behaviour and thus shadowing different types of chaotic trajectories with large (theoretically, with arbitrary) precision.
- For example, we can achieve that our pendulum goes through any specified finite sequence of gyrations by choosing the initial conditions correctly.

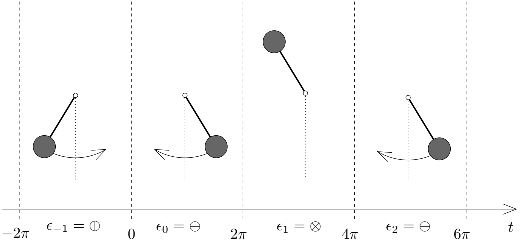


Figure: A four length part of a possible trajectory.

The model

- The search for a starting point for the expected event series was modelled as a constrained global optimization problem.
- For any arbitrary, finite series of events
  1. We compose an objective function which expresses the measure of difference between a trajectory and the expected event series.
  2. We set the searching area to a certain $x - x'$ region.
  3. The optimizer evaluates the objective function at randomly selected points in the searching area.
  4. Based on the known objective function values, the optimizer searches for global optimum points.

The model

- To provide a solution with mathematical precision, we calculated the inclusion of a solution of the differential equation with the VNODE algorithm and based on the PROFIL/BIAS interval environment.
- We applied the C version of GLOBAL algorithm for finding global minimizer points of the objective function.
    - clustering, stochastic global optimization technique
    - capable to find the global optimizer points of moderate dimensional global optimization problems, when the relative size of the region of attraction of the global minimizer points are not very small
    - successfully applied for many similar problems

Objective function

- In composition of the objective function we used the Hausdorff distance of the aimed region of the pendulum angle and speed (E), and the union of inclusions boxes of trajectories (I), which is a series of rectangle shaped, two dimensional regions, each one of them contains a part of the entire trajectories:

$$\max_{z \in I} \inf_{y \in E} d(z, y),$$

where $d(z, y)$ is a given metric, a distance between two two-dimensional points.

- We added nonnegative values proportional to how much the given conditions of the expected behaviour were hurt, plus a fixed penalty term in case at least one of the properties was not satisfied.

Expected regions

- Expected region of $\epsilon = \otimes$ event:

$$E_{\otimes} = \left\{ (x, x'),\ \text{ahol } 0 < x < 2\pi \right\}.$$

- Expected region of $\epsilon = \ominus$ event:

$$\begin{aligned} E_{\ominus} = \ & \left\{ \begin{array}{l} (x, x'),\ \text{where } 0 < x < 2\pi, \\ \text{before the intersection} \end{array} \right\}, \\ \cup & \left\{ \begin{array}{l} (x, x'),\ \text{where } -2\pi < x < 2\pi \text{ and } x' < 0, \\ \text{during the intersection} \end{array} \right\}, \\ \cup & \left\{ \begin{array}{l} (x, x'),\ \text{where } -2\pi < x < 0, \\ \text{after the intersection} \end{array} \right\}. \end{aligned}$$

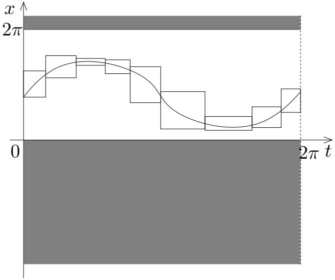
- Expected region of $\epsilon = \oplus$ event: on the analogy of $\epsilon = \ominus$ event.

Determination of the Hausdorff distance

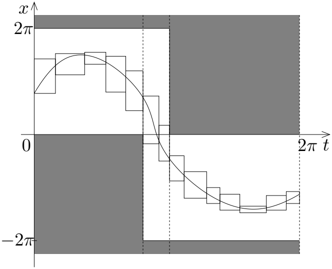
Algorithm 1 $\epsilon = \ominus$ case

$k = max_a = max_b = max_d = max_e = 0$
while $0 \notin I_k$ **and** $k \neq max_k$ **do**
    $max_b = \max(\min_{x \in I_k}(d(x,]0, 2\pi[)), max_b),\ k{+}{+}$
end while
while $0 \in I_k$ **and** $k \neq max_k$ **do**
    $max_d = \max(min_{x' \in I_k, x' < 0}(x'), max_d),\ k{+}{+}$
end while
while $k \neq max_k$ **do**
    $max_a = \max(min_{x \in I_k} d(x,]-2\pi, 0[), max_a),\ k{+}{+}$
end while
if $\forall I_k : min_{x \in I_k}(x) \geq 0$ **then**
    $max_e = d(I_{max_k}, 0)$
end if
return $\max(max_a, max_b, max_d, max_e)$

Determination of the Hausdorff distance



(a) $\epsilon = \otimes$ case.



(b) $\epsilon = \ominus$ case.

Figure: Illustration of the expected regions and the inclusions of trajectories.

Objective function for trajectories of unit length

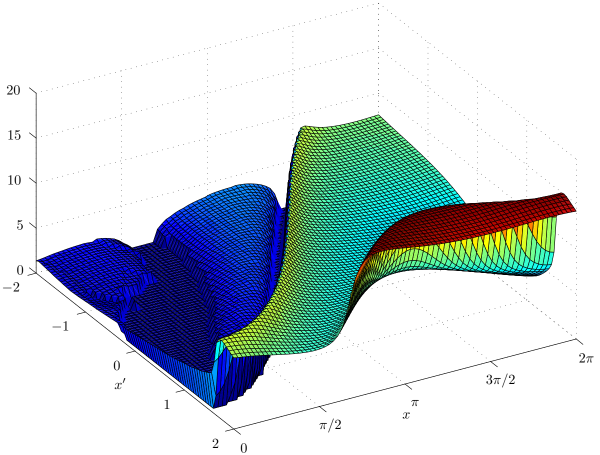


Figure: $\epsilon_0 = \oplus$

Objective function for trajectories of unit length

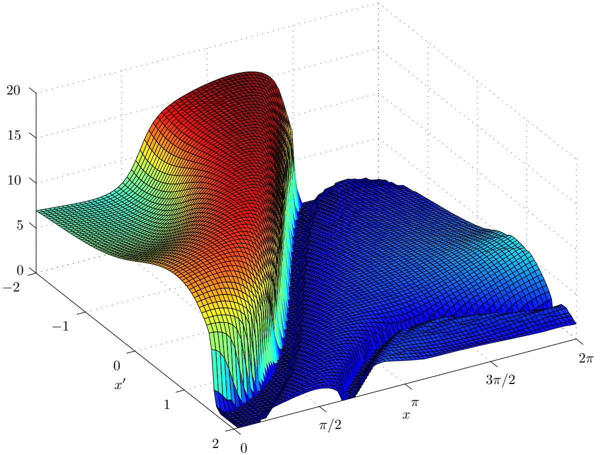


Figure: $\epsilon_0 = \ominus$

Objective function for trajectories of unit length

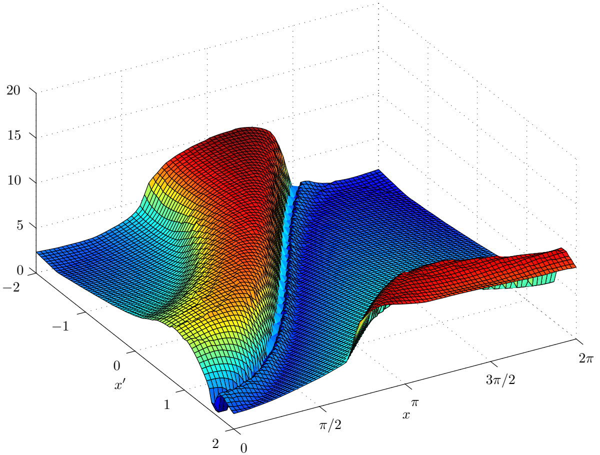


Figure: $\epsilon_0 = \otimes$

Length three expected behaviours where $\epsilon_0 = \oplus$

B	X	ZO	FE
$\nabla \oplus \oplus \oplus$	$(3.5145566; 1.1854134)$	3	666
$\nabla \oplus \oplus \otimes$	$(3.541253; 1.1780008)$	1	965
$\nabla \oplus \oplus \ominus$	$(4.1354217; 1.1146838)$	9	431
$\nabla \oplus \otimes \oplus$	$(3.4500625; 1.2046848)$	1	862
$\nabla \oplus \otimes \otimes$	$(3.6355882; 1.1519576)$	1	2 089
$\nabla \oplus \otimes \ominus$	$(4.1873482; 1.1159454)$	2	723
$\nabla \oplus \ominus \oplus$	$(4.3271325; 1.1040739)$	3	858
$\nabla \oplus \ominus \otimes$	$(3.9656183; 1.0787189)$	2	931
$\nabla \oplus \ominus \ominus$	$(3.7628911; 1.096835)$	7	540

Table: B: expected behaviour, X: a suitable starting point, ZO: number of zero optimum values, FE: number of function evaluations.

Length three expected behaviours where $\epsilon_0 = \ominus$

B	X	ZO	FE
$\nabla \ominus \oplus \oplus$	$(1.3103648; -0.45392754)$	8	1568
$\nabla \ominus \oplus \otimes$	$(1.3957671; -0.73793841)$	1	5063
$\nabla \ominus \oplus \ominus$	$(1.4709218; -0.63161865)$	12	1055
$\nabla \ominus \otimes \oplus$	$(1.0277603; -0.20910021)$	1	2294
$\nabla \ominus \otimes \otimes$	$(1.4672718; -0.61087661)$	1	9873
$\nabla \ominus \otimes \ominus$	$(1.5116331; -0.66239224)$	2	2404
$\nabla \ominus \ominus \oplus$	$(1.6396628; -0.62997909)$	7	1527
$\nabla \ominus \ominus \otimes$	$(1.4479849; -0.5786194)$	2	2303
$\nabla \ominus \ominus \ominus$	$(1.3920852; -0.37132957)$	12	1131

Table: B: expected behaviour, X: a suitable starting point, ZO: number of zero optimum values, FE: number of function evaluations.

Length three expected behaviours where $\epsilon_0 = \otimes$

B	X	ZO	FE
$\nabla \otimes \oplus \oplus$	$(2.6045829; 0.056101674)$	2	3680
$\nabla \otimes \oplus \otimes$	$(2.6558599; 0.004679824)$	1	11882
$\nabla \otimes \oplus \ominus$	$(2.5851486; 0.081902247)$	6	2054
$\nabla \otimes \otimes \oplus$	$(2.6840309; -0.024118557)$	1	8940
$\nabla \otimes \otimes \otimes$	–	0	8885
$\nabla \otimes \otimes \ominus$	$(2.4871575; 0.17213042)$	1	2782
$\nabla \otimes \ominus \oplus$	$(2.6099677; 0.043887032)$	1	2347
$\nabla \otimes \ominus \otimes$	$(2.7034849; -0.050274764)$	2	5313
$\nabla \otimes \ominus \ominus$	$(2.7717467; -0.11932383)$	5	2078

Table: B: expected behaviour, X: a suitable starting point, ZO: number of zero optimum values, FE: number of function evaluations.

Expected behaviours consist of only $\otimes$ events

L	X	ZO	BE	FE
1	$(2.7108515; -0.030099507)$	12	0	1055
2	$(2.6469962; 0.013297356)$	4	0	3254
3	$(2.6342106; 0.026105974)$	1	0	2225
4	$(2.634239; 0.026077512)$	1	0	2491
5	$(2.6342733; 0.026043083)$	0	2.3247517	7620

Table: L: number of $\otimes$ events, X: a suitable starting point, ZO: number of zero optimum values, BE: best objective function value, FE: number of function evaluations.

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