ISOMETRICAL EMBEDDINGS OF LATTICES INTO GEOMETRIC LATTICES

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Abstract. A lattice is said to be *finite height generated* if it is complete and every element is the join of some elements of finite height. Extending former results by G. Grätzer and E.W. Kiss [6] on *finite* lattices, we prove that every *finite height generated algebraic* lattice that has a pseudorank function is isometrically embeddable into a geometric lattice.

1. Introduction

Given a lattice L with a lower bound 0, the height of an element $a \in L$ is defined to be the supremum of lengths of chains in $[0, a]$. Let a function $p: L \to \mathbb{N}_\infty = \{0, 1, \dots, \infty\}$ be called a *pseudorank function* if it has the following properties:

(i) $p(0) = 0$;
(ii) $a \le b$ implies $p(a) \le p(b)$ for all $a, b \in L$;
(iii) $a < b$ implies $p(a) < p(b)$ for all $a, b \in L$ of finite height;
(iv) $p(a \wedge b) + p(a \vee b) \le p(a) + p(b)$ for all $a, b \in L$;
(v) $p(a) < \infty$ iff a is of finite height.

Note that a function $L \to \mathbb{N}_\infty$ is called *submodular* or *semimodular* if it satisfies condition (iv). Also note that if L is of finite length, our definition of pseudorank function coincides that of D.T. Finkbeiner [4] and M. Stern [9].

It is well known that the height function of a semimodular lattice of finite length is a pseudorank function. It is an easy consequence of the Jordan-Hölder Chain Condition that the elements of finite height in a semimodular lattice form a sublattice. Hence the height function of an *arbitrary* semimodular lattice is a pseudorank function. Note that each lattice of finite length has a pseudorank function, see [4, Theorem 2.1].

Let p be a pseudorank function on a lattice L, and let S be a semimodular lattice. Then an embedding $\varphi: L \to S$ is said to be *isometrical* if $p = h \circ \varphi$, where h denotes the height function of S. (We compose mappings from right to left.) An embedding is said to be *cover-preserving*, if it preserves the covering relation. E.g. if L and S are of finite length, L is also semimodular and p denotes the height function of L then φ is isometrical iff it is cover-preserving.

A result of R.P. Dilworth (around 1950) states that every finite lattice can be embedded into a geometric lattice [2, Theorem 14.1]. D.T. Finkbeiner [4] showed that every lattice of finite height with a pseudorank function p can be embedded

Date: August 30, 2012.
2000 *Mathematics Subject Classification.* Primary: 06C10; Secondary: 06B15.
Key words and phrases. Semimodular lattice, geometric lattice, isometrical embedding, cover-preserving embedding.
This research was supported by TÁMOP-4.2.2/B-10/1-2010-0012.

into a semimodular lattice isometrically with respect to p. Blending these results, G. Grätzer and E.W. Kiss [6] managed to prove that every finite lattice with a pseudorank function p can be embedded into a geometric lattice isometrically with respect to p.

The main goal of our paper is an extension of the result of G. Grätzer and E.W. Kiss for algebraic lattices that have sufficiently many elements of finite height. A lattice is said to be *finite height generated* if it is complete and every element is the join of some elements of finite height. Note that lattices of finite length are finite height generated. To show a finite height generated lattice that is not of finite length, consider e.g. $\mathbb{N}_\infty$ with the usual ordering.

Theorem 1. *Every finite height generated algebraic lattice with a pseudorank function can be embedded isometrically into a geometric lattice.*

Corollary 2. *Every finite height generated semimodular algebraic lattice has a cover-preserving embedding into a geometric lattice.*

M. Wild [10] pointed out that the proof of R.P. Dilworth [2, Theorem 14.1] implies that every finite semimodular lattice has a cover-preserving embedding into a geometric lattice. Using the toolkit of matroid theory, he also shortened R.P. Dilworth's proof. Later G. Czédli and E.T. Schmidt [3] extended this result for lattices of finite length. Their proof is different from that of G. Grätzer and E.W. Kiss and that of R.P. Dilworth and M. Wild.

Our construction is an extension of that of R.P. Dilworth. M. Wild's matroid theoretical approach has helped us to understand R.P. Dilworth's proof better. However, using the toolkit of matroid theory does not seem helpful in our case since the theory of infinite matroids is much more difficult. Even the definition of infinite matroids is not clear since there are various reasonable ways to define them, see J.G. Oxley [7, 8]. Also note that in the finite case, R.P. Dilworth's original argument gives a completely different proof for the Grätzer-Kiss Isometrical Embedding Theorem [9, Theorem 6.2.4].

Now, let us give a short outline of our paper. In Section 2, we introduce our construction. In Section 3, we prove Theorem 1 and Corollary 2. Then in Section 4, we compare the above-mentioned embedding constructions with ours.

Finally, let us overview our notation. We will use $\cap$ resp. $\cup$ for set theoretical intersection resp. union and $\wedge, \vee$ for lattice operations. Sometimes $\wedge$ will coincide with $\cap$. In these cases, we will usually use $\cap$ in order to emphasize this coincidence. Let $(a]$ resp. $[a)$ denote the principal ideal resp. filter generated by a. For the sake of simplicity, sometimes we will write x instead of $\{x\}$, e.g. $X \cup x$ instead of $X \cup \{x\}$, if it is clear that X denotes a set and x denotes an element. $X - Y$ will denote the set theoretical difference of X and Y.

2. Basic concepts and lemmas

Given a set S, we define a collection $\mathcal{L}$ of subsets of S to be a *complete lattice of subsets* of S if $\varnothing, S \in \mathcal{L}$ and $\mathcal{L}$ is closed under arbitrary intersection. Note that in this case $\mathcal{L}$ is a complete lattice with respect to set inclusion. Also note that a collection of subsets of S is closed under arbitrary intersection iff it is the lattice of closed sets of S with respect to an appropriate closure operator, see e.g. S. Burris and H.P. Sankappanavar [1].

We say that a pseudorank function r on a complete lattice $\mathcal{L}$ of subsets of S is a *rank function* if

$$(1) \qquad r(A) - r(B) \le |A - B| \quad \text{for all } A, B \in \mathcal{L} \text{ of finite height.}$$

Note that if S is finite, our rank function on a complete lattice of subsets of S are strictly increasing rank function in sense of P. Crawley and R.P. Dilworth [2].

For every finite height generated lattice with a pseudorank function, we are going to construct a complete lattice of subsets, which is isomorphic to the lattice and on which the pseudorank function becomes a rank function. Note, that our construction is an extension of that of P. Crawley and R.P. Dilworth [2, Lemma 14.1.B].

For the rest of this section, let us fix a finite height generated lattice L with a pseudorank function p. Let $J \subseteq L$ denote the set of nonzero join-irreducible elements of finite height. For $f \in J$, let f_0 denote the unique lower cover of f. The set J is a poset with respect to the restriction of the partial ordering of L. Since L is finite height generated, the set $\{(a] \cap J : a \in L\}$ of order-ideals of J forms a complete lattice of subsets that is isomorphic to L. However, (1) does not necessarily hold. To avoid this problem, we need sufficiently many elements in the ground set of the required complete lattice of subsets.

Let $\{X_f : f \in J\}$ be a collection of pairwise disjoint sets such that $|X_f| = p(f) - p(f_0)$. Set $S = \bigcup(X_f : f \in J)$. For each $a \in L$, define $\wr a = \bigcup(X_f : f \in J, f \le a)$. Then $\wr 0 = \varnothing$, $\wr 1 = S$ and $\wr(\bigwedge A) = \bigcap(\wr a : a \in A)$ for all $A \subseteq L$. Consequently, the collection $\mathcal{L} = \{\wr a : a \in L\}$ forms a complete lattice of subsets of S. Since L is finite height generated, the map $\varphi : L \to \mathcal{L}, a \mapsto \wr a$ is an isomorphism. For each $a \in L$, define $r(\wr a) = p(a)$.

Lemma 3. *The above defined r is a rank function on $\mathcal{L}$ with $p = r \circ \varphi$.*

Proof. Certainly, r is a pseudorank function. This fact will be used hereafter without further reference. To prove (1), observe that it suffices to show, that

$$(2) \qquad r(\wr a) - r(\wr b) \le |\wr a - \wr b| \quad \text{for all } a, b \in L, a \ge b \text{ of finite height.}$$

Indeed, if (2) holds then for any $c, d \in L$ of finite height, we have $r(\wr c) - r(\wr d) \le r(\wr c) - r(\wr c \cap \wr d) \le |\wr c - (\wr c \cap \wr d)|$ and $\wr c - (\wr c \cap \wr d) = \wr c - \wr d$.

We prove (2) by induction on $h(a)$, where h denotes the height function of L. Let $a, b \in L, a \ge b$ be arbitrary elements of finite height. The case $h(a) = 0$ is trivial. Suppose that $h(a) > 0$.

If $a = b$ then (2) holds trivially. If $a \succ b$ and a is join-irreducible then $b = a_0$ and (2) holds by definition. If $a \succ b$ and a is not join-irreducible then there exists an element $f \in J$ such that $a = b \vee f$ and $f < a$. Using the induction hypothesis and the submodularity of r, we obtain $r(\wr a) - r(\wr b) \le r(\wr f) - r(\wr b \cap \wr f) \le |\wr f - (\wr b \cap \wr f)| \le |\wr a - \wr b|$.

If $a > b$ and $a \nsucc b$ then there is an element $c \in L$ such that $b < c \prec a$. Hence the induction hypothesis and the previous paragraph yields $r(\wr a) - r(\wr b) = r(\wr a) - r(\wr c) + r(\wr c) - r(\wr b) \le |\wr a - \wr c| + |\wr c - \wr b| = |\wr a - \wr b|$. $\square$

Let $F \subseteq L$ denote the set of elements of finite height. Since L has a pseudorank function, F is a sublattice. Notice that

$$(3) \qquad \text{for any finite } A \subseteq S \text{ there is } x \in F \text{ such that } A \subseteq \wr x.$$

For each $x \in F$, we define r_x to be the map

$$r_x : 2^S \to \mathbb{N} = \{0, 1, \dots\}, \quad A \mapsto \min\{r(\wr y) + |(A \cap \wr x) - \wr y| : y \in F\}.$$

Note that the above definition is an extension of that of P. Crawley and R.P. Dilworth [2, Lemma 14.1.C]. Given a set $A \subseteq S$ and an element $x \in F$, we say that $y \in F$ *represents* $r_x(A)$ if $r_x(A) = r(\wr y) + |(A \cap \wr x) - \wr y|$. Some important properties of r_x can be found in the following statements.

Lemma 4.

(i) $r_x(A) = \min\{r(\wr y) + |(A \cap \wr x) - \wr y| : y \in [0,x]\}$ *for all* $A \subseteq S$.
(ii) $0 \leq r_x(A) = r_x(A \cap \wr x) \leq \min\{|A \cap \wr x|, r(\wr x)\}$ *for all* $A \subseteq S$.
(iii) $A \subseteq B$ *implies* $r_x(A) \leq r_x(B)$ *for all* $A, B \subseteq S$.
(iv) $r_x(A) = r_y(A)$ *for all* $A \subseteq S$ *and* $x, y \in F$ *satisfying* $A \subseteq \wr x \cap \wr y$.
(v) *If* $x \geq y$ *then* $r_x(\wr y) = r(\wr y)$ *for all* $x, y \in F$.

Proof. The first four statements follow easily from the definition. To prove (v), let $x \geq y$ be elements of F. By (iv) and (ii), we have $r_x(\wr y) = r_y(\wr y) \leq r(\wr y)$. To prove the opposite direction, let $u \in F$ represent $r_y(\wr y)$. Then by (1), we obtain $r_y(\wr y) = r(\wr u) + |\wr y - \wr u| \geq r(\wr u) + r(\wr y) - r(\wr u) = r(\wr y)$. □

Lemma 5. *Let* $A \subseteq B \subseteq S$ *and* $x \in F$. *Suppose that* $u \in F$ *represents* $r_x(A)$ *and* $v \in F$ *represents* $r_x(B)$. *Then* $u \wedge v$ *represents* $r_x(A)$ *and* $u \vee v$ *represents* $r_x(B)$, *that is*

$$r_x(A) = r(\wr u \cap \wr v) + |(A \cap \wr x) - (\wr u \cap \wr v)| \text{ and} \tag{4}$$

$$r_x(B) = r(\wr u \vee \wr v) + |(B \cap \wr x) - (\wr u \vee \wr v)|. \tag{5}$$

Proof. First, we need some elementary calculations.

$$\begin{aligned} |(A \cap \wr x) - (\wr u \cap \wr v)| &= |(A \cap \wr x) - \wr u| + |(A \cap \wr x \cap \wr u) - \wr v| \leq \\ &\leq |(A \cap \wr x) - \wr u| + |(B \cap \wr x \cap \wr u) - \wr v| = \\ &= |(A \cap \wr x) - \wr u| + |(B \cap \wr x) - \wr v| - |(B \cap \wr x) - (\wr u \cup \wr v)| \leq \\ &\leq |(A \cap \wr x) - \wr u| + |(B \cap \wr x) - \wr v| - |(B \cap \wr x) - (\wr u \vee \wr v)|. \end{aligned}$$

Now, using the definition of r_x, the submodularity of r and the above calculations, we obtain the following inequalities

$$\begin{aligned} r_x(A) &\leq r(\wr u \cap \wr v) + |(A \cap \wr x) - (\wr u \cap \wr v)| \leq \\ &\leq r(\wr u) + r(\wr v) - r(\wr u \vee \wr v) + |(A \cap \wr x) - (\wr u \cap \wr v)| \leq \\ &\leq r(\wr u) + r(\wr v) - r(\wr u \vee \wr v) + \\ &+ |(A \cap \wr x) - \wr u| + |(B \cap \wr x) - \wr v| - |(B \cap \wr x) - (\wr u \vee \wr v)| = \\ &= r_x(A) + \underline{r_x(B) - r(\wr u \vee \wr v) - |(B \cap \wr x) - (\wr u \vee \wr v)|} \leq r_x(A). \end{aligned}$$

Therefore the above inequalities are equalities. Thus the underlined part is zero, which gives (5), while (4) is the first inequality. □

Corollary 6. *For any* $A \subseteq S$ *and* $x \in F$, *there exists a smallest and a largest element in* $[0, x]$ *that represents* $r_x(A)$.

Lemma 7. *Let* $A \subseteq S$ *and* $a \in S$. *If* $r_x(A \cup a) = r_x(A)$ *and* $y \in F$ *represents* $r_x(A \cup a)$ *then the following hold:*

(i) y *also represents* $r_x(A)$ *and*
(ii) $a \notin \wr x$ *or* $a \in \wr y$.

Proof. $r_x(A\cup a)=r(\downarrow y)+|((A\cup a)\cap\downarrow x)-\downarrow y|\geq r(\downarrow y)+|(A\cap\downarrow x)-\downarrow y|\geq r_x(A)=r_x(A\cup a)$ implies that y also represents $r_x(A)$ and $|((A\cup a)\cap\downarrow x)-\downarrow y|=|(A\cap\downarrow x)-\downarrow y|$. Hence $a\notin\downarrow x$ or $a\in\downarrow y$. □

Lemma 8. *Let $A\subseteq S$ and $B=\{b\in S: r_x(A\cup b)=r_x(A)\}$. Then $r_x(B)=r_x(A)$.*

Proof. By the monotonicity of r_x, that is Lemma 4(iii), we know that $r_x(A)\leq r_x(B)$. By Corollary 6, there exists a largest element $y\in[0,x]$ that represents $r_x(A)$. Let $b\in B-A$ be an arbitrary element. Let $z\in[0,x]$ represent $r_x(A\cup b)$. Then Lemma 7(i) implies that z represents $r_x(A)$, and Lemma 7(ii) implies that $b\notin\downarrow x$ or $b\in\downarrow z$. We also have $\downarrow z\subseteq\downarrow y$, because y is the largest element that represents $r_x(A)$. Hence $b\notin\downarrow x$ or $b\in\downarrow y$. Consequently, $(B-A)\cap\downarrow x\subseteq\downarrow y$, which yields that

$$r_x(A)=r(\downarrow y)+|(A\cap\downarrow x)-\downarrow y|=r(\downarrow y)+|(B\cap\downarrow x)-\downarrow y|\geq r_x(B).$$ □

Note that geometric lattices and closure operators are closely related. That is, a lattice is a geometric lattice iff it is the lattice of closed sets with respect to an algebraic closure operator cl that satisfies the so-called *Exchange Property*: $b\in\mathrm{cl}(A\cup\{a\})-\mathrm{cl}(A)$ implies $a\in\mathrm{cl}(A\cup\{b\})$. For more details, see G. Grätzer [5] and M. Stern [9].

Using r_x, we define two kinds of closure operators on S: cl_x for each $x\in F$ and cl. Namely, for any $A\subseteq F$,

$$\mathrm{cl}_x(A)=\{a\in S: r_y(A\cup a)=r_y(A)\text{ for all }y\in F\cap[x)\},$$
$$\mathrm{cl}(A)=\bigcup\{\mathrm{cl}_y(A): y\in F\}.$$

Notice that

$$\mathrm{cl}_x(A)\subseteq\mathrm{cl}_y(A)\text{ if }x\leq y. \tag{6}$$

Lemma 9. *The functions $\mathrm{cl}_x\colon 2^S\to 2^S, A\mapsto\mathrm{cl}_x(A)$ and $\mathrm{cl}\colon 2^S\to 2^S, A\mapsto\mathrm{cl}(A)$ are algebraic closure operators. Moreover,* cl *satisfies the Exchange Property.*

Proof. The extensivity of cl_x is immediate from the definition. To prove the monotonicity, let $A\subseteq B\subseteq S$. By the definition of cl_x, it is enough to prove that $r_y(B\cup a)=r_y(B)$ for all $a\in\mathrm{cl}_x(A)$ and all $y\in F\cap[x)$. Suppose indirectly that there are elements $a\in\mathrm{cl}_x(A)$ and $y\in F\cap[x)$ such that $r_y(B\cup a)>r_y(B)$. Then $r_y(B\cup a)=r_y(B)+1$. By Corollary 6, there exists a smallest element $u\in[0,y]$ that represents $r_y(A\cup a)$. Let $v\in[0,y]$ represent $r_y(B)$. Then v also represents $r_y(B\cup a)$, thus $a\in\downarrow y$ and $a\notin\downarrow v$ must hold. Using Lemma 5 for $A\cup a\subseteq B\cup a$, we obtain that $u\wedge v$ represents $r_y(A\cup a)$. Then $u\leq u\wedge v$ gives $\downarrow u\subseteq\downarrow v$. By Lemma 7(ii) for $r_y(A\cup a)$ and u, we have that $a\notin\downarrow y$ or $a\in\downarrow u\subseteq\downarrow v$, which contradicts the fact that $a\in\downarrow y$ and $a\notin\downarrow v$. Consequently, cl_x is monotone.

To prove that cl_x is idempotent, let $A\subseteq S$. For any $y\in F\cap[x)$, we have $A\subseteq\mathrm{cl}_x(A)\subseteq B_y=\{b\in S: r_y(A\cup b)=r_y(A)\}$. By Lemma 8 and the monotonicity of r_y, we also have $r_y(A)=r_y(\mathrm{cl}_x(A))=r_y(B_y)$. Now, for any $a\in\mathrm{cl}_x(\mathrm{cl}_x(A))$ and any $y\in F\cap[x)$,

$$r_y(A)\leq r_y(A\cup a)\leq r_y(\mathrm{cl}_x(A)\cup a)=r_y(\mathrm{cl}_x(A))=r_y(A).$$

Hence $a\in\mathrm{cl}_x(A)$ and $\mathrm{cl}_x(\mathrm{cl}_x(A))\subseteq\mathrm{cl}_x(A)$. The other direction follows immediately from the extensivity of cl_x. Consequently, cl_x is idempotent. We conclude that cl_x is a closure operator.

To prove that cl_x is algebraic, let $A \subseteq S$ and $a \in \mathrm{cl}_x(A)$. Let $y = \bigwedge\{z \in F \cap [x) : a \in \lightning z\}$. By (3), we obtain $y \in F$. First, let $A_0 \subseteq A \cap \lightning y$ be a finite subset such that $r_y(A_0)$ is maximal. Then $r_y(A_0) = r_y(A)$. Indeed, the maximality of $r_y(A_0)$ implies that $r_y(A_0 \cup b) = r_y(A_0)$ for all $b \in A$. Using Lemma 8 for $B = \{b \in S : r_y(A_0 \cup b) = r_y(A_0)\}$, we obtain $r_y(A_0) = r_y(B)$. Therefore $r_y(A_0) = r_y(A) = r_y(B)$ by the monotonicity of r_y. Now, we have $r_y(A_0) \le r_y(A_0 \cup a) \le r_y(A \cup a) = r_y(A) = r_y(A_0)$, hence $r_y(A_0 \cup a) = r_y(A_0)$. Finally, let $z \in F \cap [x)$. If $a \notin \lightning z$ then $r_z(A_0) = r_z(A_0 \cup a)$ trivially holds. If $a \in \lightning z$ then $y \le z$ by the definition of y. Using Lemma 4(iv) for $A_0 \cup a \subseteq \lightning y \subseteq \lightning z$, we obtain $r_z(A_0) = r_y(A_0) = r_y(A_0 \cup a) = r_z(A_0 \cup a)$. Hence $a \in \mathrm{cl}_x(A_0)$. We conclude that cl_x is an algebraic closure operator.

The extensivity and monotonicity of cl follow immediately from those of cl_x. To prove the idempotency of cl, let $A \subseteq S$ and suppose that $a \in \mathrm{cl}(\mathrm{cl}(A))$. By definition, $a \in \mathrm{cl}_x(\mathrm{cl}(A))$ for some $x \in F$. Since cl_x is algebraic, there is a finite subset $A_0 \subseteq \mathrm{cl}(A)$ such that $a \in \mathrm{cl}_x(A_0)$. By (6) and the definition of cl, $A_0 \subseteq \mathrm{cl}_y(A)$ for some $y \in F$. By (6) and the monotonicity and idempotency of cl_x, we have $a \in \mathrm{cl}_x(\mathrm{cl}_y(A)) \subseteq \mathrm{cl}_{x\vee y}(\mathrm{cl}_{x\vee y}(A)) = \mathrm{cl}_{x\vee y}(A) \subseteq \mathrm{cl}(A)$. Hence $\mathrm{cl}(\mathrm{cl}(A)) \subseteq \mathrm{cl}(A)$. The other direction follows immediately from the extensivity of cl. Consequently, cl is a closure operator. It is algebraic since cl_x is algebraic for all $x \in F$.

To prove that cl satisfies the Exchange Property, let $A \subseteq S$ and $a, b \in S$ such that $a \in \mathrm{cl}(A \cup b) - \mathrm{cl}(A)$. Since cl is algebraic, there is a finite subset $A_0 \subseteq A$ such that $a \in \mathrm{cl}(A_0 \cup b) - \mathrm{cl}(A_0)$. Hence $a \in \mathrm{cl}_x(A_0 \cup b) - \mathrm{cl}_x(A_0)$ for some $x \in F$. By (3) and (6), we can assume that $A_0 \cup \{a, b\} \subseteq \lightning x$. By the definition of cl_x, there are $u, v \in F \cap [x)$ such that $r_u(A_0 \cup a) = r_u(A_0) + 1$ and $r_v(A_0 \cup b) = r_v(A_0) + 1$, since $a, b \notin \mathrm{cl}_x(A_0)$. Using this and Lemma 4(iv) for $A_0 \cup \{a, b\} \subseteq \lightning x$, we obtain that $r_y(A_0 \cup a) = r_y(A_0 \cup b) = r_y(A_0) + 1$ for all $y \in F \cap [x)$. The assumption $a \in \mathrm{cl}_x(A_0 \cup b)$ implies $r_y(A_0 \cup \{a, b\}) = r_y(A_0 \cup b) = r_y(A_0 \cup a)$ for all $y \in F \cap [x)$. Now, $b \in \mathrm{cl}_x(A_0 \cup a) \subseteq \mathrm{cl}(A \cup a)$ follows immediately. Hence cl satisfies the Exchange Property. □

3. The main proofs

Before the proof of Theorem 1, we need a short technical lemma about finite height generated *algebraic* lattices.

Lemma 10. *If L is a finite height generated algebraic lattice and the elements of finite height form a sublattice then its elements of finite height are exactly its compact elements.*

Proof. Suppose that $a \in L$ is compact. Then $a = \bigvee B$ for some elements $B \subseteq L$ of finite height, because L is finite height generated. Since a is compact, there is a finite $B_0 \subseteq B$ with $a = \bigvee B_0$. Hence a is of finite height. Now, suppose that $b \in L$ is of finite height. Then $b = \bigvee A$ for some compact elements $A \subseteq L$, because L is algebraic. Since b is of finite height, there is a finite $A_0 \subseteq A$ with $b = \bigvee A_0$. Hence b is compact. □

Proof of Theorem 1. Given a finite height generated algebraic lattice L with a pseudorank function p, define $\mathcal{L}$ and r as we did in the previous section. We will also use S for the ground set of $\mathcal{L}$ and $F \subseteq L$ for the set of elements of finite height. Recall that F is a sublattice, since L has a pseudorank function. Denote $\mathcal{L}_{\mathrm{cl}}$ the complete lattice of subsets that corresponds to the closure operator cl. By Lemma 9, $\mathcal{L}_{\mathrm{cl}}$ is a geometric lattice. It is enough to prove that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\mathrm{cl}}$ such that r and the height function of $\mathcal{L}_{\mathrm{cl}}$ coincide on $\mathcal{L}$. Then $\overline{\varphi}\colon L \to \mathcal{L}_{\mathrm{cl}}, x \mapsto \wr x$ is an isometrical embedding.

First, we show that $\mathcal{L} \subseteq \mathcal{L}_{\mathrm{cl}}$. Let $x \in L$ and $a \in S - \wr x$. Suppose, for a contradiction, that $a \in \mathrm{cl}(\wr x)$. Then, by definition, there is a $y \in F$ with $a \in \mathrm{cl}_y(\wr x)$. By (3) and (6), we can assume that $a \in \wr y$. Notice that $a \in \mathrm{cl}_y(\wr x)$ implies that $r_y(\wr x \cup a) = r_y(\wr x)$. Let $z \in F$ represent $r_y(\wr x \cup a)$. Then Lemma 7(ii) and $a \in \wr y$ implies that $a \in \wr z$. Since $a \in \wr z - \wr x$, we have that $\wr z \neq \wr x \cap \wr z = \wr (x \wedge z)$, hence $x \wedge z < z$. However, $(\wr x \cap \wr y) - \wr (x \wedge z) = (\wr x \cap \wr y) - (\wr x \cap \wr z) = (\wr x \cap \wr y) - \wr z = ((\wr x \cup a) \cap \wr y) - \wr z$. Since r is strictly monotone for elements of finite height, we obtain that $r_y(\wr x) \leq r(\wr (x \wedge z)) + |(\wr x \cap \wr y) - \wr (x \wedge z)| < r(\wr z) + |(\wr x \cap \wr y) - \wr (x \wedge z)| = r(\wr z) + |((\wr x \cup a) \cap \wr y) - \wr z| = r_y(\wr x \cup a)$, which contradicts $r_y(\wr x) = r_y(\wr x \cup a)$. This proves that $\wr x \in \mathcal{L}_{\mathrm{cl}}$ and $\mathcal{L} \subseteq \mathcal{L}_{\mathrm{cl}}$. Moreover, the meet operation both on $\mathcal{L}$ and $\mathcal{L}_{\mathrm{cl}}$ is the intersection, therefore $\mathcal{L}$ is a meet-subsemilattice of $\mathcal{L}_{\mathrm{cl}}$.

To prove that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\mathrm{cl}}$, observe that the join of two elements $\wr x, \wr y \in \mathcal{L}$ in the larger lattice $\mathcal{L}_{\mathrm{cl}}$ is $\mathrm{cl}(\wr x \cup \wr y)$. Since $\mathrm{cl}(\wr x \cup \wr y) \subseteq \wr (x \vee y)$, it is enough to prove that $\mathrm{cl}(\wr x \cup \wr y) \supseteq \wr (x \vee y)$. Let $a \in \wr (x \vee y)$. By definition, it means that $a \in X_b$ for some $b \in J \cap (x \vee y]$. Since L is finite height generated, $x = \bigvee X$ and $y = \bigvee Y$ for some $X, Y \subseteq F$. By Lemma 10, b is compact, hence $b \leq \bigvee X_0 \vee \bigvee Y_0$ for some finite $X_0 \subseteq X$, $Y_0 \subseteq Y$. Let $x_0 = \bigvee X_0$ and $y_0 = \bigvee Y_0$. By Lemma 10, $x_0, y_0 \in F$. Now, $b \leq x_0 \vee y_0$ implies $a \in \wr (x_0 \vee y_0)$. In order to prove that $a \in \mathrm{cl}(\wr x \cup \wr y)$, it is enough to show that $a \in \mathrm{cl}(\wr x_0 \cup \wr y_0)$. We prove that $a \in \mathrm{cl}_{x_0 \vee y_0}(\wr x_0 \cup \wr y_0)$, which yields $a \in \mathrm{cl}(\wr x_0 \cup \wr y_0)$. As a preparation for this, we show that

$$(7) \qquad r_z(\wr x_0 \cup \wr y_0) = r_z(\wr (x_0 \vee y_0)) \text{ for all } z \in F \cap [x_0 \vee y_0).$$

Since $r_z(\wr x_0) = r(\wr x_0)$ by Lemma 4(v), x_0 represents $r_z(\wr x_0)$. Similarly, y_0 represents $r_z(\wr y_0)$. Assume that z_0 represents $r_z(\wr x_0 \cup \wr y_0)$. Using Lemma 5 twice for $\wr x_0 \subseteq \wr x_0 \cup \wr y_0$ and $\wr y_0 \subseteq \wr x_0 \cup \wr y_0$, we obtain that $x_0 \vee y_0 \vee z_0$ represents $r_z(\wr x_0 \cup \wr y_0)$. However, $\wr x_0 \cup \wr y_0 \subseteq \wr (x_0 \vee y_0 \vee z_0)$, hence $r_z(\wr x_0 \cup \wr y_0) = r(\wr (x_0 \vee y_0 \vee z_0)) + |((\wr x_0 \cup \wr y_0) \cap \wr z) - \wr (x_0 \vee y_0 \vee z_0)| = r(\wr (x_0 \vee y_0 \vee z_0))$. On the other hand, $\wr x_0 \cup \wr y_0 \subseteq \wr (x_0 \vee y_0)$, which implies $r_z(\wr x_0 \cup \wr y_0) \leq r(\wr (x_0 \vee y_0)) + |((\wr x_0 \cup \wr y_0) \cap \wr z) - \wr (x_0 \vee y_0)| = r(\wr (x_0 \vee y_0)) \leq r(\wr (x_0 \vee y_0 \vee z_0))$. Together with $r(\wr (x_0 \vee y_0 \vee z_0)) = r_z(\wr x_0 \cup \wr y_0)$, we obtain that $r_z(\wr x_0 \cup \wr y_0) = r(\wr (x_0 \vee y_0))$. Lemma 4(v) yields that $r(\wr (x_0 \vee y_0)) = r_z(\wr (x_0 \vee y_0))$, which finishes the proof of (7). Now, $\wr x_0 \cup \wr y_0 \subseteq \wr x_0 \cup \wr y_0 \cup a \subseteq \wr (x_0 \vee y_0)$, the monotonicity of r_z and (7) implies that $r_z(\wr x_0 \cup \wr y_0 \cup a) = r_z(\wr x_0 \cup \wr y_0)$ for all $z \in F \cap [x_0 \vee y_0)$. Hence $a \in \mathrm{cl}_{x_0 \vee y_0}(\wr x_0 \cup \wr y_0)$. We conclude that $\mathcal{L}$ is a sublattice of $\mathcal{L}_{\mathrm{cl}}$.

To prove that the embedding is isometrical, we have to show that $r(\wr x) = h(\wr x)$ for all $x \in L$, where h denotes the height function of $\mathcal{L}_{\mathrm{cl}}$. By the definition of finite height generated lattices, it suffices to prove that $r(\wr x) = h(\wr x)$ for all $x \in F$. We use induction on the height of $x \in F$. If $x = 0$ then $r(\wr 0) = 0 = h(\wr 0)$.

Suppose that $0 \leq x \prec y$ and $r(\wr x) = h(\wr x)$. Since r is a rank function on $\mathcal{L}$, we have a set $A = \{a_1, \ldots, a_{k-1}, a_k\} \subseteq \wr y - \wr x$ with $k = r(\wr y) - r(\wr x)$ distinct elements. Let $A_0 = \wr x$ and $A_i = \wr x \cup a_1 \cup \cdots \cup a_i$ for all $i \in \{1, \ldots, k\}$. Assume that $z \in F \cap [y)$. Clearly, $r_z(A_i) \leq r(\wr x) + i$. By Lemma 4(v), $r_z(\wr x) = r(\wr x)$. Hence $r_z(A_i) \leq r_z(\wr x) + i$. The opposite direction is also true. Suppose, for a contradiction, that $r_z(A_i) < r(\wr x) + i$ for some $i \in \{1, \ldots, k\}$. Let i be minimal for this property, that is $r_z(A_i) < r(\wr x) + i$ and $r_z(A_{i-1}) = r(\wr x) + i - 1$. Note that such i exists, since $r_z(A_0) = r(\wr x)$. Then, by the monotonicity of r_z, $r_z(A_i) = r(\wr x) + i - 1 = r_z(A_{i-1})$. Let u_i represent $r_z(A_i)$. Applying Lemma 7(ii) for $r_z(A_i)$ and u_i, we obtain that

$a_i \in \downarrow u_i$, since $a_i \in \downarrow y \subseteq \downarrow z$. Notice that x represents $r_z(\downarrow x)$ by Lemma 4(v). Using Lemma 5 for $\downarrow x \subseteq A_i$, we obtain that $x \vee u_i$ represents $r_z(A_i)$. We conclude from $\downarrow x \subset A_i \subseteq \downarrow y, x \prec y, a_i \in \downarrow u_i$ and the construction of S and $\mathcal{L}$ that $x \vee u_i \geq y$. Now, we have

$$r(\downarrow x) + i > r_z(A_i) = r(\downarrow(x \vee u_i)) + |(A_i \cap \downarrow z) - \downarrow(x \vee u_i)| = \\ = r(\downarrow(x \vee u_i)) \geq r(\downarrow y) = r(\downarrow x) + k \geq r(\downarrow x) + i,$$

which is a contradiction. Therefore

$$r_z(A_i) = r_z(\downarrow x) + i \text{ for all } i \in \{1, \dots, k\} \text{ and all } z \in F \cap [y). \tag{8}$$

Hence for every $f \in F$, we have that $r_z(A_{i-1}) \neq r_z(A_i)$ for $z = f \vee y$, which shows that $a_i \notin \mathrm{cl}_f(A_{i-1})$ for all $f \in F$, that is $a_i \notin \mathrm{cl}(A_{i-1})$. This gives that $\mathrm{cl}(A_{i-1}) \neq \mathrm{cl}(A_i)$. Clearly, $\mathrm{cl}(A_{i-1}) \preceq \mathrm{cl}(A_{i-1} \cup a_i) = \mathrm{cl}(A_i)$. Thus

$$\downarrow x = \mathrm{cl}(A_0) \prec \mathrm{cl}(A_1) \prec \cdots \prec \mathrm{cl}(A_k). \tag{9}$$

We know from (8) and the definition of k that $r_z(A_k) = r_z(\downarrow y)$ for all $z \in F \cap [y)$. Since r_z is monotone, $r_z(A_k) = r_z(A_k \cup b)$ for all $b \in \downarrow y - A_k$. Hence $b \in \mathrm{cl}_y(A_k) \subseteq \mathrm{cl}(A_k)$ for all $b \in \downarrow y - A_k$, and we obtain that $\downarrow y \subseteq \mathrm{cl}(A_k)$. This, together with $A_k \subseteq \downarrow y$, yields that $\mathrm{cl}(A_k) = \downarrow y$. Consequently, we conclude from (9), the semimodularity of $\mathcal{L}_{\mathrm{cl}}$ and the induction hypothesis that $h(\downarrow y) = h(\downarrow x) + k = r(\downarrow x) + k = r(\downarrow y)$. □

Proof of Corollary 2. Let L be a finite height generated semimodular algebraic lattice. Consider the height function $h_L : L \to \mathbb{N}_\infty$. We conclude from Theorem 1 that L has an isometrical embedding ψ into a geometric lattice G with respect to h_L. Assume that $x \prec y$ in L and choose a minimal element f of finite height in $(y] - (x]$. Let g be a lower cover of f. Then $x = x \vee g$ and $y = x \vee f$. Now, $\psi(f)$ covers $\psi(g)$, since $h_G(\psi(f)) - h_G(\psi(g)) = h_L(f) - h_L(g) = 1$, where h_G denotes the height function of G. Hence the semimodularity of G implies that $\psi(y) = \psi(x) \vee \psi(f)$ covers $\psi(x) = \psi(x) \vee \psi(g)$. Therefore ψ is cover-preserving. □

Remark 11. If L is of finite length, the construction of $\mathcal{L}_{\mathrm{cl}}$ becomes more simple: we need only r_1 since $\mathrm{cl} = \mathrm{cl}_1$. Note that in this case r_1 is a rank function on $\mathcal{L}_{\mathrm{cl}}$.

4. Concluding remarks

As to isometrical embeddings, let L be a finite lattice. Then our construction is completely different from that of G. Grätzer and E.W. Kiss [6]. First of all, we embed $L \cong \mathcal{L}$ directly into the geometric lattice $\mathcal{L}_{\mathrm{cl}}$, while they did their embedding in two steps: first, they embedded L into a semimodular lattice, and second, they embedded this semimodular lattice into a geometric lattice.

On the other hand, our $\mathcal{L}$ is uniquely determined, while they provided a technique, where they used a so-called *semimodular construction scheme*. This scheme consists of the *frame* L and finitely many lattices, the so-called *blocks*. These blocks can be chosen infinitely many different ways, and the geometric lattice depends on their choice.

As to cover-preserving embeddings, let L be a semimodular lattice of finite length and let $p = h$ be the height function of L. Then our construction yields the same geometric lattice as the entirely different method of G. Czédli and E.T. Schmidt [3].

For more details about different cover-preserving embeddings of semimodular lattices, see the historical comments in G. Czédli and E.T. Schmidt [3].

Acknowledgment. The author is grateful to the anonymous referee, who improved the paper with a thorough report. Several comments and suggestions are acknowledged.

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