"Broadening the knowledge base and supporting the long term professional sustainability of the Research University Centre of Excellence at the University of Szeged by ensuring the rising generation of excellent scientists.

Doctoral School of Mathematics and Computer Science

Stochastic Days in Szeged

26.07.2012.

Relaxed Sector Condition and Random Walk in Divergence Free Random Drift Field

Bálint Tóth

(Budapest University of Technology and Economics)

TÁMOP-4.2.2/B-10/1-2010-0012 project

A projekt az Európai Unió támogatásával, az Európai Szociális Alap társfinanszírozásával valósul meg.

BÁLINT TÓTH
(TU Budapest and U of Bristol)

Relaxed Sector Condition and Random Walk in Divergence Free Random Drift Field

KRÁMLI-FEST, Szeged, 26-27 July 2013

Notation:

$(\Omega, \pi, \tau_z : z \in \mathbb{Z}^d)$	probabability space with ergodic $\mathbb{Z}^d$-action
$\mathcal{E} = \{e \in \mathbb{Z}^d : \|e\| = 1\}$	possible steps of the rw
$v_e : \Omega \to [-1, 1], \quad e \in \mathcal{E}$	
∘ $v_e(\omega) + v_{-e}(\tau_e \omega) \equiv 0$	vector field
∘ $\sum_{e \in \mathcal{E}} v_e(\omega) \equiv 0$	divergence-free
∘ $\int_\Omega v_e(\omega) d\pi(\omega) = 0,$	no overall drift
$V_e(\omega, x) := v_e(\tau_x \omega)$	the vector field over $\mathbb{Z}^d$

Helmholtz's Theorem:

$\mathbf{d}=2$: The **height function**: $\mathbb{Z}^2_* := \mathbb{Z}^2 + (1/2, 1/2)$.

There exists $H : \Omega \times \mathbb{Z}^2_* \to \mathbb{R}$ scalar field with stationary increments such that

$$V = \operatorname{curl} H, \qquad V_e(x) = H(x + \frac{e + \tilde{e}}{2}) - H(x + \frac{e - \tilde{e}}{2})$$

$\mathbf{d}=3$: The **stream field**: $\mathbb{Z}^3_* := \mathbb{Z}^2 + (1/2, 1/2, 1/2)$.

There exists $H_e : \Omega \times \mathbb{Z}^3_* \to \mathbb{R}$, $e \in \mathcal{E}$, vector field with stationary increments such that

$$V = \operatorname{curl} H, \qquad V_e(\omega, x) = \ldots \text{explain in plain words}$$

Examples and essentially different cases:

∘ H stationary + ergodic + bounded [S.M. Kozlov (1985)]:

$$h \in \mathcal{L}^\infty, \qquad H(\omega, x) = h(\tau_x \omega) - h(\omega)$$

∘ H stationary + ergodic + unbounded + curl H bounded:

$$h \in \mathcal{L}^2 \setminus \mathcal{L}^\infty, \qquad H(\omega, x) = h(\tau_x \omega) - h(\omega)$$

this is the case discussed today. H_{-1}-condition ✓

∘ H {stationary + ergodic} increments (but not stationary).

+ curl H bounded. No H_{-1}-condition, superdiffusive.

∘∘ randomly oriented Manhattan-lattice

∘∘ six-vertex / square ice ($d = 2$)

∘∘ dimer tiling ($d = 2$)

The **random walk in random environment**: $0 < \varepsilon < 1/(2d)$, fixed

$$\mathbf{P}_\omega\Big(X_{n+1} = x + e \,\Big|\, X_0^n,\ X_n = x \Big) = \frac{1}{2d} + \varepsilon V_e(\omega, x)$$

The **environment process**:

$$\eta_n := \tau_{X_n}\omega$$

Stationary and ergodic Markov process on (Ω, π). (Due to div-freeness.)

Some **operators** on the Hilbert space $\mathcal{L}^2(\Omega, \pi)$:

$\mathcal{L}^2(\Omega, \pi)$-gradient :
$$\nabla_e f(\omega) := f(\tau_e \omega) - f(\omega)$$
$$\nabla_e^* = \nabla_{-e}$$

$\mathcal{L}^2(\Omega, \pi)$-Laplacian :
$$\Delta f(\omega) := \frac{1}{d} \sum_{e \in \mathcal{E}} \left(f(\tau_e \omega) - f(\omega) \right)$$
$$\Delta^* = \Delta \leq 0$$

multiplication ops. :
$$M_e f(\omega) := v_e(\omega) f(\omega)$$
$$M_e^* = M_e$$

A commutation relation – due to div-freeness of v:

$$\sum_{e \in \mathcal{E}} M_e \nabla_e + \sum_{e \in \mathcal{E}} \nabla_e M_{-e} = 0$$

The **transition operator / infinitesimal generator** of the environment process:

$$L = P - I = \frac{1}{2}\Delta + \varepsilon \sum_{e \in \mathcal{E}} M_e \nabla_e =: -S + A$$

Martingale decomposition of the displacement:

$$\varphi : \Omega \to \mathbb{R}^d, \qquad \varphi(\omega) := \sum_{e \in \mathcal{E}} v_e(\omega) e$$

Then:

$$Y_n := X_n - \varepsilon \sum_{k=0}^{n-1} \varphi(\eta_k)$$

is a martingale with stationary+ergodic increments.

$$X_n = Y_n + \varepsilon \sum_{k=0}^{n-1} \varphi(\eta_k)$$

Gaol: understand diffusive behaviour of $\sum_{k=0}^{n-1} \varphi(\eta_k)$.

Relaxed Sector Condition [I. Horváth, B. Tóth, B. Vető (2012)]

Theorem: Efficient martingale approximation (a la Kipnis-Varadhan) holds for $\sum_{k=0}^{n-1} \varphi(\eta_k)$ if

(1)	" $S^{-1/2}AS^{-1/2}$ "	is *skew self-adjoint* (not just skew symmetric).
(2)	$\varphi \in \mathtt{Ran}(S^{-1/2})$	H_{-1}-condition

Remarks:

(1) Extends Varadhan et al.'s *Graded Sector Condition*.

(2) Proof: partly reminiscent of Trotter-Kurtz.

Two possible definitions of " $S^{-1/2}AS^{-1/2}$ ":

$$B := \sum_{e\in\mathcal{E}} \left((-\Delta)^{-1/2}\nabla_e\right) M_e\,(-\Delta)^{-1/2} = \text{"}\; S^{-1/2}AS^{-1/2}\;\text{"}$$

on $\qquad \mathcal{C} := \mathtt{Dom}(-\Delta)^{-1/2} = \mathtt{Ran}(-\Delta)^{1/2}$

$$\widetilde{B} := (-\Delta)^{-1/2} \sum_{e\in\mathcal{E}} M_e \left((-\Delta)^{-1/2}\nabla_e\right) = \text{"}\; S^{-1/2}AS^{-1/2}\;\text{"}$$

on $\qquad \widetilde{\mathcal{C}} := \{f \in \mathcal{L}^2 : \sum_{e\in\mathcal{E}} M_e \left((-\Delta)^{-1/2}\nabla_e\right) f \in \mathtt{Dom}(-\Delta)^{-1/2}\}$

Facts (easy): (1) B is *skew symmetric* on $\mathcal{C}$.

(2) $\mathcal{C} \subset \widetilde{\mathcal{C}}$ and $\widetilde{B}\Big|_{\mathcal{C}} = B$.

(3) $\widetilde{B} = \overline{\widetilde{B}}$ and $\widetilde{B} = -B^*$.

Wanted: $\overline{B} = \widetilde{B}$, or, equivalently $\overline{B} = -B^*$

What is missing from skew self-adjointmess of " $S^{-1/2}AS^{-1/2}$ "?

von Neumann's criterion:

$$\begin{pmatrix} B \text{ skew symmetric, and} \\ \overline{\texttt{Ran}(B \pm I)} = \mathcal{H} \end{pmatrix} \Leftrightarrow \begin{pmatrix} B \text{ essentially} \\ \text{skew self-adjoint} \end{pmatrix}$$

Needed:

$$\sum_{e \in \mathcal{E}} M_e \Big((-\Delta)^{-1/2} \nabla_e \Big) \psi = (-\Delta)^{1/2} \psi \qquad \Rightarrow \qquad \psi = 0.$$

Warning: Formal manipulation deceives: $\psi \notin \texttt{Dom}(-\Delta)^{-1/2}$!

Raise it to the lattice $\mathbb{Z}^d$: change of notation: from now on: $\nabla, \Delta, \ldots$ = lattice gradient, lattice Laplacian, ...

Wanted:
NO nontrivial scalar field $\Psi : \Omega \times \mathbb{Z}^d \to \mathbb{R}$ with stationary increment, and $\mathbf{E}\Big(\Psi \Big) = 0$ solves the PDE

$$\Delta \Psi = V \cdot \nabla \Psi. \tag{1}$$

$$\text{"}\ \Psi = (-\Delta)^{-1/2} \psi \text{ ''}, \qquad \nabla \Psi(\omega, x) = (-\Delta)^{-1/2} \nabla \psi(\tau_x \omega)$$

Note similarity: No sublinearly growing harmonic function on $\mathbb{Z}^d$.

$\mathbf{H}_{-1}$ **assumed**:
The height function/stream field is stationary, $\mathcal{L}^2$. Hence

$$\overline{\lim_{n\to\infty}}\, n^{-1}\mathbf{E}\Big(X_n^2 \Big) < \infty.$$

Let Ψ be solution of (1). Then $n \mapsto R_n := \Psi(X_n)$ is a martingale with stationary and ergodic increments.

$$\rho^2 := \mathbf{E}\Big((R_{n+1} - R_n)^2 \Big),$$

$$(1 - 2d\varepsilon)\,\|\,\psi\,\|^2 \le \rho^2 \le (1 + 2d\varepsilon)\,\|\,\psi\,\|^2$$

and

$$n^{-1/2} R_n \Rightarrow \mathcal{N}(0, \rho^2)$$

IF

$$\lim_{|x|\to\infty} |x|^{-1} |\Psi(x)| = 0 \qquad \text{a.s.} \tag{2}$$

then $\rho = 0$, ✓ .

Naïve guess: (2) holds for 0-mean fields with stationary increments. ($d = 1$: ergodic theorem. Generally no true in $d \geq 2$.)

Proof of (2) *in* $d = 2$:

Maximum principle: $$\max_{x \in \Lambda} |\Psi(x)| = \max_{x \in \partial\Lambda} |\Psi(x)|$$

By ergodic thm . . . : $$N^{-1} \max_{x \in \partial[-N,N]^2} |\Psi(x)| \xrightarrow{\mathbf{P}} 0,$$

Thus: $$N^{-1} \max_{x \in [-N,N]^2} |\Psi(x)| \xrightarrow{\mathbf{P}} 0,$$

Altogether:

$$\mathbf{P}\Big(|R_n| > \varepsilon\sqrt{n} \Big)$$

$$= \mathbf{P}\Big(|R_n| > \varepsilon\sqrt{n},\ |X_n| > K\sqrt{n} \Big) + \mathbf{P}\Big(|R_n| > \varepsilon\sqrt{n},\ |X_n| \le K\sqrt{n} \Big)$$

$$\le \mathbf{P}\Big(|X_n| > K\sqrt{n} \Big) + \mathbf{P}\Big(\max_{|x| \le K\sqrt{n}} |\Psi(x)| > \varepsilon\sqrt{n} \Big) \to 0$$