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Comparision of quenched and annealed invariance principles for random conductance model

Ádám Tímár

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Comparison of quenched and annealed invariance principles for random conductance model

Ádám Timár (U. of Szeged)

joint work with **Martin Barlow** (UBC) and **Krzysztof Burdzy** (UW)

Outline

Ω = A∪B
A
B

The random conductance model

Consider the d dimensional integer lattice $\mathbb{Z}^d$ with edge set E_d (nearest neighbor).
Let $\{\mu_e\}_{e \in E_d} = \omega$ be random nonnegative weights (conductances) on the edges.
Define $\mu_x = \sum_{xy \in E_d} \mu_{xy}$, and consider random walk with transition probabilities:

$$P_\omega(x, y) = P(x, y) = \frac{\mu_{xy}}{\mu_x},$$

whenever $\mu_x \neq 0$. This is random walk in random environment (RWRE).

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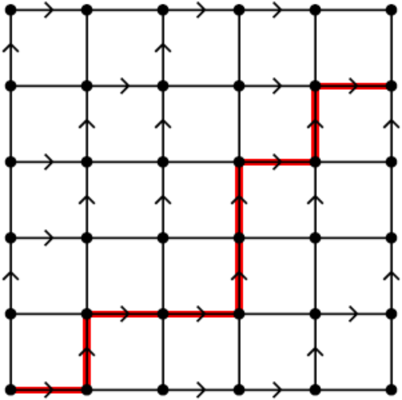
Typical assumption: The environment is *shift-invariant*, or more generally *symmetric*, i.e., $\{\mu_e\}_{e \in E_d}$ is invariant under graph automorphisms of $\mathbb{Z}^d$.

Question:

Does RWRE behave similarly to simple random walk on $\mathbb{Z}^d$? What is the limit behavior?



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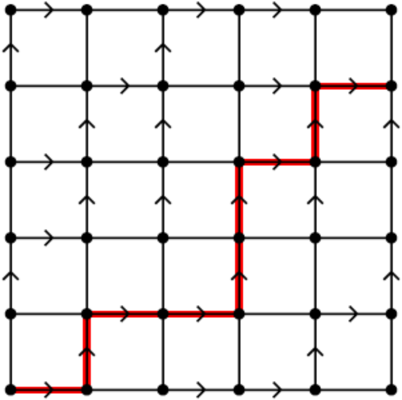

However, in "decent" models almost sure and averaged behaviour are usually similar after scaling.

Question:

Does RWRE behave similarly to simple random walk on $\mathbb{Z}^d$? What is the limit behavior?

That is, consider continuous time random walk $X = \{X_t,\ t \geq 0\}$ on $\mathbb{Z}^d$ in the random environment started from $\mathbf{0}$, with transition probabilities $P_\omega(x, y)$ and exponential waiting times with mean $1/\mu_x$. Let

$$X_t^{(\epsilon)} := \epsilon X_{t/\epsilon^2}.$$

Does $X^{(\epsilon)} := \{X_t^{(\epsilon)},\ t \geq 0\}$ converge to BM in the Skorokhod space $\mathcal{D}_T$? In what sense?

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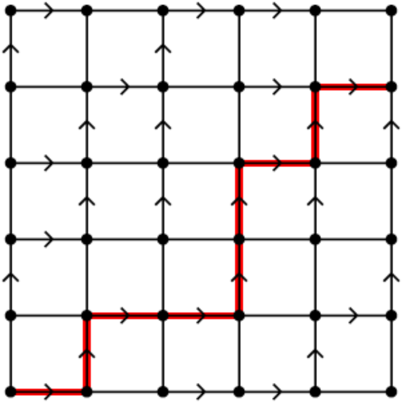
Quenched or annealed invariance principle. Convergence for *almost every environment* or in *averaged* sense.



Definitions

F a bounded continuous function on $\mathcal{D}_T$, Σ a constant matrix, W standard Brownian motion.

(i) The Quenched Functional CLT (QFCLT) holds for X if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $E_\omega F(X^{(\epsilon)}) \to E_{\text{BM}} F(\Sigma W)$ as $\epsilon \to 0$, with $\mathbb{P}$-probability 1.

(ii) The Averaged (or Annealed) Functional CLT (AFCLT) holds for X if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\mathbb{E}\, E_\omega F(X^{(\epsilon)}) \to E_{\text{BM}} F(\Sigma W)$.
This is the same as standard weak convergence with respect to the probability measure $\mathbb{E}\, P_\omega$.

Observe that Σ has to be σ times the identity for some constant σ, by invariance.

Lemma: QFCLT $\Rightarrow$ AFCLT.

General question: Does AFCLT imply QFCLT?

Andres-Barlow-Deuschel-Hambly: If the μ_e are i.i.d., and $\mathbb{P}(\mu_e > 0) > p_c$, then the QFCLT holds.

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De Masi-Ferrari-Goldstein-Wick: If $\mathbb{E}\,\mu_e < \infty$ holds for an ergodic symmetric stationary environment the AFCLT holds.

Question: How about QFCLT? Open.

Our main result

Theorem (Barlow-Burdzy-T.)

There exists a symmetric, stationary and ergodic environment such that for a subsequence $\epsilon_n \to 0$
(a) the AFCLT holds for $X^{(\epsilon_n)}$ with limit W,
but
(b) the QFCLT does not hold for $X^{(\epsilon_n)}$ with limit ΣW for any Σ.

Furthermore, the environment $\{\mu_e\}_{e \in E_d}$ satisfies
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Remark: with slightly weaker condition on the moments we have the full AFCLT (not just for a subsequence).

Results when QFCLT holds

For symmetric, ergodic environments:

Biskup: If d=2, $\mathbb{E}(\mu_e^{-1} \vee \mu_e) < \infty$ then QFCLT holds with $\sigma \neq 0$.

Andres-Deuschel-Slowik: If $d \geq 2$, $\mathbb{E}\,\mu_e^p < \infty$ and $\mathbb{E}\,\mu_e^{-q} < \infty$ with $p^{-1} + q^{-1} < 2/d$, then the QFCLT holds.

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Recall, our environment satisfies: $\{\mu_e\}_{e \in E_d}$ with $\mathbb{E}(\mu_e^p \vee \mu_e^{-p}) < \infty$ for any $p < 1$.

The construction

We do it for $d = 2$.
Fix sequences a_n and b_n.
Choose

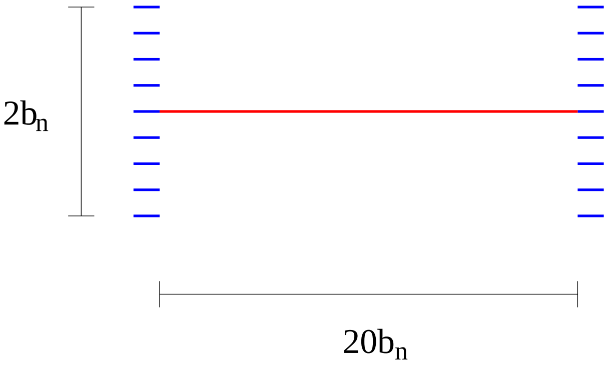
$$\frac{b_n}{a_n} \approx \frac{1}{\sqrt{n}}$$

and

$$a_n \ll a_{n+1}.$$

For $n = 1, 2, \ldots$, we will define *obstacles of level n*, that is, sets of edges with nonunit conductance.
The union of obstacles of level n will be called $\mathcal{D}_n$.

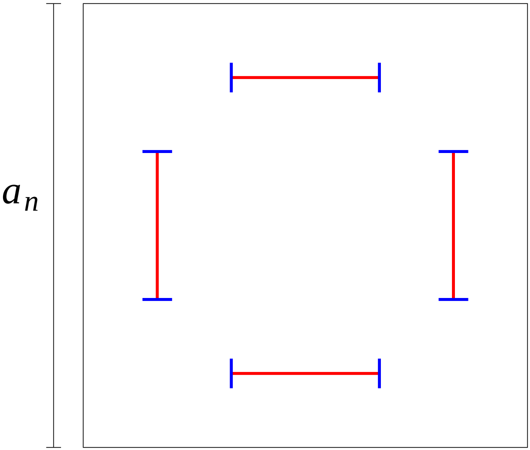
The shape of one obstacle is:



Blue edges have very low conductance η_n. The red line represents edges with very high conductance K_n.

$\eta_n := b_n^{-(1+1/n)}$, $K_n \approx b_n$

At level n, we tile the plane with tiles containing obstacles as follows.



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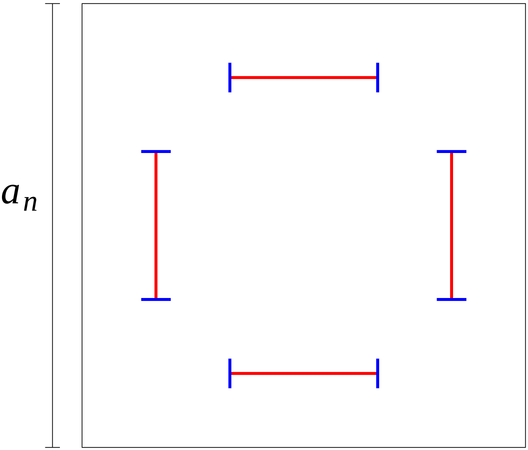
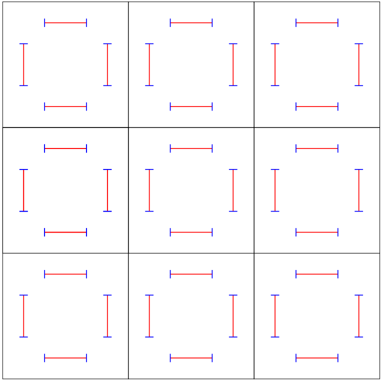


Then shift it randomly, to make the environment symmetric.

Do similarly for level $n+1$, with bigger "tiles" that are unions of tiles from level n. Redefine edge conductances if necessary.



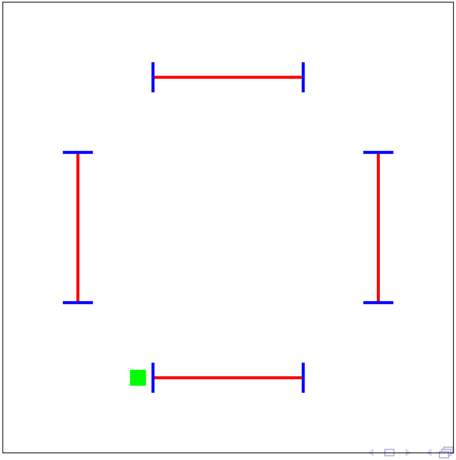
The resulting random conductance is μ_e.

If only $\cup_{m=1}^{n} \mathcal{D}_m$ is taken, we call the conductance μ_e^n.

QFCLT does not hold

From now on $T = 1$.

What is the probability that $\mathbf{0}$ is in the green box for one of the tiles? It is a $\frac{b_n}{4} \times \frac{b_n}{4}$ box, whose center is at distance $b_n/8$ from the blue part.

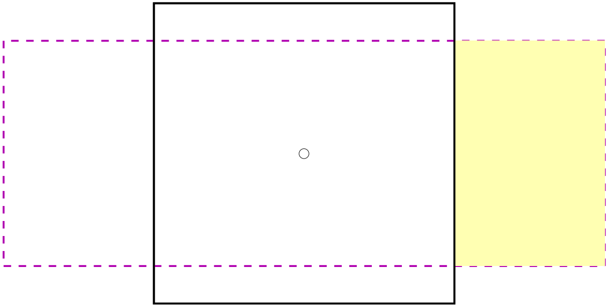


Hence, there are *infinitely many* n's almost surely such that $\mathbf{0}$ is contained in a green box .

Moreover, the same is true if we also require that no $\mathcal{D}_m$ intersects the b_n-neighborhood of the green box, $m > n$.

For a 2-dimensional process $Z = (Z^1, Z^2)$, define the event

$$F(Z) = \left\{ |Z_s^2| < 3/4, |Z_s^1| \leq 2, 0 \leq s \leq 1, Z_1^1 > 1 \right\}.$$



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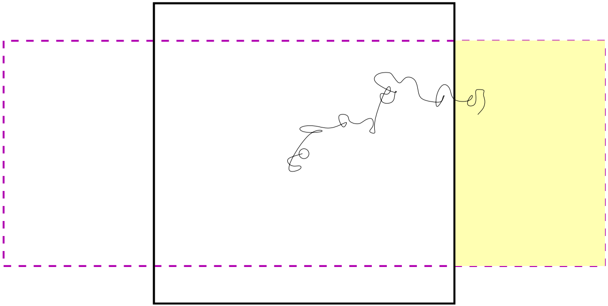
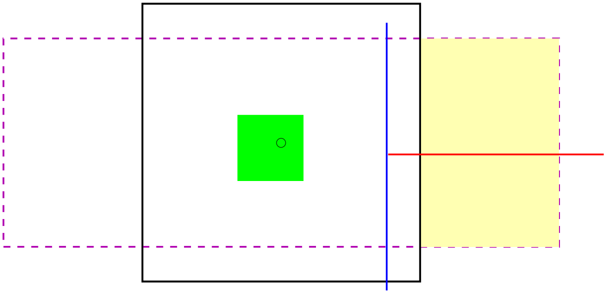


The support theorem implies that $P_{\mathrm{BM}}(F(W)) > 0$.

However, for $\epsilon_n := 1/b_n$, we have $\mathbb{P}(F(X^{(\epsilon_n)})) < cb_n^{-1/n}$ *whenever* $\mathbf{0}$ *is in a green box for level* n.



This happens for infinitely many n's almost surely, hence the QFCLT fails.

AFCLT holds

As before, $\epsilon_n = 1/b_n$.
Recall: the environment $\{\mu_e^n\}$ is the union of the *first n levels of obstacles*.
For $\{\mu_e^n\}$ QFCLT is known (Barlow-Deuschel), since μ_e^n and μ_e^{-n} are bounded away from 0.
By periodicity of $\{\mu_e^n\}$, we can compute effective resistances in boxes, and choose η_n and K_n of the orders mentioned, and so that the limit is indeed $\sum = I$.
This is where the choice of "red" conductances becomes important.

So choosing a_n and $b_n \approx a_n/\sqrt{n}$ large enough, RW in $\{\mu_e^{n-1}\}$ is $1/n$ close to BM.
We can couple RW in $\{\mu_e\}$ with RW in $\{\mu_e^{n-1}\}$ until the first time we hit an obstacle in $\cup_{m \geq n} \mathcal{D}_m$.
The probability of hitting such an obstacle can be bounded using $b_n^2/a_n^2 \approx 1/n$, by a geometric argument as before.

Thank you, András!