k-sets in $\text{PG}(3,q)$ of type $(m, n)$ with respect to planes

Vito Napolitano
Department of Mathematics and Physics
Seconda Università di NAPOLI
• $\mathcal{P} = \mathbb{P}G(d, q)$

• $1 \leq h \leq d-1$ $\mathcal{P}_h = \text{the family of all } h\text{-dimensional subspaces of } \mathbb{P}$

• $0 \leq m_1 < \ldots < m_s$

$K \subseteq \mathbb{P}$ is of class $[m_1, \ldots, m_s]_h$ if $|K \cap \Pi| \in \{m_1, \ldots, m_s\}$ for all $\Pi \in \mathcal{P}_h$

$m_1, \ldots, m_s$ are the intersection numbers of $K$
$K$ is of type $(m_1, \ldots, m_s)_h$ if for every intersection number $m_j$ there is a subspace $\Pi \in \mathcal{P}_h$ such that

$$|K \cap \Pi| = m_j$$
$k$-sets in $P$ via intersection numbers

- **CHARACTERIZATION PROBLEM** (B. Segre point of view in (finite) geometry)

- **EXISTENCE PROBLEM** (codes theory, strongly regular graphs)

- **CLASSIFICATION PROBLEM** (for small values of $q$)
A \((q+1)\)-set of \(\text{PG}(2, q)\), \(q\) odd, of type \((0, 1, 2)_1\) is a (non-degenerate) conic. (B. Segre 1954)

A \((q^2+1)\)-set of \(\text{PG}(3, q)\), \(q\) odd, of type \((0, 1, 2)_1\) is an elliptic quadric. (A. Barlotti 1955, Panella 1955)
A $k$-set of $P$ of type $(m, n)_{d-1}$ spanning $P$ gives rise to a two weight $[k, d+1]$ code with weights $k - m$, $k - n$, and

and

a $k$-set of $P$ of type $(m, n)_{d-1}$ spanning $P$ derives from a two weight $[k, d+1]$ code with weights $k - m$, $k - n$
Intersection with lines

$K$ is of class $[0, 1, q+1]_1 \iff K$ is a subspace of $P$

$K$ is of type $(m)_1 \iff K$ is the empty set $(m= 0)$ or $P$ $(m = q+1)$

$K$ is of type $(0,1)_1 \iff K$ is a point of $P$

There is no $k$-set of type $ (0, q+1)_1$
Intersection with lines

\[ d \geq 3, \ K \text{ of type } (m, n)_1 \text{ in } \mathbb{P}, \ S \text{ a } h\text{-dimensional subspace of } \mathbb{P} : \]

\[ S \cap K \text{ is of type } (m, n)_1 \text{ in } \text{PG}(h, q) \]
**k-sets in \( \text{PG}(2, q) \)**

\[ n \leq q \]

\[ K \text{ of type } (0,n) \Rightarrow n \mid q \]

and (maximal arc)

\[ k = q \cdot n - q + n \]

A hyperoval is a maximal arc with \( n = 2 \).

If \( n > 2 \), for every pair \( (n, q) = (2^a, 2^b) \), \( 0 < a < b \), there are maximal arcs (Denniston (1969); Thas (1974), (1980); Mathon (2002))
“The most wanted research problem”*

Conjecture (J. A. Thas 1975): For $q$ odd there is no maximal arc

The conjecture is true (Ball, Blokhuis, Mazzocca 1997 in Combinatorica)

* T. Penttila, G.F. Royle “Sets of type $(m,n)$ in the affine and projective planes of order 9” (1995 in Designs Codes and Cryptography).
**k-sets in** $\text{PG}(2, q)$

*K a k-set of type $(1, n)$ $\Rightarrow q = p^{2h}$,
$q = (n-1)^2$ and $K$ is a Baer subplane or a Hermitian arc.*

$2 \leq m < n < q+1$ :

*K a k-set of type $(m, m+s)$, $s^2 \geq q$ , $(m, m+s) = (m-1, m+s-1) = 1$ $\Rightarrow q = s^2$
and $k = m(s^2 +s +1)$ or $k = s^3 +s(s-1)(m-1)+m$*
A $k$-set in $\text{PG}(2, q^2)$ of type $(m, m+q)$ with $k = m(q^2+q+1)$

The set of points of the union of $m$ pairwise disjoint Baer subplanes $m < q^2-q+1$
**k-sets in PG(2, q)**

*K a k-set of type \((m, n)\) in PG(2, q)*

\[2 \leq m < n \leq q-1\]

\[\downarrow\]

- \[k^2 - k[1+(q+1)(n+m-1)] + mn(q+1)(q^2+1) = 0\]
- \[n - m \mid q\]
- \[m q + n \leq k \leq (n-1)q + m\]
**Intersection with lines** ($d \geq 3$)

A $k$-set of $P$, $d \geq 3$, of type $(0, n)_1$, $n \leq q$, either is a point ($n = 1$) or $P$ less a hyperplane ($n = q$). (M. Tallini Scafati 1969)

A $k$-set of $P$, $d \geq 3$, of type $(m, q+1)_1$ is $P$ less a point ($m = q$) or a hyperplane ($m = 1$).
Intersection with lines

\[ d \geq 3 \quad n \geq 2, \quad m > 0: \]

\[ K \text{ is a } k\text{-set of type } (1, n)_{1} \text{ of } P \Rightarrow K \text{ is a hyperplane} \]

**Proof:**

- \( H = K \cap S_{3} \text{ is a } k\text{-set of type } (1, n)_{1} \text{ in } PG(3, q) \)
Intersection with lines \((d \geq 3)\)

- \(\text{PG}(3,q)\) has no set of type \((1, n)_1\) and \(n < q\) \((\text{PG}(2,q)\) plays a special rôle (as in Tallini Scafati (1969))\)

- \(\text{P}\) has no set of type \((1, q)_1\)

  \(K\) is a \(k\)-set of type \((m, q)_1\) of \(\text{P}\) \(\Rightarrow\) \(K\) is \(\text{P}\) less a hyperplane
Intersection with lines \((d = 3)\)

\(K\) a \(k\)-set of type \((m, n)\) in \(\text{PG}(3, q)\)

\((2 \leq m < n \leq q-1)\)

\[\implies q\text{ is a odd square}\]

\[k = \left[1 + (q^2+1)(q + \varepsilon \cdot q^{1/2}) \pm q \cdot q^{1/2}\right]/2\]

\((\varepsilon = \pm 1)\)

\[m = \left[q+1 - q^{1/2} (1- \varepsilon)\right]/2\]

\[n = \left[q+1 + q^{1/2} (1- \varepsilon)\right]/2\]
**Intersection with lines** \((d \geq 3)\)

*\(S\) a 3-dimensional subspace of \(\mathbb{P}^3\) \(K\) of type \((m, n)\) in \(\mathbb{P}\)

\(S \cap K\) is of type \((m, n)\) in \(\mathbb{P}G(3, q)\) \(\Rightarrow\)

- \(m = \left\lfloor (q+1- q^{\frac{1}{2}}(1- \varepsilon)) \right\rfloor /2\)
- \(n = \left\lfloor (q+1+ q^{\frac{1}{2}}(1- \varepsilon)) \right\rfloor /2\)
- \(k = \left\lfloor 1 + (q^{d-1} + \cdots + q+1)(q + \varepsilon \cdot q^{\frac{1}{2}}) \pm (q^{\frac{1}{2}})^d \right\rfloor /2\)
  
  \((\varepsilon = \pm 1)\)
Intersection with lines \((d \geq 3)\)

Characterizations of Quadrics and Hermitian varieties as sets of class \([0, 1, n, q+1]_1\) with some extra regularity condition (e.g. at each point \(p\) the set of 1-secant lines is a subspace, on each point there is at least one \(n\)-line):

**quadratic sets and \(n\)-varieties.**
Intersection with lines

K with set of line-intersection numbers
\( \mathcal{I} = \{0, 1, \ldots, s\} \)

\( m = \min \mathcal{I} \setminus \{0\} \) and \( n = \max \mathcal{I} \setminus \{0\} \)

\( b_i = \# \text{i-secant lines, } i \in \{0, 1, \ldots, s\} \)

\( \Theta_r = q^r + q^{r-1} + \ldots + q + 1 \)

Then
Intersection with lines \((d \geq 2)\)

\[b_0 \geq [k2 - k(1+(m+n-1) \Theta_{r-1}) + mn \Theta_r \Theta_{r-1} / \Theta_1]/mn\]

with equality iff \(K\) is of class \([0, m, n]_1\).

Moreover

\(b_0\) and the above ratio are both \(= 0\) iff \(K\) is of type \((m, n)_1\).
**k-sets of type \((m, n)_h\) in \(P\)**

A k-set of type \((m, n)_h\), \(h \leq d-2\) is of type \((r, s)_i\) for every 
\(i \in \{h+1, \ldots, d-1\}\)

\(K\) a k-set of type \((m, n)_{d-1}\) \(\Rightarrow n - m \mid q^{d-1}\) [Tallini Scafati 1969]

\(K\) a \(q^t\)-set \((m, n)_{d-1}\) is either a point or \(P\) less a hyperplane  [L. Berardi – T. Masini On sets of type \((m,n)_{r-1}\) in PG(r,q) Discrete Math 2009]
**k-sets of type \((m, n)_2\) in \(PG(3, q)\)**

- Hyperbolic quadrics,
- \((q^2+1)\)-caps
- non-singular Hermitian varieties,
- subgeometries
- sets of points on \(m\) pairwise skew lines
- any subset of the ovoidal partition of \(PG(3, q)\)
- a subgeometry \(G\) union a family of pairwise skew lines external to \(G\)

\[(n - m = q)\]
**k-sets of type \((m, n)_2\) in \(PG(3, q)\)**

**J.A. Thas (1973):**

The only sets in \(P\) of type \((1, n)\) w.r. to hyperplanes are the lines or the ovoids of \(PG(3, q)\).

The proof uses an algebraic argument.

Such result has been proved in a more general setting and in a geometric way:
**k-sets of type \((m, n)_2 in PG(3, q)\)**

(S = a 3-dimensional locally projective planar space of order q)

\(K \subseteq S \text{ and meeting every plane in either 1 or } n \ (n > 1) \text{ points is a line (with } q + 1 \text{ points) or a set of } q^2 + 1 \text{ points no three of which are collinear.}\)
**k-sets of type \((m, n)_2\) in PG(3, q)**


\(S = \) a 3-dimensional locally projective planar space of order \(q\)

A set \(K\) of points of \(S\) meeting every plane in either 2 or \(n\) \((n > 2)\) points is a pair of skew lines \(\text{both of size } q + 1\)
**$k$-sets of type $(m, n)_2$ in $\text{PG}(3, q)$**

O. Ferri (1980):

* $K$ cap of type $(m, n)_2$ in $\text{PG}(3, q)$

$\Downarrow$

*K is an ovoid ($m=1$) or $q=2$ $m=0$ and $K$ is $\text{PG}(3, 2)$ less a plane*
$k$-sets of class $[3, n]_2$ in $\text{PG}(3, q)$

The union of three pairwise skew lines in $\text{PG}(3, q)$,

A plane in $\text{PG}(3, 2)$

$\text{PG}(3, 2)$

$\text{PG}(3, 2)$ embedded in $\text{PG}(3, 4)$
k-sets of type \((3, n)_2\) in \(PG(3, q)\)

- \(q > 2 \Rightarrow (n - 3)| q\) (i.e. \(n \leq q + 3\))

- either \(n = q + 3\) or \(s \leq 3\) for each \(s\)-line
\textit{k-sets of type } (3, q+3) \_2 \textit{ in } PG(3, q)

V.N. - D. Olanda (2012): n = q+3

\begin{itemize}
  \item If K contains no line then q = 3 or 4.
  \item If q = 4 then K = PG(3, 2).
  \item If q = 3 then k = 12 or k = 15 and K is one of the following three Examples:
\end{itemize}
$k$-sets of type $(3, 6)_2$ in $PG(3, 3)$

$K_1 (k = 12)$
A (1 0 0 0) , B(0 1 0 0), C (0 1 1 1), D (0 0 1 0),
E (0 1 0 1), F (0 0 0 1), G (1 0 0 1), H (1 1 0 1) ,
I (1 0 2 0), L (1 2 2 0), M (1 0 2 1), N (0 1 1 0)

$K_2 (k = 15)$
A (1 1 2 1) , B(1 0 0 0), C (0 1 0 0), D (0 0 1 0),
E (0 0 0 1), F (0 0 1 2), G (1 1 1 1), H (1 1 1 2) ,
I (1 0 2 0), L (1 2 2 0), M (0 1 2 2), N (0 1 1 0),
O(1 0 2 2) ,P(1 2 1 1), Q(1 2 1 2)
$k$-sets of type $(3, 6)_2$ in $PG(3, 3)$

$K_3 (k = 15)$

A (1 0 0 0), B (0 1 1 0), C (0 1 0 0),
D (0 0 1 0), E (0 0 0 1), F (1 1 2 1),
G (1 1 1 1), H (1 0 1 2), I (1 1 1 2),
L (1 2 2 0), M (0 1 2 2), N (1 1 2 2),
O (0 1 2 1), P (1 0 1 1), Q (1, 0, 2, 0)
**k-sets of type (3, 6)₂ in PG(3, 3)**

**K₁**: [12, 4, 9]₃-code with second weight 9  
(“subcode” of the ternary extended Golay code)

**K₂ and K₃**: two different [15, 4, 6]₃-codes with second weight 12

**K₁K₂ and K₃**: an exhaustive research obtained by adapting a program in MAGMA contained in

[S. Marcugini, F. Pambianco, Minimal 1-saturating sets in PG(2, q), q ≤ 16, Austral. J. Combin. 28 (2003), 161-169]
$k$-sets of type $(3, q+3)_2$ in $PG(3, q)$

V.N. - D. Olanda (2012): $n = q+3$

*If $K$ contains a line then $K$ is the set of the points of the union of three skew lines.*
$k$-sets of type $(3, n)_2$ in $\text{PG}(3, q)$

$n < q+3$:

- There is a 3-line $L$ such that all planes on $L$ are $h$-plane:

$q = 8$, $n = 7 = q/2 + 3$ and $k = 39$. 
$k$-sets of type $(3, n)_2$ in $PG(3, q)$

- on each 3-line there is at least one 3-plane:

  either

  there is a $n$-plane with a point on no 2-line and $q = 2^t$, $n = 2^s + 3$, $2 \leq s \leq t - 1$

  or

  $$(q+1)n - 4q \leq k \leq qn - 3q + 3$$
$k$-sets of type $(m, n)_2$ in $PG(3, q)$

$m \leq 3 \Rightarrow n \leq m + q$ and $k \geq m(q+1)$

$m \leq 3 \Rightarrow m \leq q + 1$
**k-sets of type \((m, n)_2\) in PG(3, q) with \(m \leq q+1\)**

Let \(K_1\) be the only 12-set of type \((3, 6)_2\) in PG(3, 3).

Let \(L\) be an external line to \(K_1\): \(\Omega = K_1 \cup L\).

\(\Omega\) is a 16-set of type \((4, 7)_2\) in PG(3, 3):

\([16, 4, 9]_3\) code with second weight 12.
**k-sets of type \((m, n)\) in \(PG(3, q)\) with \(m \leq q+1\)**

**Planes:** \(m = q + 1\) \(k = q^2 + q + 1 < m(q+1)\)

\[\text{m} \leq q \Rightarrow k \geq m(q+1)\]

**Theorem (V.N. 20??)**

A k-set \(K\) in \(PG(3,q)\) of type \((q+1, n)\), \(m \leq q+1\) is a plane or \(k \geq (q+1)^2\) and at least one external line exists. If \(k = (q+1)^2\) and \(K\) contains at least \(q - q^{1/2}\) pairwise skew lines then either
$k$-sets of type $(m, n)_2$ in $PG(3, q)$ with $m \leq q+1$

$K$ is the set of points of $q+1$ pairwise skew lines or $q = s^2$ and $K$ is the set of points of $PG(3,s)$ union the points of $s^2$ – $s$ pairwise skew lines.
**k-sets of type \((m, n)_2\) in \(PG(3, q)\)
with \(m \leq q+1\)

\(\Omega = K_1 \cup L\) the 16-set of type \((4, 7)_2\) in \(PG(3, 3)\):

\[4 = 3 + 1 = q + 1\] and \[16 = m(q+1)\] \(\Rightarrow\) an external line \(M\) to exist \(\Omega\)

\(\Omega \cup M\) is a 20-set of type \((5, 8)_2\) in \(PG(3, 3)\)

\([20, 4, 12]_3\) code with second weight 15
k-sets of type \((m, n)_2\) in PG(3, q) with \(m \leq q\)

Theorem

Let \(K\) be a set of points of PG(3, q) of type \((m, n)_2\) with \(m \leq q\) and \(k = m(q+1)\). If \(s \geq m\) for every \(s\)-line with \(s \geq 3\) then either \(K\) is the set of points of \(m\) pairwise skew lines, or \(q = (m-1)^2\) and \(K\) is the subgeometry PG(3, \(m-1\)) or \(m = q = 3\) and \(K\) is one of the sets \(K_1\) and \(K_2\).
**k-sets of type \((m, n)_2\) in \(PG(3, q)\) with \(m \leq q\)**

**Theorem**

A k-set \(K\) in \(PG(3,q)\) of type \((m, n)_2\) \(m \leq q\) is and \(k = m(q+1)\). If contains at least \(q - q^{1/2} - 1\) pairwise skew lines then either \(K\) is the set of points of \(m\) pairwise skew lines or \(m = q = s^2\) and \(K\) is the set of points of \(PG(3,s)\) union the points of \(s^2 \setminus s\) -1 pairwise skew lines.
A non-singular Hermitian variety $H(3, q^2)$ in $PG(3, q^2)$ is of type $(1, q+1, q^2+1)_1$ and $(q^3+1, q^3+q^2+1)_2$.

L. Berardi – T. Masini On sets of type $(m,n)_{r-1}$ in $PG(r,q)$ Discrete Math 2009:

A $k$-set of type $(m,n)_2$ in $PG(3, q^2)$ is of *Hermitian type* if $k = q^3+q^2+q+1$, $m = q^3+1$, $n = q^3+q^2+1$. 
Hermitian variety in $\text{PG}(3, q^2)$

Theorem (Berardi-Masini 2009)

A $(q^3+q^2+q+1)$-set of type $(m, n)_2$ in $\text{PG}(3, q^2)$ is of Hermitian type.

J. Schillewaert-J.A. Thas Characterizations of hermitian varieties by intersection numbers
Designs Codes and Cryptography (2008):
Hermitian variety in $\text{PG}(3, q^2)$

Theorem (B. Schillewaert-J. Thas 2008)

A $k$-set of types $(1, q+1, q^2+1)_1$ and $(q^3+1, q^3+q^2+1)_2$ in $\text{PG}(3, q^2)$ is a Hermitian variety $H(3, q^2)$

Moreover, they characterize $H(d, q^2)$ with respect to planes and solids for any dimension $d \geq 4$

(First solve the case $d = 4$, then study $K \cap S$ and $K \cap T$ with $S, T$ a 3-space and a 4-space of $\mathbb{P}$ respectively)
Hermitian variety in $PG(3, q^2)$

Theorem (V.N.20??) Let $K$ be a $m(q+1)$-set of $PG(3, q)$, of types $(1, s+1, q+1)_1$ and $(m, n)_2$, $1 \leq s \leq q-1$, then $q = s^2$ and $K$ is a Hermitian variety $H(3, q^2)$