On Topology Preservation for Triangular Thinning Algorithms

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Abstract. Thinning is a frequently used strategy to produce skeleton-like shape features of binary objects. One of the main problems of parallel thinning is to ensure topology preservation. Solutions to this problem have been already given for the case of orthogonal and hexagonal grids. This work introduces some characterizations of simple pixels and some sufficient conditions for parallel thinning algorithms working on triangular grids (or hexagonal lattices) to preserve topology.

Keywords: Triangular grids, Topology preservation, Thinning.

1 Introduction

The concept of skeletonization serves as a useful shape descriptor in various areas of image processing and pattern recognition [13]. Thinning is a widely used iterative technique for producing the skeletons of binary objects [8,16]. Thinning algorithms are composed of reductions (i.e., some object points having value of "1" in a binary picture that satisfy certain topological and geometric constrains are changed to "0" ones simultaneously).

A fundamental requirement of these algorithms is topology preservation [7]. At first, sufficient conditions for the topological correctness of reduction operators working on orthogonal grids were proposed by Ronse and Kong [6,12].

2D digital pictures on hexagonal and triangular grids have been studied by a number of authors [7,10]. There were also various thinning algorithms working on hexagonal and triangular grids proposed in [1,2,5,14,15,17]. For the hexagonal case, Kardos and Palágyi established some sufficient conditions for topology preserving reductions [5]. A triangular grid, which is formed by a tessellation of regular triangles, corresponds, by duality, to the hexagonal lattice, where the points are the centers of that triangles, see Fig. 1. The geometry of triangular grids has been already investigated (see for example [3,11]), however, their topological properties have been poorly dealt with.

In this paper we study reductions on triangular grids in the view of topology preservation. For this purpose, first we discuss some characterizations of so-called simple pixels which play a key role in this field, and we also present some sufficient conditions for topology preserving reductions. Our result can be applied to construct topologically correct triangular parallel thinning algorithms.

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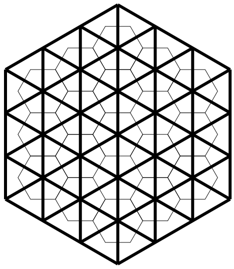


Fig. 1. A triangular grid and the hexagonal lattice dual to it. Triangular pixels are bounded by thick line segments. Pixel centers are joined with thin line segments.

The rest of this paper is organized as follows. Section 2 briefly introduces some basic notions of digital topology. Then, in Section 3 we introduce two kinds of characterizations of simple pixels. Section 4 discusses our sufficient conditions mentioned above. To illustrate the usefulness of these conditions, we define a pair of reductions in Section 5 and validate its topological correctness.

2 Basic Notions and Results

We use the fundamental concepts of digital topology as reviewed by Kong and Rosenfeld [7].

Let us consider the *digital space* V, and let us call the elements of V *pixels*. We refer with the notation $N_k(p)$ to the set of pixels that are *k-adjacent* to pixel $p \in V$ and let $N_k^*(p) = N_k(p) \setminus \{p\}$. Note that reflexive and symmetric adjacency relations are generally considered (i.e., $p \in N_k(p)$ and if $q \in N_k(p)$, then $p \in N_k(q)$). The sequence of distinct pixels $\langle x_0, x_1, \ldots, x_s \rangle$ is called a *k-path* of length s from pixel x_0 to pixel x_s in a non-empty set of pixels $X \subseteq V$ if each pixel of the sequence is in X and x_i is k-adjacent to x_{i-1} $(i = 1, \ldots, s)$. Note that a single pixel is a k-path of length 0. In the special case when $x_0 = x_n$, we call the k-path an *k-cycle*. Two pixels $p, q \in V$ are said to be *k-connected* in the set $X \subseteq V$ if there is a k-path from p to q in X. A set of pixels $X \subseteq V$ is *k-connected* in the set of pixels $Y \supseteq X$ if any two pixels in X are k-connected in Y. A set of pixels $X \subseteq V$ is a *k-component* in the set of pixels $Y \supseteq X$ if X is k-connected in Y, but the set $X \cup \{y\}$ is not k-connected in Y for any $y \in Y \setminus X$ (i.e., X is a maximal k-connected set of pixels in Y).

An (m, n) *digital picture* is a a quadruple $\mathcal{P} = (V, m, n, B)$. Each pixel in $B \subseteq V$ is called a *black pixel* and has a value of 1 assigned to it. Picture $\mathcal{P}$ is finite if it contains finitely many black pixels. Each pixel in $V \setminus B$ is called a *white pixel* and has a value of 0 assigned to it. Adjacency relation m is assigned to black pixels, and a *black component* or an *object* is an m-connected set of pixels in B. Adjacency relation n is assigned to white pixels, and a *white component* is an n-connected set of pixels in $V \setminus B$. In a finite picture there is a unique white component that is called the *background*. A finite white component is called a *cavity*.

A *reduction* transforms a binary picture only by changing some black pixels to white ones (which is referred to as the deletion of 1's). A 2D reduction does *not* preserve topology [6] if any black component is split or is completely deleted, any white component is merged with another white component, or a new white component is created.

A black pixel is called a *border pixel* in an (m, n) picture if it is m-adjacent to at least one white pixel. A *simple pixel* is a black pixel whose deletion is a topology preserving reduction [7]. Let $\mathcal{P}$ be an (m, n) picture. The set of black pixels $D = \{d_1, \ldots, d_k\}$ is called a *simple set* of $\mathcal{P}$ if D can be arranged in a sequence $\langle d_{i_1}, \ldots, d_{i_k}\rangle$ in which d_{i_1} is simple and each d_{i_j} is simple after $\{d_{i_1}, \ldots, d_{i_{j-1}}\}$ is deleted from $\mathcal{P}$, for $j = 2, \ldots, k$. (By definition, let the empty set be simple.)

In this paper our attention is focussed on pictures sampled on the triangular grid denoted by T (see Fig. 1). Each pixel in T is a regular triangle and a point in the hexagonal lattice is associated to it [10]. Two kinds of adjacency relations have been considered in the triangular grid T: two triangles (pixels) are 3-*adjacent* if they share an edge, and two triangles are 12-*adjacent* if they share at least one vertex. It is easy to see that $N_3(p) \subset N_{12}(p)$ for any $p \in T$. The considered adjacencies on the hexagonal lattice is shown in Fig. 2. The set composed by six pairwise 12-adjacent pixels of T is called a *unit hexagon* (see Fig. 2c).

In order to avoid connectivity paradoxes [7] and verify the discrete Jordan's theorem [10], $(m, n) = (12, 3)$ and $(m, n) = (3, 12)$ pictures are considered on the triangular grid T (where $m \neq n$). That is why we intruduce characterizations of simple points and give suficient conditions for topology preserving reductions for $(12, 3)$ and $(3, 12)$ pictures in the next two sections.

3 Characterizations of Simple Pixels

If we want to verify whether a reduction preserves topology or not, first we must be able to determine, which pixels in an object are simple. For this aim, we present some characterizations of simple pixels in both (12,3) and (3,12) pictures.

Theorem 1. *Let p a black pixel in a picture (T, m, n, B) $((m, n) = (12, 3)$, $(3, 12))$. Pixel p is simple if and only if all the following conditions are satisfied:*

1. *Pixel p is m-adjacent to exactly one m-component of $N^*_{12}(p) \cap B$.*
2. *Pixel p is n-adjacent to exactly one n-component of $N_{12}(p) \setminus B$.*

Proof. First, we prove that if p is simple in picture (T, m, n, B), then Conditions 1 and 2 are fulfilled. Let us denote $C(p)$ and $\overline{C}(p)$ the number of m-components of $N^*_{12}(p) \cap B$ and the number of n-components of $N_{12}(p) \setminus B$, respectively. Thus we want to prove that $C(p) = \overline{C}(p) = 1$. We give an indirect proof for this, therefore, let us suppose that p is simple but at least one of Conditions 1 and 2 does not hold.

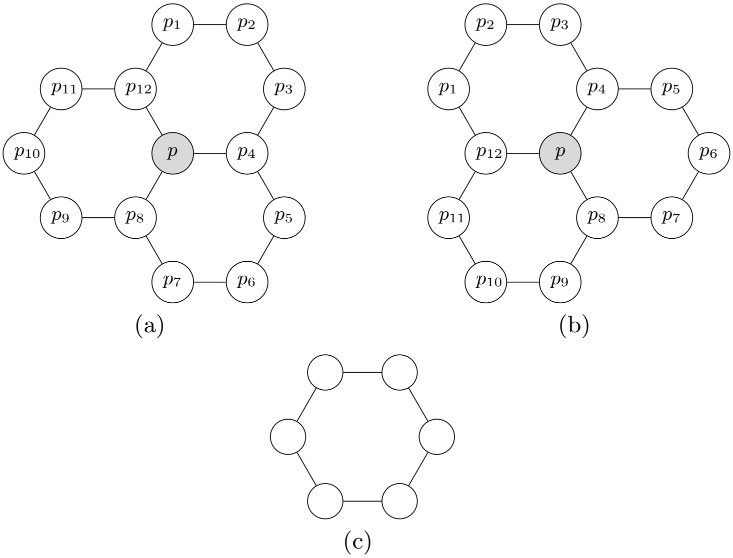


Fig. 2. Indexing schemes for $N_3(p) = \{p_4, p_8, p_{12}\}$ and $N_{12}(p) = \{p_1, \ldots, p_{12}\}$ (a)-(b), and a unit hexagon (c)

First, we will see that in this case none of Conditions 1 and 2 is fulfilled, i.e., there exist some pixels $p_{i_1}, p_{i_2} \in N_{12}^*(p) \cap B$ and $p_{j_1}, p_{j_2} \in N_{12}(p) \setminus B$ such that p_{i_1} and p_{i_2} are not m-connected in $N_{12}^*(p) \cap B$, and the pixels p_{j_1}, p_{j_2} are not n-connected in $N_{12}(p) \setminus B$:

- The assumption $C(p) = 0$ implies that p is an isolated pixel, which contradicts the simplicity of p since an object is completely deleted if an isolated point is deleted. Hence, $C(p) \neq 0$.
- The relationship $\overline{C}(p) = 0$ means that p is not a border pixel, thus again p could not be simple since deletion of a non-border pixel creates a new cavity. Therefore, $\overline{C}(p) \neq 0$.
- Let us examine the case when $C(p) \geq 2$. This directly implies that there exist two pixels $p_{i_1}, p_{i_2} \in B$ such that p_{i_1} and p_{i_2} are not m-connected in $N_{12}^*(p) \cap B$.
  - If $(m, n) = (12, 3)$, then p must have two white 3-neighbors, or else any pixel in $N_{12}^*(p) \cap B$ would be 12-adjacent to at least one black 3-neighbor of p, and as any two 3-neighbors of p are 12-adjacent, as well, p_{i_1} and p_{i_2} would be 12-connected in $N_{12}^*(p) \cap B$. Let us denote the mentioned two white 3-neighbors p_{j_1} and p_{j_2}. For the sake of clarity, let for example $p_{j_1} = p_4$, $p_{j_2} = p_8$. (Any other possible case could be examined similarly because of the symmetrical structure of $N_{12}(p)$.)
  If p_{12} is a black pixel, then at least one of the pixels p_5, p_6, and p_7 must be black, and if p_{12} is white, then at least two of the sets $\{p_1, p_2, p_3\}$, $\{p_5, p_6, p_7\}$, and $\{p_9, p_{10}, p_{11}\}$ must contain a black pixel, or else p would

be 12-adjacent to exactly one black component in $N^*_{12}(p) \cap B$, which is a contradiction with our assumption. Hence, p_4 and p_8 are not 3-connected in $N^*_{12}(p) \setminus B$. Consequently, there exist two black pixels $q, r \in N^*_{12}(p) \cap B$ such that q and r are not 12-adjacent in $N^*_{12}(p) \cap B$ and they are not contained in the same 3-path between p_{j_1} and p_{j_2} in $N^*_{12}(p)$, and thus, p_{j_1} and p_{j_2} are not contained in the same 3-path between q and r in $N^*_{12}(p)$. Without loss of generality, let $p_{i_1} = q$ and $p_{i_2} = r$.

- If $(m, n) = (3, 12)$, then p_{i_1} and p_{i_2} are 3-adjacent to p. Let, for example, $p_{i_1} = p_4$ and $p_{i_2} = p_8$. Each of the sets $\{p_3, p_5\}$ and $\{p_7, p_9\}$ contains at least one black pixel, or else a black component would be split by the removal of p. If p_{12} is a white pixel, then at least one of the pixels p_5, p_6, and p_7 must be white, and if p_{12} is black, then at least two sets of $\{p_1, p_2, p_3\}$, $\{p_5, p_6, p_7\}$, and $\{p_9, p_{10}, p_{11}\}$ must contain a white pixel, or else p would be 3-adjacent to exactly one 3-component of $N^*_{12}(p) \cap B$. By the latter four relationships it can be shown that p is 12-adjacent to at least two white 12-components of $N^*_{12}(p) \setminus B$. This implies that there exist two white pixels $q, r \in N^*_{12}(p) \setminus B$ such that q and r are not 12-adjacent in $N^*_{12}(p) \setminus B$ and they are not contained in the same 3-path between p_{i_1} and p_{i_2} in $N^*_{12}(p)$. It is also easy to see that q and r are not 12-connected in $N^*_{12}(p) \setminus B$. Without loss of generality, let $p_{j_1} = q$ and $p_{j_2} = r$.

– Now, let us consider the case when $\overline{C}(p) \geq 2$, i.e., there exist two pixels $p_{j_1}, p_{j_2} \in T \setminus B$ such that p_{j_1} is not n-connected to p_{j_2} in $N_{12} \setminus B$. We can get a similar result to the one derived in the previous case, if we follow the same train of thoughts for the situations $(m, n) = (3, 12)$ and $(m, n) = (12, 3)$, as in the previous case for $(m, n) = (12, 3)$ and $(m, n) = (3, 12)$, respectively. To do this, we only need to interchange the adjectives "white" and "black", the set notations $N^*_{12}(p) \cap B$ and $N_{12}(p) \setminus B$, and the indices i_1, i_2 and j_1, j_2, respectively.

Let us take such pixels $p_{i_1}, p_{i_2} \in N^*_{12}(p) \cap B$ and $p_{j_1}, p_{j_2} \in N_{12}(p) \setminus B$ that have the property shown above. Obviously, p_{i_1} and p_{i_2} are m-connected in $B \setminus \{p\}$, furthermore, p_{j_1} and p_{j_2} are n-connected in $T \setminus B$, or else a black component would be split by the removal of p, or two white components would be merged, which contradicts the simplicity of p.

Let P_1 be the set of the pixels of an m-cycle that we get if we extend an m-path between p_{i_1} and p_{i_2} with p, furthermore, let P_2 be the set of the pixels of an n-cycle that we get if we extend an n-path between p_{j_1} and p_{j_2} in $T \setminus B$ with p.

It is easy to see that picture $(T, n, m, T \setminus P_1)$ contains two objects. Due to the assumptions on p_{j_1} and p_{j_2}, it follows that p_{j_1} and p_{j_2} are not n-connected in $T \setminus P_1$ (i.e., p_{j_1} and p_{j_2} are contained in distinct objects in picture $(T, n, m, T \setminus P_1)$), or else we came to a contradiction with our assumption that p_{j_1} and p_{j_2} are not contained in the same 3-path between p_{i_1} and p_{i_2} in $N^*_{12}(p)$.

An object of picture $(T, n, m, T \setminus P_1)$ must contain all elements of $P_2 \setminus \{p\}$, or else p_{j_1} and p_{j_2} would not be n-connected in $T \setminus B$, thus the number of white components would be reduced by the removal of p. However, this leads to a contradiction with the above stated fact that p_{j_1} and p_{j_2} are not contained in the same object in $(T, n, m, T \setminus P_1)$.

Hence from this follows that if p is simple, then both of Conditions 1 and 2 hold.

Now, let us suppose that Conditions 1 and 2 are satisfied. We will see that the removal of p preserves topology:

– Let q_1 and q_2 be two black pixels in (T, m, n, B) such that q_1 and q_2 are both m-connected to p, i.e., these pixels are contained by the same black component. It is easy to verify that in this case, there exists an m-path in B between q_1 and q_2 that contains a subpath $\langle p_{i_1}, p, p_{i_2} \rangle$ $(i_1, i_2 \in \{1, 2, \ldots, 12\}, i_1 \neq i_2)$. However, by Condition 1, p_{i_1} and p_{i_2} are also m-connected in $N^*_{12}(p) \cap B$, which means that there is also an m-path between q and r that does not contain p. This implies that after the removal of p, q, and r will still fall into the same black component, i.e., no black component is split by the removal of p.
– Now, let r_1 and r_2 be two white pixels belonging to distinct white components in (T, m, n, B). If r_1 is n-connected to a pixel $p_j \in N_3(p) \setminus B$, then r_2 cannot be n-connected to any pixel in $N_{12}(p) \setminus B$, or else r_1 and r_2 would fall into the same white component, as, by Condition 2, p_j is n-connected to all pixels of $N_3(p) \setminus B$. Therefore, after the removal of p, r_1 and r_2 will still fall into two distinct white components, which means that no cavity is merged with the background nor with another cavity by the removal of p.
– Furthermore, no object is completely removed by the deletion of p, or else this would mean that p is an isolated object pixel, but this would contradict Condition 1. Even new cavities cannot be created by the removal of p, because this could only arise, if $N^*_m(p)$ contained only black pixels, which contradicts Condition 2.

By the above observations we can state that if Conditions 1 and 2 hold, then p is simple. $\square$

Some configurations of simple and non-simple pixels in (12,3) pictures are shown in Fig. 3.

We remark that Theorem 4.1 in [7] states a similar relationship as above for (8,4) and (4,8) pictures sampled on the orthogonal grid $\mathbb{Z}^2$.

The following lemma points out to a kind of a duality between (12,3) and (3,12) pictures concerning simple pixels.

Lemma 1. *Pixel p is simple in picture (T, m, n, B) $((m, n) = (12, 3), (3, 12))$, if and only if p is simple in picture $(T, n, m, (T \setminus B) \cup \{p\})$.*

Proof. Let us suppose that p is simple in picture (T, m, n, B). Note that we get picture $(T, n, m, T \setminus B)$ from (T, m, n, B) by recoloring all its pixels and switching the applied types of adjacency relation between black and white pixels. From this follows, that each white/black component of $(T, y, x, T \setminus B)$ coincides with a black/white component of (T, x, y, B).

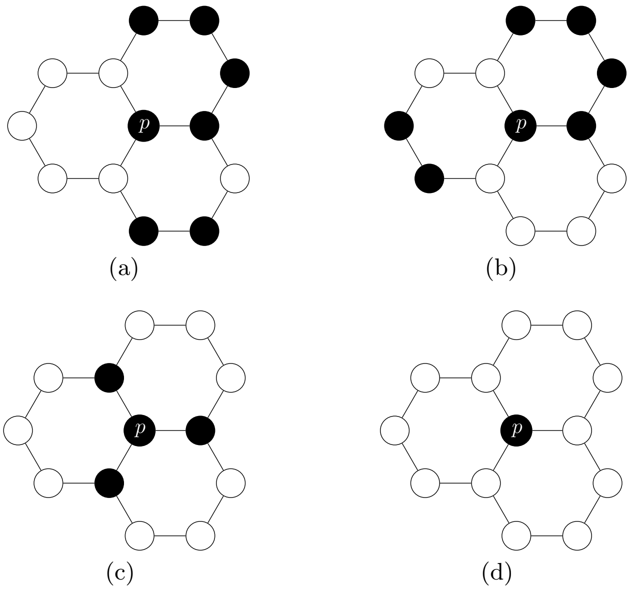


Fig. 3. Examples for a simple (a) and three non-simple (b-d) pixels in (12,3) pictures. Pixel p is not simple in (b) since its deletion may split an object (Condition 1 of Theorem 1 is violated). A white 3-component (singleton cavity) is created by deletion of p in (c) (Condition 2 of Theorem 1 is violated). Pixel p is also not simple in (d), since its deletion completely deletes a (singleton) object (Condition 1 of Theorem 1 is violated).

By Theorem 1, p is m-adjacent to exactly one m-component of $N_{12}^*(p) \cap B$ in (T, m, n, B), which we will hereafter denote by $\mathcal{C}_1$. Furthermore, p is n-adjacent to exactly one n-component of $N_{12}(p) \setminus B$, which will be denoted by $\mathcal{C}_2$.

According to the above observations, it is easy to show that in picture $(T, n, m, T \setminus B)$ and thus in picture $(T, n, m, (T \setminus B) \cup \{p\})$ the white component coinciding with $\mathcal{C}_1$ is the only m-component of $N_{12}(p) \setminus (T \setminus B)$ being m-adjacent to p, and the black component coinciding with $\mathcal{C}_2$ is the only n-component of $N_{12}^*(p) \cap (T \setminus B)$ being n-adjacent to p. Hence, p is simple in picture $(T, n, m, (T \setminus B) \cup \{p\})$, as well.

The proof in the opposite direction can be carried out similarly. □

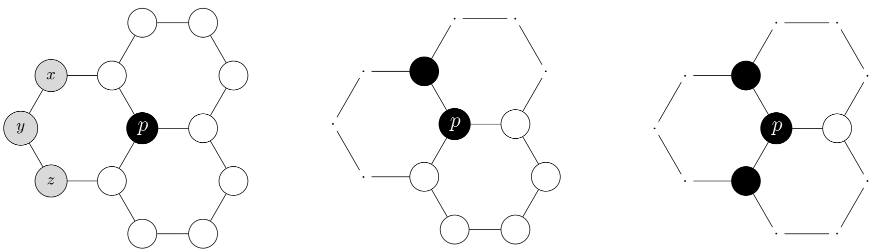
For a better understanding of the concept of Lemma 1 , let us again examine the configurations in Fig. 3. It can be easily verified that if we "invert" those configurations by switching the black and white pixels in $N_{12}(p)$, then we get some other configurations such that the first example ("inverted" version of Fig. 3a) represents a simple pixel in a (3,12) picture, and the remaining three examples ("inverted" versions of Figs. 3b-d) refer to non-simple pixels in (3,12) pictures.

Using the relationship formulated in Theorem 1 to check simple pixels may not be convenient for implementational purposes. Here we give some configurations (so-called matching templates) to decide whether an object pixel p is simple or not, which make possible an efficient implementation of the verification of simplicity. We define a *matching template* as a pair (τ_b, τ_w), where τ_b and τ_w are both subsets of the set $\{1, 2, \ldots, 12\}$ such that $\tau_b \cap \tau_w = \emptyset$. We say that an object pixel p in a picture (T, m, n, B) $((m, n) = (12, 3), (3, 12))$ *matches* a matching template $\tau = (\tau_b, \tau_w)$, if the following two conditions hold:

- if $k \in \tau_b$ $(k \in \{1, 2, \ldots, 12\})$, then $p_k \in N^*_{12}(p) \cap B$, and
- if $k \in \tau_w$ $(k \in \{1, 2, \ldots, 12\})$, then $p_k \in N^*_{12}(p) \setminus B$.

In the lattice representation of the matching template $\tau = (\tau_b, \tau_w)$ with central pixel p, p_k is depicted in black (white) if $k \in \tau_b$ $(k \in \tau_w)$, furthermore, p_k is denoted by "." if $k \notin \tau_b \cup \tau_w$.

Let $\mathcal{T}_i^{(m,n)}$ denote the set of matching templates composed by the matching template $T_i^{(m,n)}$ in Figs. 4-5 and its rotations by $k \cdot 60$ degrees $((m, n) = (12, 3), (3, 12);\ i = 1, 2, 3;\ k = 1, \ldots, 5)$, and let $\mathcal{T}^{(m,n)} = \mathcal{T}_1^{(m,n)} \cup \mathcal{T}_2^{(m,n)} \cup \mathcal{T}_3^{(m,n)}$.



$T_1^{(12,3)}$ $T_2^{(12,3)}$ $T_3^{(12,3)}$

Fig. 4. Matching templates for characterizing the simplicity of a pixel p in (12,3) pictures. Notations: each position marked ● matches a black pixel; ○ matches a white point; at least one of the pixels x, y, and z is black; positions denoted by "." refer to pixels of arbitrary color.

Theorem 2. *A black pixel p is simple in an (m, n) picture, if and only if it matches an element of the set of matching templates $\mathcal{T}^{(m,n)}$ $((m, n) = (12, 3)$, $(3, 12))$.*

Proof. It can be easily verified that we can get the templates of $\mathcal{T}_{4-i}^{(3,12)}$ $(i = 1, 2, 3)$ from the elements of $\mathcal{T}_i^{(12,3)}$ by recoloring their pixels and by switching the types of adjacency relations applied on their black and white pixels. Hence, by Lemma 1 it is sufficient to carry out the proof for $(m, n) = (12, 3)$.

By observing the possible configurations we can state that if p has more than one white 3-neighbors, then there exists a white 3-path between them, and any pixel coinciding with a position denoted by "." is 12-connected with any black pixel. From these follows that p represents a simple pixel in each of the mentioned templates.

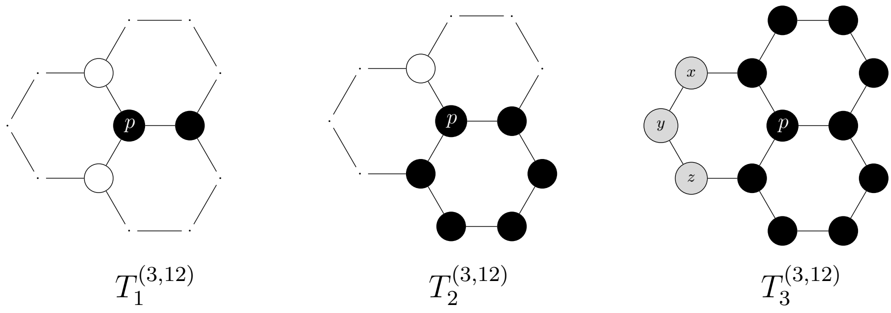


Fig. 5. Matching templates for characterizing the simplicity of a pixel p in (3,12) pictures. Notations: each position marked • matches a black pixel; ◦ matches a white point; at least one of the pixels x, y, and z is white; positions denoted by "." refer to pixels of arbitrary color.

Let us suppose that p is simple but it does not match any template of $\mathcal{T}_i^{(12,3)}$ $(i = 1, 2, 3)$. Even in this case, p must be 3-adjacent to at least one white pixel, or else p would not be simple. Hence, the neighborhood of p differs in some pixels from the ones shown in $\mathcal{T}_i^{(12,3)}$. If we changed the color of a white pixel in $N_{12}^*(p) \setminus N_3^*(p)$ to black, then this would obviously break the 3-connectedness of two white 3-neighbors of p, which, by Theorem 1, would cause that p is not simple after the recoloring. Furthermore, if we changed every black pixels in $\{x, y, z\}$ to white in the templates of $\mathcal{T}_1^{(12,3)}$, then p would be an isolated object pixel, i.e., a non-simple pixel. Hence, if p does not match any of the templates of $\mathcal{T}^{(12,3)}$, then p is not simple. □

4 Sufficient Conditions for Topology Preserving Reductions

Based on Theorems 1 and 2 we present some sufficient conditions for topology preservation on both (12,3) and (3,12) pictures.

Before we formulate the main theorem of this paper, first we provide two important relationships.

Lemma 2. *Let $\mathcal{O}$ be a reduction, and let S be the set of black pixels removed by $\mathcal{O}$ from picture $\mathcal{P} = (T, m, n, B)$ $((m, n) = (12, 3), (3, 12))$. $\mathcal{O}$ is topology-preserving, if for any pixel $p \in S$ and for any set $Q \subseteq S \cap N_{12}^*(p)$, p is simple in picture $(T, m, n, B \setminus Q)$.*

Proof. We note that the proof of the alternative version of this lemma for (6,6) pictures given in Lemma 2 of [5] does not rely on any special property of the hexagonal grids, therefore the proof for the triangular case can be done exactly the same way, only some notations will change. □

Lemma 3. *Let us suppose that* p *and* q *are simple pixels in picture* (T, m, n, B), *where* $(m, n) = (12, 3), (3, 12)$. *Then the following statements are equivalent:*

1. *Pixel* q *is simple in picture* $(T, m, n, B \setminus \{p\})$.
2. *Pixel* p *is simple in picture* $(T, m, n, B \setminus \{q\})$.

Proof. Let $x, y \in \{p, q\}$, where $x \neq y$. Let us suppose that x is simple in picture $(T, m, n, B \setminus \{y\})$. From this follows that $\{p, q\}$ is a simple set. If y would not be a simple pixel in $(T, m, n, B \setminus \{x\})$, then this would imply that the removal of y is not a topology-preserving reduction in $(T, m, n, B \setminus \{x\})$, thus even the removal of $\{x, y\} = \{p, q\}$ is not topology-preserving in (T, m, n, B). However, this would contradict the simplicity of $\{p, q\}$. □

Now we are ready to discuss our sufficient conditions for topology preservation.

Theorem 3. *A reduction* $\mathcal{O}$ *is topology-preserving in picture* $\mathcal{P} = (T, m, n, B)$ $((m, n) = (12, 3), (3, 12))$, *if all of the following conditions hold:*

1. *Only simple pixels are deleted by* $\mathcal{O}$.
2. *If* $\mathcal{O}$ *removes two* n*-adjacent pixels* p *and* q, *then* p *is simple in* $(T, m, n, B \setminus \{q\})$, *or* q *is simple in* $(T, m, n, B \setminus \{p\})$.
3. $\mathcal{O}$ *does not delete completely any object contained in a unit hexagon.*

Proof. Let us suppose that $\mathcal{O}$ fulfills Conditions 1-3, and let us denote by S the set of black pixels removed by $\mathcal{O}$. By Condition 1, each member of S is a simple pixel. By Lemma 2, it is sufficient to show that for any pixel $p \in S$ and for any set $Q \subseteq S \cap N^*_{12}(p)$, p is simple in picture $(T, m, n, B \setminus Q)$.

Let $p \in S$ and $Q \subseteq S \cap N^*_{12}(p)$. As p is simple, it matches a template $X \in \mathcal{T}^{(m,n)}$ in (T, m, n, B) by Theorem 2. If $X \neq \mathcal{T}_3^{(3,12)}$, then let $Q' \subseteq Q$ the set of black pixels coinciding with a position denoted by "." in X, else let Q' be the set of black pixels coinciding with one of the positions denoted by x, y, and z. Obviously, if $Q' = Q$, then p will match X even in picture $(T, m, n, B \setminus Q)$.

Let us suppose that $X \in \mathcal{T}_1^{(m,n)}$. If $(m, n) = (3, 12)$, then $Q' = Q$, as if we removed any black 3-neighbor of p, then p would not match any template of $\mathcal{T}^{(m,n)}$, which, by Theorem 2, would contradict Condition 2.

Hence, in this case the property to be proved holds by the above observation on Q'. Let us examine the case $(m, n) = (12, 3)$ and let us introduce the set $R = \{x, y, z\} \cap B$. It is obvious that $Q \subseteq R$. If each element of R is simple, then it is easy to see by Condition 2 and Theorem 1 that this can only occur if p and the elements of R constitute an object that can be covered by a unit hexagon. This and Condition 3 implies that p is not removed by $\mathcal{O}$, i.e., $p \notin S$. If any element of R is simple, then p will obviously match X in picture $(T, 12, 3, B \setminus Q)$, hence by Theorem 2, it remains simple after the removal of Q.

Let us suppose that $X \in \mathcal{T}_2^{(m,n)} \cup \mathcal{T}_3^{(m,n)}$. If $(m, n) = (3, 12)$, then p would not match X (nor any other possible templates), hence, p would not be simple by Theorem 2. However, this would contradict Condition 2 (see Lemma 3), therefore

$\mathcal{O}$ does not remove any pixel in $N_{12}^*(p) \setminus (N_3^*(p) \cup Q')$. Consequently, if $\mathcal{O}$ does not delete any black 3-neighbor of p, then $Q = Q'$ must hold, thus the property to be proved holds by the above observation on Q'.

Let $q, r \in N_3^*(p) \cap B$. The following statements can be formulated on q and r, depending on the possible values of X:

- If $X \in \mathcal{T}_2^{(12,3)}$, then $q = r$, as p has only one black 3-neighbor.
- If $X \in \mathcal{T}_3^{(12,3)}$ or $X \in \mathcal{T}_2^{(3,12)}$, then $N_3^*(p) \cap B = \{q, r\}$. As both q and r have at least two black 3-neighbors, none of them matches any template of $\mathcal{T}_1^{(m,n)}$. By careful examination of the templates of $\mathcal{T}_2^{(m,n)}$ and $\mathcal{T}_3^{(m,n)}$, we can state that r may even not match these templates after the removal of q, and the same goes for q after the deletion of r, i.e., $\{q, r\}$ is not a simple set. Hence, by Condition 2, $\mathcal{O}$ may only remove at most one black 3-neighbor of p.
- If $X \in \mathcal{T}_3^{(3,12)}$, then $\mathcal{O}$ may not remove the black 3-neighbor of p being not 3-adjacent to x nor to z in X, or else p would not remain simple by Theorem 1, hence Condition 2 would fail (see Lemma 3). Therefore, $\mathcal{O}$ may remove at most two black 3-neighbors of p, namely the common black 3-neighbors of pixels p, x and p, z. Let these pixels be q and r, respectively. By applying Theorems 1 and 2, we can easily show that if $\mathcal{O}$ removes q, then $x \in T \setminus B$, and if $\mathcal{O}$ deletes r, then $z \in T \setminus B$ must be satisfied. Hence, if $\{q, r\}$ is a simple set, then, according to the above observations on $N_{12}^*(p) \setminus (N_3^*(p) \cup Q')$, $Q = \{q, r\}$ or $Q = \{q, r, y\}$ holds, furthermore, after the removal of $\{q, r\}$, p will match a template of $\mathcal{T}_1^{(3,12)}$. If $y \in Q$, then we can easily verify by Theorem 2 that the colors of q, r does not influence the simplicity of y, and in the latter template, the color of y does not influence the simplicity of p. From this we can conclude that p is simple even in picture $(T, 3, 12, B \setminus Q)$. If set $\{q, r\}$ is not simple, then $\mathcal{O}$ can only remove at most one black 3-neighbor of p.

Hereafter, it is sufficient to examine the case when $\mathcal{O}$ removes exactly one black 3-neighbor of p, as all the other cases are previously discussed. For the sake of simplicity, let $q \in N_3^*(p) \cap B$ be this removed pixel, i.e., let $q \in Q$. By Condition 2 and Lemma 3, p is simple after the removal of q, hence, p matches a template $Y \in \mathcal{T}_1^{(m,n)} \cup \mathcal{T}_2^{(m,n)}$ after q is deleted.

Let us examine the situation after the removal of q depending on the value of Y. We will prove that for any possible value of Y all the pixels of $Q \setminus \{q\}$ are simple in $(T, m, n, B \setminus \{q\})$, hereafter we can reduce the remaining part of the proof to one of the previously discussed situations.

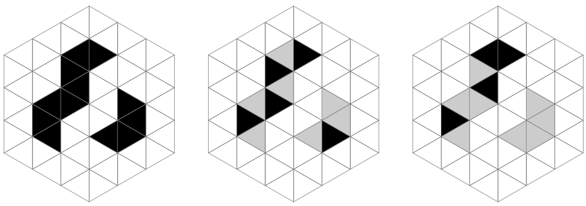
- If $Y \in \mathcal{T}_1^{(12,3)}$, then obviously $Q \subseteq \{q, x, y, z\}$. It is easy to verify that the color of p does not influence the properties of pixels x, y, z formulated in Theorem 1, i.e., the pixels of $\{x, y, z\} \cap Q$ remain simple after the removal of q. From this point, the proof can be reduced to the case when $X \in \mathcal{T}_1^{(12,3)}$.

- If $Y \in \mathcal{T}_1^{(3,12)}$, then, by Condition 2, Lemma 3, and Theorem 1, each element of the sets $Q \cap N_{12}^*(q)$ and $Q \setminus N_{12}^*(q)$ remains simple after q is deleted. From this point, the proof can be reduced to the case when $X \in \mathcal{T}_1^{(3,12)}$.
- If $Y \in \mathcal{T}_2^{(12,3)}$, then, by Theorem 1, the elements of $Q \setminus N_{12}(q)$ remain simple after the removal of q. Obviously, the pixels of $Q \cap (N_{12}(q) \setminus N_3(q))$ fulfill Condition 2 of Theorem 1, furthermore, as they with q are elements of a unit hexagon that contains a non-deletable black pixel, and as the elements of a unit hexagon are pairwise 12-adjacent, pixels belonging to $Q \cap (N_{12}(q) \setminus N_3(q))$ also satisfy Condition 1 of Theorem 1, independently of the color of q. Consequently, by Theorem 1, the elements of $Q \cap (N_{12}(q) \setminus N_3(q))$ are simple in $(T, m, n, B \setminus \{q\})$, as well. In Y we can note that the white element on the position corresponding to the removed pixel q may have only one black 3-neighbor. Let $Q \cap N_3(q) = \{s\}$. Pixel s is simple even after q is removed, because of Condition 2 and Lemma 3. Hence, each pixel of $Q \setminus \{q\}$ remains simple after the removal of q. Hereafter, the proof can be reduced to the case when $X \in \mathcal{T}_2^{(12,3)}$ and the remaining black 3-neighbor of p is not deleted.
- If $Y \in \mathcal{T}_2^{(3,12)}$, then, by Condition 2 and Lemma 3, each element of $Q \cap N_{12}^*(q)$ remains simple after q is deleted, and the pixels of $Q \setminus N_{12}^*(q)$ were not even deletable in the initial picture by the previous observations. From this point, we can reduce the proof to the situation where $X \in \mathcal{T}_2^{(3,12)}$ and none of the remaining black 3-neighbors of p are deleted.

Herewith, we have examined all possible cases. □

We note that the above result is similar to Ronse's sufficient conditions for topology preserving reduction on (8,4) and (4,8) pictures on the 2D orthogonal grid $\mathbb{Z}^2$ [12].

Figure 6 illustrates two examples for reductions on (12,3) pictures. The first one (see Fig. 6b) satisfies all the conditions of Theorem 3, therefore it is topology preserving. The second one (see Fig. 6c) violates Conditions 2 and 3 of Theorem 3, and it is topologically incorrect.



(a) (b) (c)

Fig. 6. The original picture (a) and the results produced by two reductions (b-c) on it. Deleted pixels are depicted in gray. The first reduction satisfies all conditions of Theorem 3, while the second one violates Conditions 2 and 3 of Theorem 3.

5 Validating Topology Preservation

Here we introduce a pair of reductions relying on the so-called subfield-based thinning strategy, which rests on the decomposition of the digital space into several subfields [4]. During an iteration step, the subfields are alternatively activated, and only pixels in the active subfield may be deleted. We propose the partitioning of the triangular grid T into two subfields, $SF(0)$ and $SF(1)$ (see Fig. 7.

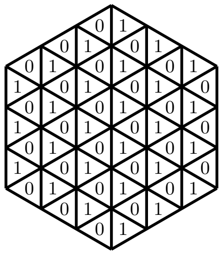


Fig. 7. Partition of T into two subfields. The pixels marked i are in $SF(i)$ $(i = 0, 1)$.

By carefully examining the pattern in Fig. 7 we can observe the following property of this partitiong:

Proposition 1. *If* $p \in SF(i)$ $(i = 0, 1)$*, then* $N_3^*(p) \cap SF(i) = \emptyset$.

Thinning algorithms must care about not only preserving the topology but also the shape of the original object. The latter requirement is usually fulfilled by adding some geometric constraints to the deleting conditions of the algorithms. For this purpose, here we give a definition of so-called end pixels.

Definition 1. *A black pixel* p *in a* $(12, 3)$ *picture is called an end pixel if there are at most two black pixels 12-adjacent to* p*, and at most one of them is 3-adjacent to* p*.*

We define our reductions on (12,3) pictures as follows.

Definition 2. *Let R-SF-i be the reduction that deletes a black pixel* $p \in SF(i)$ *from a* $(12, 3)$ *picture if* p *is simple and not an end pixel* $(i = 0, 1)$*.*

Now, using our sufficient conditions introduced in the previous section, we prove that the above reductions are topology preserving.

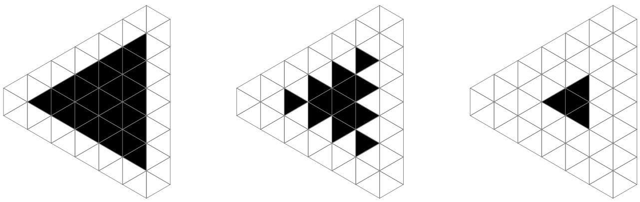
Theorem 4. *Both reductions R-SF-*0 *and R-SF-*1 *are topology-preserving.*

Proof. As *R-SF-i* $(i = 0, 1)$ deletes only simple pixels, Condition 1 of Theorem 3 is satisfied. Furthermore, Proposition 1 implies that if *R-SF-i* deletes p, then p does not remove any $q \in N_3(p)$, which means that *R-SF-i* fulfills Condition 2 of Theorem 3.

Let S be a set of the black pixels of an object contained in a unit hexagon, and let us denote by $|S|$ the number of elements in S. If there is a black pixel $p \in S$ such that $p \notin SF(i)$, then it is obvious that p will not be removed. Let us suppose that $S \subset SF(i)$. It can be readily seen that in this case $|S| \leq 3$. If $|S| = 1$, then the only object pixel in S is a non-simple pixel, which will not be deleted by *R-SF-i*. If $|S| = 2$ or $|S| = 3$, then the pixels of S are all end pixels, which are also retained by *R-SF-i*. Having examined all the possible cases for the content of S, we can conclude that Condition 3 of Theorem 3 also holds.

Hence, *R-SF-i* ($i = 0, 1$) is topology-preserving by Theorem 3. □

Let us perform these two reductions succesively in a (12,3) picture shown in Figure 8a. The effects of operators *R-SF*-0 and *R-SF*-1 can be observed in Figs. 8b-c.



(a) (b) (c)

Fig. 8. A picture of a triangle (a) and the results produced by successively applying the reductions *R-SF*-0 (b) and *R-SF*-1 (c) in it

6 Conclusions

This paper has concerned itself with the topological properties of (12,3) and (3,12) pictures. We have given some characterizations of simple pixels in these types of pictures, and as the main novelty of our work, we have presented some sufficient conditions to ensure topology preservation for reductions on (12,3) pictures. As an illustration for the usefulness of these conditions, we have defined a pair of subfield-based reductions, and we have verified their topological correctness. As a future work we intend to give further conditions that are capable of constructing triangular thinning algorithms whose topological correctness is automatically guaranteed.

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