KOVÁCS BÉLA,

MATHEMATICS I.

1

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A Műszaki Földtudományi Alapszak tananyagainak kifejlesztése a TÁMOP 4.1.2-08/1/A-2009-0033 pályázat keretében valósult meg.

I. Sets

1. Notations

The concept and properties of the **set** are often used in mathematics. We do not define *set*, as it is an *intuitive concept*.

This is not unusual. You may recall that in geometry we do not attempt to define what we mean by a point or a line.

A set may be thought of as a collection of objects. These objects are called *elements* or the *members* of the set.

In mathematics you may have discussed some of the following:

- The set of counting numbers,
- The set of points,
- The set of solutions for an equation.

The elements of a set

We do not have to describe a set by a name as we did in the previous section, we can also describe it by listing or naming its elements. Brackets, { }, are used to enclose the members of a set when we list them. [1]

Sets are usually denoted by *capital letters,* such as $A, B,$ or $C.$ Therefore, we could write

$$A = \{a, e, i, o, u\}$$

to indicate that the set A contains the elements: $a, e, i, o, u.$ We shall use symbol $\in$ to indicate that elements are members of a set. The symbol $\in$ is read "*is a member of*" or "*is an element of*" and the notation $\notin$ is read "*is not an element of*". Using our previous example $A = \{a, e, i, o, u\}$ we may say that $a \in A, e \in A, u \in A, 2 \notin A.$

A set is uniquely determined by its elements. This means that two sets are not different if they have the same elements. In other words, if the two sets have the same elements, then the two sets are equal to each other.

Order of elements, repeated elements

In a set the *order* of members is not significant. This means that if we change the order of the elements in a set then we do not have a new set. For example, $\{1,2,3\}$ and $\{1,3,2\}$ are the same set. So we may write $\{1,2,3\} = \{2,1,3\}$. These sets contain exactly the same elements, namely $1, 2,$ and $3.$

In a set the order of elements is unessential. This means that if we repeat one element in the set once again, then we do not have a new set. For example, the set $\{1,2,2,3,3\}$ is equal to the set $\{1,2,3\}$. So we can write $\{1,2,2,3,3\} = \{1,2,3\}$.

Describe of a set using the properties of the elements

We may also use curly brackets { } when a set is described by the help of the *common properties* of the elements.

The third method, which is very often used and the correct method, is the following: first, after a curling bracket {, we write a letter or symbol which denotes an element of the set. After this there is a colon and then we write a statement for the symbol on the right side. This statement can be expressed in terms of formula or words. It is possible to describe a set. For example, instead of the $\{1, 2, 3\}$ set we can write $\{x : x$ natural number and $1 \le x \le 3\}$.

$$\{1,2,3\} = \{x : x \text{ natural number and } 1 \le x \le 3\}$$

If those elements which we would like to list in a set are already elements of another set which has a name, then we write this set on the left side of the colon: $\{x \in \mathbb{Z} : 1 \le x \le 3\}$. This means that: " *The set of those elements of* $\mathbb{Z}$ for which 1 is less than or equal to x and x is less than or equal to 3."

Number of a set; empty set

A set may possess finite elements as well as infinite ones. It is also possible that a set has no elements at all. In this case the set also has a well-defined number of elements, namely zero.

Consider the set of dolphins that live in Lake Balaton. This set has no elements: there are no dolphins living in Lake Balaton.

If we were to list the elements for this set, we would have to put nothing between the braces. We would have { }. This is an **empty set**, also called the *null set.* The empty set is usually denoted by the symbol { }, but another common symbol is ϕ.

Some important sets

Some important sets have particular symbols.

We list five:

- $\mathbb{N} = \mathbb{Z}^+ =$ set of natural numbers $= \{1,2,3,\ldots\}$
- $\mathbb{Z} =$ ring of integers $= \{\ldots,-2,-1,0,1,2,3,\ldots\}$
- $\mathbb{Q} =$ field of rational numbers $= \{a/b : a,b \in \mathbb{Z}, b \neq 0\}$
- $\mathbb{R} =$ field of real numbers
- $\mathbb{C} =$ field of complex numbers $= \{a+bi : a,b \in \mathbb{R}\}$ $(i^2 = -1)$

2. Set operations

Subset

$A \subseteq B$

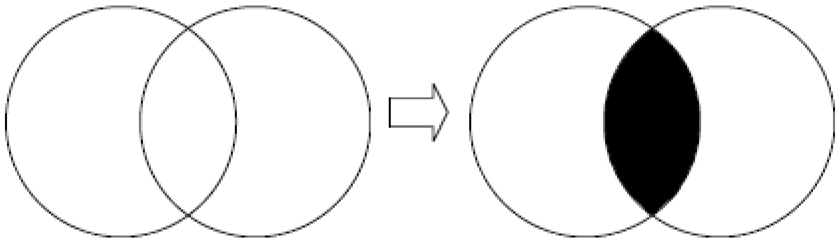
We say that the set A is a **subset** *of* the set B, when each element of A is also an element of B, and we use the notation $A \subseteq B$. The empty set is a subset of all sets, $\phi \subseteq B$.

Intersection

$A \cap B$

Let A and B be sets, then the **intersection** of A and B is the set of all elements which belong to both set A and B, with the notation

$$A \cap B = \{x : x \in A \text{ and } x \in B\} = \text{each } x \text{ such that } x \in A \text{ and } x \in B.$$



For example

$$\{1,2\} \cap \{2,3\} = \{2\}$$

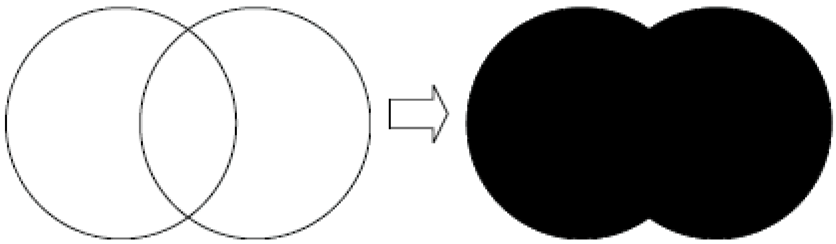
and

$$\{1\} \cap \{2\} = \{\}.$$

Union

$A \cup B$

Let A and B be sets. The **union** of A and B is a set of all elements which belong to at least one of A and B.



For example:

$$\{1,2\} \cup \{2,3\} = \{1,2,3\}$$

and

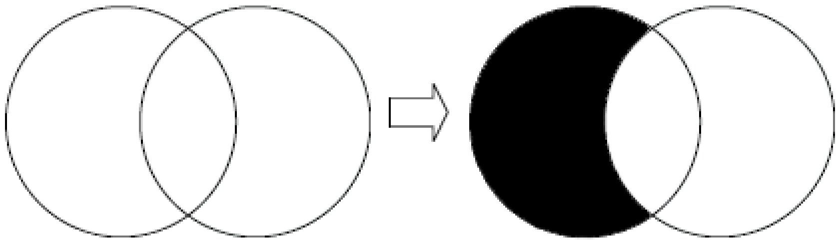
$$\{1\} \cup \{2\} = \{1,2\}.$$

$A \cup B = \{x : x \in A$ or $x \in B\} =$ each x such that $x \in A$ or $x \in B$.

Set difference

$A - B$

Let A and B be sets. The **difference** of A and B is a set whose elements belong to A but do not belong to B (notation: $A - B$).



So

$$\{1,2\} - \{2,3\} = \{1\}$$

and

$$\{1\} - \{2\} = \{1\}.$$

Rule of set operations

The $\bigcap$ and $\bigcup$ operations are *commutatives* and *associatives*, in the same way as the real numbers for addition and multiplication, i.e. if we have X, Y and Z sets, then $X \bigcap Y = Y \bigcap X$ and $X \bigcup Y = Y \bigcup X$.

Similarly:

$$X \bigcap (Y \bigcap Z) = (X \bigcap Y) \bigcap Z \text{ and } X \bigcup (Y \bigcup Z) = (X \bigcup Y) \bigcup Z.$$

Moreover, $\bigcap$ and $\bigcup$ are *distributive* over the other operation, i.e.,

$$X \bigcap (Y \bigcup Z) = (X \bigcap Y) \bigcup (X \bigcap Z) \text{ and } X \bigcup (Y \bigcap Z) = (X \bigcup Y) \bigcap (X \bigcup Z).$$

Similarly, for real numbers multiplication is distributive over addition, but the inverse is not true.

Desscription of rule	**Name of rule (explanation)**
$X \bigcup Y = Y \bigcup X$	Union is commutative
$X \bigcap Y = Y \bigcap X$	Intersection is commutative
$(X \bigcup Y) \bigcup Z = X \bigcup (Y \bigcup Z)$	Union is associative
$(X \bigcap Y) \bigcap Z = X \bigcap (Y \bigcap Z)$	Intersection is associative
$(X \bigcup Y) \bigcap Z = (X \bigcap Y) \bigcup (Y \bigcap Z)$	Union is distributive over intersection
$X \bigcap (Y \bigcup Z) = (X \bigcap Y) \bigcup (X \bigcap Z)$	Intersection is distributive over union

Summary of set operations

Complement set

$\overline{A}$

Let H be an arbitrary set. Assume that A is a subset of H. Then the **complement set** of A **concerning** H is a set whose elements belong to *H* but do not belong to *A*. ($\overline{A} = H \setminus A$).

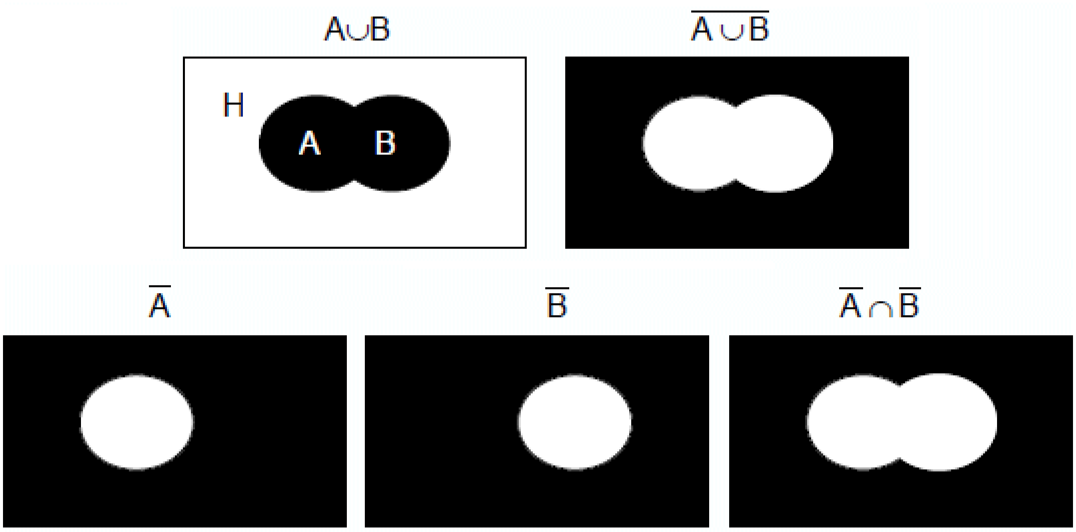
Notation: $\overline{A}$.

De Morgan's identities

De Morgan's identities for complement sets are the following:

$$\overline{A \bigcup B} = \overline{A} \bigcap \overline{B}$$

$$\overline{A \bigcap B} = \overline{A} \bigcup \overline{B}$$



Venn-diagrams to show De Morgan's identities

3. Descartes product

Ordered pair, sets

Definition An **ordered pair consists of a first and a second element.** If we denote a the first element and b the second one, then (a,b) denotes the ordered pair.

So that the $(a,b) = (a',b')$ equality means that $a = a'$ and $b = b'$.

The difference between an *ordered pair* and *a set consisting of two elements* is that the order of elements is important for the ordered pair while for sets is not.

So for a set $\{a,b\} = \{b,a\}$, but for an ordered pair $(a,b) = (b,a)$ is true if and only if $a = b$.

Definition The $A \times B := \{(a,b) : a \in A, b \in B\}$ set of ordered pairs are called the **Descartes product** of A and B sets.

In other words, this means that the Descartes product of A and B sets is a set consisting of all ordered pairs whose first element comes from A and second one from B .

The symbol $:=$ means that the mathematical quantity on the left side of the colon is first defined with this formula. In this case we don't have to search in our memory whether this quantity is known to us or not. Of course it is clear from context, but this symbol helps us to read the text more easily.

EXERCISE

Sample exercise

Given $A = \{1,2,3\}$ and $B = \{a,b\}$. , find $A \times B := \{(a,b) : a \in A, b \in B\}$ **Descartes product of** sets and .

A B

Solution

$$A \times B = \{(1,a),(1,b),(2,a),(2,b),(3,a),(3,b)\}.$$

EXAMPLE

Example:

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 = \text{ the plane}$$

Ordered n-tuples

Similarly to the definition of an ordered pair, we can define an ordered triplicate (a,b,c), and ordered *n*-tuples $(a_1,a_2,\ldots,a_n)$. If we have sets $A_1,A_2,\ldots,A_n$, then the

$$A_1 \times A_2 \times \ldots \times A_n := \{(a_1,a_2,\ldots,a_n) : a_1 \in A_1,\ldots,a_n \in A_n\}$$

set is called the **Descartes product of sets** $A_1,A_2,\ldots,A_n$.

We often use notation $\mathbb{R}^n$ for *n*-tuples **Descartes product** of $\mathbb{R}$:

$$\mathbb{R}^n := \mathbb{R} \times \ldots \times \mathbb{R}.$$

4. Family of sets

In mathematics we often meet sets whose elements are also sets. For example, we define the following set:

$$M = \{\{1,2\},\{1,2,3\},\{2,3,4\},\{4\}\},$$

which consists of four sets containing natural numbers, i.e., four subsets of $\{1,2,3,4\}$.

In order to avoid "set of sets" repetition of a word we prefer the terms **set system** or *family of sets.*

It is said that the M set system contains some subsets of $\{1,2,3,4\}$. These sets are sometimes denoted by letter script, for example : $\mathcal{M}$.

If a set consists of all subsets of a set X then this is called the **power set** of X and denoted by 2^X. There is ananother usual notation for the power set, namely $\mathcal{P}(X)$.

5. Examples

Solutions: ○ visible ○ invisible

1. Let $A = \{1,2,5,8\}$, $B = \{0,2,3,5\}$, $C = \{3,4\}$, $D = \{2\}$.

Then:

$$A \cup B = \{0,1,2,3,5,8\},\quad A \cap B = \{2,5\},\quad A \setminus B = \{1,8\},$$

$A \cap C = \phi$, $C = \{3,4\} = \{3,4,3\} = \{4,3\} = \{3,4,4,3,3\}$,

$C \cap D = \phi$, $A \times C = \{(1,3), (1,4), (2,3), (2,4), (5,3), (5,4), (8,3), (8,4)\}$,

$C \times C = C^2 = \{3,4\} \times \{3,4\} = \{(3,3), (3,4), (4,3), (4,4)\}$.

Each of the above sets is a finite set ($A \cap C = \phi$ the null-set is also a finite set). Set D is a subset of A and B but it is not a subset of C. Similarly, $A \cap B$ is a subset of A and B.

2. Let $H = \{1,2,3,4,5\}$ be the universal set, $A = \{2,4,5\}$.

Then $\overline{A} = H \setminus A = \{1,3\}$, $\overline{\overline{A}} = \{2,4,5\}$.

3. Given $A = \{1,2,3\}$, find the power set of A.

Solution. Denote $P(A)$ the power set, whose elements are all the subset of A, with the null-set and A set. So

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Note, that the number of elements of $P(A)$ is $8 = 2^3$. In general, it is also true that if the number of elements of A is *n*, then the number of elements of $P(A)$ is 2^n (see the 17th example).

4. Let $\mathbb{N} = \{1,2,3,\dots\}$ be the set of natural numbers, and $\mathbb{Z} = \{\dots,-3,-2,-1,0,1,2,3,\dots\}$ ring of integers, and $\mathbb{R}$ the real numbers.

In this case $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$, i.e. **N** is a subset of **Z** and $\mathbb{R}$, **Z** is a subset of $\mathbb{R}$. These sets are infinite sets.

5. Given $\mathbb{N} = \{1,2,3,\dots\}$ list some elements of $\mathbb{N} \times \mathbb{N} = \mathbb{N}^2$.

Solution: $\mathbb{N}^2 = \{(1,1),(1,2),(2,1),(2,2),(1,3),(2,3),(3,1),(3,2),(3,3),\dots\}$. This set can demonstrate the inner points of system of co-ordinates (see Fig. 1.1). For example the (2, 3) element corresponds to the point $P(2,3)$.

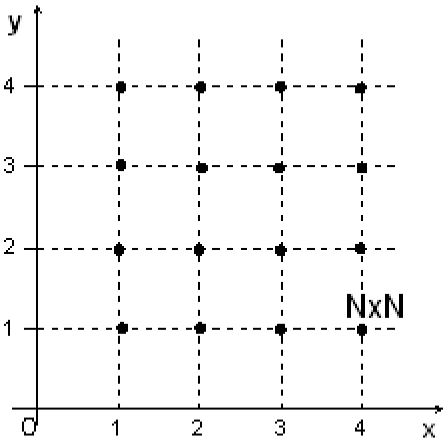


Figure 1.1

6. Let **R** be the set of real numbers. Given the sets $A = \{x : x \in \mathbf{R},\ |x| < 2\}$ and $B = \{x : x \in \mathbf{R},\ x > 0\}$ find $\overline{A}$, $\overline{B}$, $A \cap B$, $A \cup B$, $A \setminus B$ and illustrate each of them with a real line.

Solution. $\overline{A} = \{x{:}x \in \mathrm{R}, |x| \geq 2\}$, $\overline{B} = \{x{:}x \in \mathrm{R}, x \leq 0\}$. Since the common elements of A and B are those positive real numbers which are less than two, $A \cap B = \{x{:}x \in \mathrm{R}, 0 < x < 2\}$. The union of the two sets contains all elements of *A* and *B*, so $A \cup B = \{x : x \in \mathbf{R},\ x > -2\}$. The set $A \setminus B$ contains those elements of A which are not elements of B , so $A \setminus B = \{x : x \in \mathbf{R},\ -2 < x \leq 0\}$ (see Fig. 1.2).

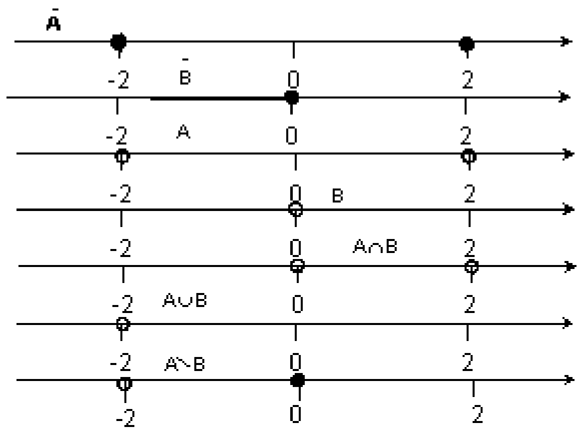


Figure 1.2

7. Show that the statement $A \setminus (B \cup C) = A \cap \overline{B} \cap \overline{C}$ is true for arbitrary A, B, C sets.

Solution. Using the formula $A \setminus B = A \cap \overline{B}$ and *second De Morgan's* we obtain

$$A \setminus (B \cup C) = A \cap (\overline{B \cup C}) = A \cap (\overline{B} \cap \overline{C}) = A \cap \overline{B} \cap \overline{C}\,.$$

8. Let H be a universal set $(A \subset H$ and $B \subset H)$. Show, that $A \cup (A \cap B) = A$.

Solution. It is well known that $A = A \cap H$, és $H \cup B = H$. So

$$A \cup (A \cap B) = (A \cap H) \cup (A \cap B) = A \cap (H \cup B) = A \cap H = A.$$

Here we take out the set A. This identity is illustrated with a Venn diagram (see *Fig. 1.3*).

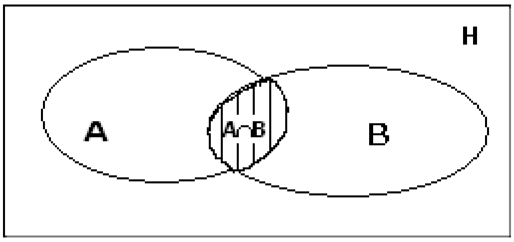


Figure 1.3

9. Let H be the universal set and $A \subset H,\ B \subset H$. Show that $(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap \overline{B}) = A \cup B$ (see *Fig. 1.4*).

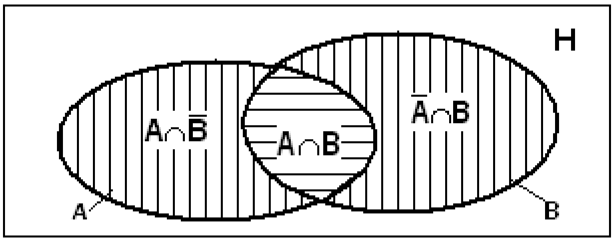


Figure 1.4

Solution. Using the formula $D = D \cup D$ for $D = A \cap B$ we obtain $A \cap B = (A \cap B) \cup (A \cap B)$. Then

$$(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap \overline{B}) = (A \cap \overline{B}) \cup (A \cap B) \cup (A \cap B) \cup (\overline{A} \cap B) =$$

$$= \left[A \cap (\overline{B} \cup B)\right] \cup \left[B \cap (A \cup \overline{A})\right] = (A \cap H) \cup (B \cap H) = A \cup B.$$

where we apply $\overline{B} \cup B = H$, $A \cup \overline{A} = H$, $A \cap H = A$, $B \cap H = B$.

6. Problems

Solutions: ⦿ visible ○ invisible

1. Describe the $A = \{n : n \in \mathbb{N},\ 3 < n \leq 7\}$ and $B = \{x : x \in \mathbb{R}, x^2 - 6x = 0\}$ sets listing their elements (**N** is the set of natural numbers, **R** is the set of real numbers).

Solution: The elements of the set A are those natural numbers which are greater than 3 and less than or equal to 7, i.e. $A = \{4, 5, 6, 7\}$. The elements of set B are the solutions of $x^2 - 6x = 0$ second order equation. Rewriting the equation $x^2 - 6x = 0$ in the new form $x(x - 6) = 0$ so the roots are obtained easily, we get $x_1 = 0$, $x_2 = 6$, i.e. $B = \{0,\ 6\}$.

2. Characterize set listing elements.

$$X = \{x : x \in \mathbf{R},\ x^2 - 6x + 25 = 0\}$$

Solution: In order to get the element of set X we must solve the second order equation $x^2 - 6x + 25 = 0$:

$$x_{1,2} = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$$

We can see that the results are not real numbers, so there are no elements of set X . Hence X is the null-set.

3. Given $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$ and $B = \{\lg\sqrt{1}, \lg\sqrt{2}, \lg\sqrt{3}, \ldots, \lg\sqrt{n}, \ldots\}$ find $A \cap B$ set listing elements.

Solution: The elements of set B are in order: $\frac{1}{2}\lg 1 = 0$, $\frac{1}{2}\lg 2$, $\frac{1}{2}\lg 3, \ \ldots, \frac{1}{2}\lg n, \ \ldots$.

The 10th element: $\frac{1}{2}\lg 10 = \frac{1}{2}$, and the 100th element $\frac{1}{2}\lg 100 = \frac{1}{2}\cdot 2 = 1$. So there are only two common elements. Hence $A \cap B = \left\{\frac{1}{2},\ 1\right\} = \left\{1,\ \frac{1}{2}\right\}$.

4. Use a Venn diagram to show that $(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = H$

(H is a universal set), then show that this formula is true using set operations.

Solution: In *Fig.1.5* we can see the sets $A \cap \overline{B}, A \cap B, \overline{A} \cap B,\ \overline{A} \cap \overline{B} = \overline{A \cup B}$. These sets are disjunctive by pairs and the union of them is equal to *H.*

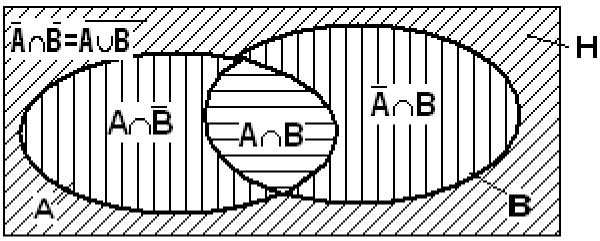


Figure 1.5

In sample example 9 we have $(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B) = A \cup B$. Using this formula:

$$(A \cap \overline{B}) \cup (A \cap B) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B}) = (A \cup B) \cup (\overline{A} \cap \overline{B}) = (A \cup B) \cup (\overline{A \cup B}) = H$$

.

We must also apply $\overline{A} \cap \overline{B} = \overline{A \cup B}$ and $A \cup \overline{A} = H$.

5. In a certain group of 100 students, 28 students are taking English, 30 students are taking German, 42 students are taking French. Of these, 8 students are taking both English and German, 10 students are taking English and French, 5 students are taking German and French and 3 students are taking English, German and French.

a) How many students in this group are not taking any of the three subjects?

b) How many of these students are taking French but not English and German?

Solution: We label the sets *A* (English), *N* (German), *F* (French). Draw the Venn diagram of these sets and the given data (see *Fig. 1.6*). Starting with the intersection of the three sets, we must put 3 in it, because three students are taking all three subjects. There are five people who are taking German and French, so in the intersection of *N* and *F* we must put 5, but 3 is already there, so 2 must be written yet. We have to similarly fill in the other spaces of the Venn diagram, so it is easy to answer the question: a) 20, b) 30.

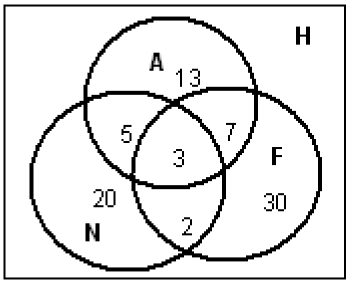


Figure 1.6

6. Given $X = \{x : x \in \mathbf{R}, x > 0\}$ $Y = \{y : y \in \mathbf{R},\ 0 < y \leq 4\}$. Illustrate the $X \times Y$ Descartes product.

Solution: The $X \times Y$ Descartes product can be illustrated at those points $P(x, y)$ of co-ordinate systems where $x > 0$, $0 < y \leq 4$ (see *Fig. 1.7*).

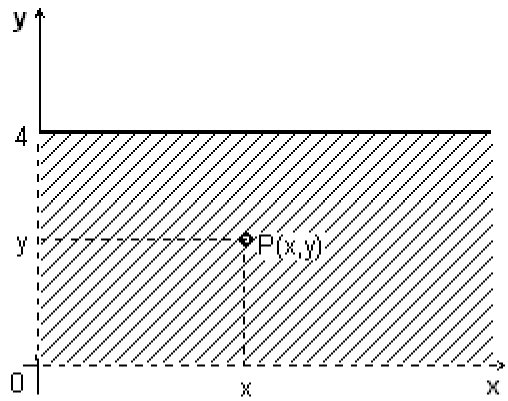


Figure 1.7

7. How many elements have the following sets got?

a) $A =$ {Vörösmarty Mihály, 2, the root of the equation $x^2 - 4 = 0$, writer of *Szózat*};

b) $B = \{x : x^2 = 0\}$;

c) $C = \{4,\ a,\ b\}$;

d) $D = \{\phi\}$

Solutions:

a) Since the writer of *Szózat* and Vörösmarty Mihály are the same person, this is one element of the set. The roots of the equation $x^2 - 4 = 0$ are $x_1 = 2$ and $x_2 = -2$. In the list 2 and x_1 are the same elements of the set. Consequently *A* has 3 elements: Vörösmarty Mihály, 2 and -2.

b) Since $x^2 = 0$ has only one root, namely 0, *B* is a singleton, i.e. there is only one element of B.

c) If $a \neq b$ and $a \neq 4, b \neq 4$, then *C* has 3 elements. If $a \neq b$ and $a = 4$, or $a \neq b$ and $b \neq 4$, then *C* has 2 elements. If $a = b \neq 4$, then *C* also has 2 elements. If $a = b = 4$, *C* has 1 element.

d) *D* has only one element, namely the null-set.

8. How many elements has the power set of $A = \{a, b, c, d, e\}$ got and how many elements have the $A \times A$ got ?

Solution: The power set of *A* contains $2^5 = 32$ elements (see sample example 3). The set of $A \times A$ Descartes product contains $5^2 = 25$ elements.

9. Are there the same elements in the sets of *A* and *B* if

$$A = \left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^n}, \ldots\right\} \text{ and } B = \{ \sin\frac{k\pi}{6}, k \text{ are the natural numbers}\}?$$

Solution: The elements of set *B* are $k = 1, 2, 3, \ldots$ in order: $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{2\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{3\pi}{6} = 1$, $\sin\frac{4\pi}{6} = \frac{\sqrt{3}}{2}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$, $\sin\frac{6\pi}{6} = 0$, $\sin\frac{7\pi}{6} = -\frac{1}{2}$, $\sin\frac{8\pi}{6} = -\frac{\sqrt{3}}{2}$, $\sin\frac{9\pi}{6} = -1$, $\sin\frac{10\pi}{6} = -\frac{\sqrt{3}}{2}$, $\sin\frac{11\pi}{6} = -\frac{1}{2}$, $\sin\frac{12\pi}{6} = 0$. The next elements are already equal to these elements. So if $B = \left\{\frac{1}{2}, \frac{\sqrt{3}}{2}, 1, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1\right\}$, it implies that there are two elements which are elements of sets of *A* and *B* , namely 1 and $\frac{1}{2}$.

10. Tell whether the statement $(A \cap B) \cup (\overline{A} \cap B) \cup (\overline{\overline{A} \cap \overline{B}}) = B$ is true or false.

Solution: Using $\overline{A} \cup \overline{B} = \overline{A \cap B}$, we obtain $\overline{\overline{A} \cup \overline{B}} = \overline{\overline{A \cap B}} = A \cap B$. So

$(A \cap B) \cup (\overline{A} \cap B) \cup (\overline{\overline{A} \cup \overline{B}}) = (A \cap B) \cup (\overline{A} \cap B) \cup (A \cap B) = (A \cap B) \cup (\overline{A} \cap B) =$

$= B \cap (A \cup \overline{A}) = B \cap H = B$, so the formula is true.

[1] Of course, this is possible only for a set which has a finite number of elements.