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Probability Theory and Mathematical Statistics

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Új Magyarország FEJLESZTÉSI TERV

A Műszaki Földtudományi Alapszak tananyagainak kifejlesztése a TÁMOP 4.1.2-08/1/A-2009-0033 pályázat keretében valósult meg.

I. Combinatorial analysis

1. Combinatorics

Definition: The collection Ω consists of n distinct objects. A permutation of order k is an ordered selection of k elements from Ω, where $0 \le k \le n$. A combination of order k is an unordered selection of k elements from Ω

The selections for both permutations and combinations can be made with or without replacement but are assumed to be made without replacement, unless otherwise stated.

Theorem: (Multiplication Principle) If a task A can be done in m different ways and, after it is completed in any of these ways, task B can be completed in n different ways, then A and B , together, can be performed in $m \times n$ ways.

We define $n!$ (pronounced "n factorial") for each non-negative integer n by $0! = 1$, $n! = n \times (n-1)!$ for $n > 0$. Thus we can write

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1, \qquad (n > 1).$$

Theorem: The number of permutations of n elements, taken k at a time, without repetitions, is

$$P(n,k) = (n)_k = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+1).$$

If repetitions are allowed the number of permutations is n^k.

Both the symbols $C(n,k)$ and $\binom{n}{k}$ are used to designate the number of combinations of *k* objects selected from a set of *n* elements.

Theorem: There are

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

combinations of n objects, taken at a time. If repetitions are allowed the number of combinations is

$$\binom{n+k-1}{k}.$$

Corollary: $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$, that is,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

PROBLEM

EXAMPLE 1 Find how many ways the first three places (with no ties) can be filled in a race with 15 contestants.

Solution: The first place can be filled in 15 ways, since any contestant can come first. When the first place has been filled, there are 14 more contestants to choose from for the second place. Hence the first two places can be filled in 15×14 ways. Finally, for each of these ways, the third place can be filled by any of the remaining 13 contestants, and the total number of ways is $15\times14\times13.$ Alternatively, from the formula

$$P(15,3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = 15\times14\times13.$$

PROBLEM

EXAMPLE 2 Eight people are to sit down at a round table. Determine the number of ways in which they can sit relative to one another.

Solution: Suppose that one person sits down first. Then there are seven places to be filled by a permutation of the other seven. Thus this can be done in $P(7,7) = 7!$ ways.

PROBLEM

EXAMPLE 3 Find how many three-letter codes can be formed from an alphabet of 26 letters (assuming that letters can be repeated).

Solution: The first letter of the code may be any of the 26 in the alphabet. The same is true for the second letter and for the third letter. Therefore, the total number of codes is $26\times26\times26 = 26^3.$

PROBLEM

EXAMPLE 4 Find the number of ways of forming a class committee of three members from a class of 20.

Solution: Begin by choosing the three members of the committee in order. The first can be chosen in 20 ways, the second in 19 ways, and the third in 18 ways. The sequence of three can be choosen from 20 in $P(20,3) = 20\times19\times18$ ways. Now a given sequence of 3 members can be permuted in $3!$ ways. Therefore, every set of 3 members will occur $3!$ times in the $P(20,3)$ ordered sets of 3. So, the required number of committees, which is the number of 3 member sets, unordered, is

$$\frac{P(20,3)}{3!} = 1140 = C(20,3).$$

PROBLEM

EXAMPLE 5 A pack of 52 cards contains "spades," "clubs," "diamonds," and "hearts" in equal numbers. In how many ways can a hand of 13 cards be drawn, so as to contain precisely 5 spades?

Solution: Any 5 of the 13 spades might be drawn, which can be done in $C(13,5)$ ways. By hypothesis, the other 7 cards may be any 7 of the 39 clubs, diamonds, and hearts. These can be drawn in $C(39,7)$ ways. By multiplication principle, the total number of hands possible is then $C(13,5)\times C(39,7)$.

2. Exercises

Solutions: ⦿ visible ○ invisible

1. Determine the number of:

i. one-,

ii. two-,

iii. three-,

iv. four-element subsets of the set $\{a,b,c,d\}$.

Show that the total number of subsets of this set is $2^4 = 16$.

Solution:
(i) 4.
(ii) 6.
(iii) 4.
(iv) 1.

2. In a tennis club of 68 members, 30 are women. Find the number of ways of forming a comitee of four members:

i. with no restrictions,

ii. with at least one woman on the committee.

Solution:

(i) $\binom{68}{4} = 814385.$

(ii) $\binom{68}{4} - \binom{38}{4} = 740570.$

3. Find the number of different ways in which a committee of two male and two female students can be selected from five male and four female students.

Solution: $\binom{5}{2}\binom{4}{2} = 60.$

4. A water-polo team consisting of a goalkeeper and six other players is to be selected from 11 players. Just two of the 11 players are goalkeepers. Find the number of ways in which the team may be selected.

Solution:

$$2\binom{9}{6} = 168.$$

5. Find how many different four-digit numbers can be formed from the seven digits 1,2,4,5,6,8,9:

a. if each digit may be used as often as desired,

b. if no digit is repeated.

In case (b), find how many of the numbers include two odd digits and two even digits and state how many such numbers are even.

Solution: (a) $7^4 = 2401$. (b) $(7)_3 = 840$, $4!\binom{3}{2}\binom{4}{2} = 432$, 216.

6. Seven people are to be seated in a row, on the platform at the school speech day. The chairman of the governors must sit in the centre and the visiting speaker and headteacher must sit next to the chairman. Find the number of ways in which the seating plan can be arranged.

Solution: $2(4!) = 48$.

7. Show that the number of permutations of n things, of which p are alike of one kind, q are alike of a different kind, and the remainder are different from these and from each other, is $\frac{n!}{p!q!}$.

8. Calculate the number of ways in which six people can form

a. a queue (that is, a single file) of six people,

b. a queue of two people and another queue of four people,

c. a group of two people and another group of four people,

d. first, second and third pairs,

e. three pairs.

Assume that the order within a group or a pair is not significant.

Solution: (a) 720. (b) 720. (c) 15. (d) 90. (e) 15.

9. Seven students are eligible for selection to a delegation of four students from a school to attend a conference. Two of them will not attend together but each is prepared to attend in the absence of the other. In how many different ways can the delegation be chosen?

Solution: 25.

10. Four persons enter a minibus in which 7 seats are vacant. In how many ways can they be seated?

Solution: $(7)_3 = 840$.

11. In how many numbers between 1,000 and 10,000 is there neither a repeated digit nor a zero?

> **Solution:** $9\binom{9}{3} = 1134.$

12. In how many circular orders may 4 men and 4 women sit, if men and women are to alternate?

> **Solution:** $3!4! = 144.$

13. In how many ways could:

a. A bodyguard of four be chosen from 12 secret service men?

b. A bowling team of five be chosen from a club of 20 men?

c. A president, a secretary, and a treasurer be elected from a club membership of 50?

d. A committee of 3 seniors and 2 juniors be chosen from a club consisting of 15 seniors and 10 juniors?

e. An arbitration board consisting of 3 employers, 3 labourers, and 1 outsider be chosen from 10 employers, 30 labourers, and 8 outsiders?

> **Solution:** (a) $\binom{12}{4}$. (b) $\binom{20}{5}$. (c) $(50)_{47}$. (d) $\binom{15}{3}\binom{10}{2}$. (e) $\binom{10}{3}\binom{30}{3}\binom{8}{1}$.

14. How many straight lines are determined by 10 points located on a circle? How many triangles have three of those points as vertices?

> **Solution:** $\binom{10}{2} = 45,\ \binom{10}{3} = 120.$

15. If we draw 5 balls at random from a bag containing 11 red and 6 white balls, in how many ways may we get 3 red and 2 white?

> **Solution:** $\binom{11}{3}\binom{6}{2}.$

16. A restaurant menu lists 3 soups, 10 meat dishes, 5 desserts, and 3 beverages. In how many ways can a meal (consisting of soup, meat dish, dessert, and beverage) be ordered?

> **Solution:** $3 \cdot 10 \cdot 5 \cdot 3 = 450.$

17. If $\binom{n}{11} = \binom{n}{7}$, find n. If $\binom{18}{r} = \binom{18}{r-2}$, find r.

Solution: $n = 18,\ r = 10.$

18. Prove that

$$\binom{n+2}{r+1} = \binom{n}{r-1} + \binom{n}{r+1} + 2\binom{n}{r},$$

$$\binom{n}{1} + 2\binom{n}{2} + \cdots + n\binom{n}{n} = 2^{n-1}n,$$

$$\binom{n}{1} - 2\binom{n}{2} + \cdots + (-1)^{n-1}n\binom{n}{n} = 0,$$

$$2\cdot 1\binom{n}{2} + 3\cdot 2\binom{n}{3} + \cdots + n(n-1)\binom{n}{n} = 2^{n-2}n(n-1),$$

$$\binom{n}{k} + \binom{n}{k-1}\binom{m}{1} + \cdots + \binom{n}{k-r}\binom{m}{r} + \cdots + \binom{m}{k} = \binom{n+m}{k}.$$

19. Show that the number of ways in which n indistinguishable objects may be arranged in M distinguishable cells is $\binom{M+n-1}{n}$.

20. Let $n > M$. Show that the number of ways in which n indistinguishable objects may be arranged in M distinguishable cells so that no cell will be empty is $\binom{n-1}{n-M}$.

21. Let $U(M,n)$ denote the number of unordered samples of size n that one may draw, by sampling with replacement, from an urn containing M distinguishable balls. Show that $U(M,n) = \binom{M+n-1}{n}$.

Hint: To prove the assertion, make use of the principle of mathematical induction.