FEGYVERNEKI SÁNDOR,

PROBABILITY THEORY AND MATHEMATICAL STATISTICS

3

Új Magyarország FEJLESZTÉSI TERV

A Műszaki Földtudományi Alapszak tananyagainak kifejlesztése a TÁMOP 4.1.2-08/1/A-2009-0033 pályázat keretében valósult meg.

III. Random Variables and their Characteristics

1. Random variables

Let (Ω, Σ, P) be a probability space. A random variable X is a real-valued function defined on Ω such that the set

$\{X < x\} = \{\omega : \omega \in \Omega \quad \text{and} \quad X(\omega) < x\}$

be an event for each real x.

If $X : \Omega \mapsto \mathbf{R}$ is a random variable, then its distribution function F is given by

$F(x) = P(X < x).$

Theorem: A function F is the distribution function of a random variable iff F is nonnegative, nondecreasing, left continuous and

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} F(x) = 1.$$

The probability mass function of X is defined for all real x by

$p(x) = P(X = x).$

A random variable X is said to be discrete if

$$\sum_{x \in T} p(x) = 1,$$

where $T = \{x : p(x) > 0\}$ and if T is either a finite set or an countable infinite set. The elements of T are called the mass points of X.

A random variable X is continuous if $p(x) = 0$ for all real x. If there exists a nonnegative, real function $f(x)$ such that

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

then f is the probability density function of X.

Theorem: A function f is the probability density function of a random variable iff f is nonnegative and

$$\int_{-\infty}^{+\infty} f(t)dt = 1.$$

Let $h(X)$ be a function of a random variable X. The *expected value* of $h(X)$, denoted by $E(h(X))$, is defined by

$$E(h(X)) = \sum_{x \in T} h(x)p(x), \quad \text{if } X \text{ is} \quad \text{discrete},$$

or

$$E(h(X)) = \int_{-\infty}^{+\infty} h(x)f(x)dx, \quad \text{if } X \text{ is} \quad \text{continuous},$$

and $E(|h(X)|)$ exists.

Some important parameters of a random variable X:

- $m = E(X)$, the mean or expectation,
- $\sigma^2 = Var(X) = E((X - E(X))^2)$, the variance,
- $\sigma = \sqrt{Var(X)}$, the standard deviation,
- $E(X^k)$, the k **th moment,**
- $E((X - E(X))^k)$, the k **th central moment.**

PROBLEM

EXAMPLE 1 For a circular disc of radius 4 cm, what is the probability density function of the distance (r) from the centre of a point chosen at random on the disc? Calculate the probability that a point lies between 2 and 3 cm from the centre of the disc.

Solution: In this case it is easier first of all to calculate the distribution function F.

$$F(r) = P(\text{distance from centre} \leq r) = \text{fraction of total area whose distance} \leq r. \text{ i.e.}$$

$$F(r) = \frac{\pi r^2}{\pi \times 4^2} = \frac{r^2}{16}.$$

Therefore the probability density function is

$$f(r) = F'(r) = \frac{r}{8}$$

and

$$P(2 \leq r < 3) = F(3) - F(2) = \frac{5}{16}.$$

PROBLEM

EXAMPLE 2 The density function of the lifetime X, in minutes, of atoms of a radioactive element is given by

$$f(t) = 3e^{-3t}.$$

Find the expected lifetime of an atom, its variance, and the half-life of the element (the time it takes for half of the material to decay.)

Solution: The expected lifetime is given by

$$E(X) = \int_0^{\infty} 3te^{-3t}dt.$$

Integrating by parts,

$$E(X) = [-te^{-3t}]_0^{\infty} + \int_0^{\infty} e^{-3t}dt = 0 + \frac{1}{3},$$

and

$$Var(X) = \int_0^{\infty} 3t^2e^{-3t}dt - \frac{1}{9} = \frac{1}{9}.$$

Now if T is the half-life, then

$$\int_0^{T} 3e^{-3t}dt = \frac{1}{2},$$

that is, $T = \frac{1}{3}\ln 2.$

2. Exercises

Solutions: ⊙ visible ○ invisible

1. Describe the probability distribution of the random variable given

i. the number of aces in a hand of 13 cards drawn without replacement from a bridge deck.

ii. the sum of numbers on 2 balls drawn with replacement (without replacement) from an urn containing 6 balls, numbered 1 to 6.

iii. the maximum of the numbers on 2 balls drawn with replacement (without replacement) from an urn containing 6 balls, numbered 1 to 6.

iv. the number of white balls drawn in a sample of size 2 drawn with replacement (without replacement) from an urn containing 6 balls, of which 4 are white.

v. the second digit in the decimal expansion of a number chosen on the unit interval at random.

vi. the number of times a fair coin is tossed until heads appears.

vii. the number of cards drawn without replacement from a deck of 52 cards until (a) a spade appears, (b) an ace appears.

Solutions:

(i) $p(x) = \dfrac{\binom{4}{x}\binom{48}{13-x}}{\binom{52}{13}}, x = 0, 1, 2, 3, 4,$

(ii) with: $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36},$

without: $\frac{2}{30}, \frac{2}{30}, \frac{4}{30}, \frac{4}{30}, \frac{6}{30}, \frac{4}{30}, \frac{4}{30}, \frac{2}{30}, \frac{2}{30},$

(iii) without: $p(x)=\frac{2}{30}(x-1)$, with: $p(x)=\frac{2x-1}{36}$, $x=1,2,3,4,5,6$,

(iv) with: $\frac{1}{9}, \frac{4}{9}, \frac{4}{9}$, without: $\frac{1}{15}, \frac{8}{15}, \frac{6}{15}$,

(v) $p(x)=\frac{1}{10}$, $x=0,1,2,3,4,5,6,7,8,9$,

(vi) $p(n)=2^{-n}$, $n\in\mathbb{N}$,

(vii) (a) $13\binom{39}{x-1}/(53-x)\binom{52}{x-1}$, (b) $4\binom{48}{x-1}/(53-x)\binom{52}{x-1}$.

---

2. Verify that the following functions are probability mass functions and determine the expectation and variance:

Bernoulli-distribution $(0\le p\le 1)$

$$p(x)=\begin{cases} p, & \text{if } x=1,\\ 1-p, & \text{if } x=0,\\ 0, & \text{otherwise.}\end{cases}$$

binomial distribution denoted by $b(k;n,p)$ $(n\in\mathbb{N}, 0\le p\le 1)$

$$p(x)=\begin{cases} \binom{n}{x}p^x(1-p)^{n-x}, & \text{for } x=0,1,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

Poisson distribution denoted by $p(k,\lambda),\lambda>0)$

$$p(x)=\begin{cases} e^{-\lambda}\frac{\lambda^x}{x!}, & \text{for } x=0,1,2\dots,\\ 0, & \text{otherwise.}\end{cases}$$

geometric distribution $(0\le p\le 1)$

$$p(x)=\begin{cases} p(1-p)^{x-1}, & \text{for } x=0,1,2,\dots,\\ 0, & \text{otherwise.}\end{cases}$$

hypergeometric distribution $(N\in\mathbb{N},\ n\in\{1,2,\dots N\}$ and $p\in\{0,1/N,2/N,\dots,1\}, q=1-p)$

$$p(x)=\begin{cases} \binom{Np}{x}\binom{Nq}{n-x}, & \text{for } x=0,1,\dots,n,\\ 0, & \text{otherwise.}\end{cases}$$

negative binomial distribution $(r\in\mathbb{N}, 0\le p\le 1, q=1-p)$

$$p(x)=\begin{cases} \binom{r+x-1}{x}p^r q^x, & \text{for } x=0,1,2,\dots,\\ 0, & \text{otherwise.}\end{cases}$$

Solutions:

(i) p, pq, (ii) np, npq, (iii) λ,λ, (iv) $\frac{1}{p}, \frac{q}{p^2}$, (v) $np, npq\frac{N-n}{N-1}$.

3. A random variable X is called a Bernoulli random variable if it has Bernoulli-distribution. The Bernoulli random variable is often used to describe a random experiment with only two outcomes, "success" or "failure." X is 1 for a success and 0 for a failure. Such an experiment is called a Bernoulli trial. A sequence of n such trials is called a Bernoulli sequence of trials if the probability of success does not change from trial to trial.

Consider a Bernoulli sequence of trials where the probability on each trial is p. Prove that the random variable X which counts

the number of successes in the n trials has binomial distribution with parameters n and p.

the number of trials before the trial at which the first success occurs has geometric distribution with parameter p.

the number of trials before the trial at which the r th success occurs has negative binomial distribution with parameters r and p.

---

4. Prove that the random variable X which counts the number of white balls contained in a sample of size n drawn without replacement from an urn containing N balls, of which Np are white, has hypergeometric distribution with parameters N, n and p.

---

5. Verify that the following functions are probability density functions and determine the expectation and variance:

i. uniform distribution over the interval a to b denoted by $U(a,b)$

$$f(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a < x < b, \\ 0, & \text{otherwise.} \end{cases}$$

ii. normal distribution $(m \in \mathbf{R}, \sigma > 0)$ denoted by $N(m, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-m)^2}{2\sigma^2}), \qquad x \in \mathbf{R}.$$

iii. exponential distribution $(\lambda > 0)$

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{for } x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

iv. Cauchy distribution $(-\infty < \alpha < +\infty, \beta > 0)$

$$f(x) = \frac{1}{\pi\beta\{1+(\frac{x-\alpha}{\beta})^2\}}, \qquad x \in \mathbf{R}$$

Solutions:

(i) $\frac{a+b}{2}$, $\frac{(a-b)^2}{12}$, (ii) m, σ^2, (iii) $\frac{1}{\lambda}$, $\frac{1}{\lambda^2}$, (iv) $E(X)$ does not exist.

---

6. Suppose that the amount of money (in dollars) that a person in a certain social group has saved is found to be a random variable, with a probability function specified by the distribution function

$$F(x)=\begin{cases}\frac{1}{2}e^{-(\frac{x}{50})^2}, \text{ for } x\le 0,\\ 1-\frac{1}{2}e^{-(\frac{x}{50})^2}, \text{ for } x\ge 0.\end{cases}$$

Note that a negative amount of savings represents a debt.

(i) Sketch the distribution function.

(ii) Is the distribution function continuous? If so, give a formula for its probability density function.

What is the probability that the amount of savings possessed by a person in the group will be (a) more than 50 dollars, (b) less than -50 dollars, (c) between -50 dollars and 50 dollars, (d) equal to 50 dollars?

What is the probability that the amount of savings possessed by a person in the group will be (a) less than 100 dollars, given that it is more than 50 dollars, (b) more than 50 dollars, given that it is less than 100 dollars?

Solutions:

(ii) $f(x)=\frac{|x|}{2500}\exp\left[-\left(\frac{x}{50}\right)^2\right]$, (iii) (a) 0.184, (b) 0.184, (c) 0.632, (d) 0,

(iv) (a) $1-\frac{1}{e}$, (b) $\frac{e^{-1}-e^{-4}}{2-e^{-4}}$.

---

7. Suppose that the time in minutes that a man has to wait at a certain subway station for a train is found to be a random variable with distribution function

$$F(x)=\begin{cases}0, \text{ if } x\le 0,\\ \frac{1}{2}x, \text{ if } 0\le x\le 1,\\ \frac{1}{2}, \text{ if } 1\le x\le 2,\\ \frac{1}{4}x, \text{ if } 2\le x\le 4,\\ 0, \text{ if } x\ge 4.\end{cases}$$

(i) Sketch the distribution function.

(ii) Is the distribution function continuous? If so, give a formula for its probability density function.

What is the probability that the time the man will have to wait for a train will be (a) more than 3 minutes, (b) less than 3 minutes, (c) between 1 and 3 minutes?

What is the probability that the time the man will have to wait for a train will be (a) more than 3 minutes given that it is more than 1 minute, (b) less than 3 minutes given that it is more than 1 minute?

Solutions:

(ii) $f(x)=\frac{1}{2}$, for $0<x<1$; $=\frac{1}{4}$, for $2<x<4$; $=0$ otherwise,

(iii) (a) $\frac{1}{4}$, (b) $\frac{3}{4}$, (c) $\frac{1}{4}$,

(iv) (a) $\frac{1}{2}$, (b) $\frac{1}{2}$.

8. The probability that a transistor in a radio lasts less than t hours is $1-e^{-t/2000}$. Find the probability density function for the lifetime of a transistor.

i. What is the probability that a transistor lasts more than 4,000 hours?

ii. What is the probability that a transistor ceases functioning after 2,000 hours of use but before 3,000 hours.

iii. If a radio contains 8 transistors, what is the probability that none of them fails before 1,000 hours of use?

Solutions:
$f(t)=\frac{1}{2000}\exp\left[-\frac{t}{2000}\right]$,
(i) 0.135, (ii) 0.145, (iii) 0.018.

9. Give formulas for, and identify, the probability law of each of the following numerical valued random phenomena:

i. The number of defectives in a sample of size 20, chosen without replacement from a batch of 200 articles, of which 5% are defective.

ii. The number of baby boys in a series of 30 independent births, assuming the probability at each birth that a boy will be born is 0.51.

iii. The minimum number of babies a woman must have in order to give birth to a boy (ignore multiple births, assume independence, and assume the probability at each birth that a boy will be born is 0.51).

iv. The number of patients in a group of 35 having a certain disease who will recover if the long-run frequency of recovery from this disease is 75% (assume that each patient has an independent chance to recover).

Solutions:
(i) Hypergeometric with parameters $N=200,\ n=20,\ p=0.05$,
(ii) binomial with parameters $n=30,\ p=0.51$,
(iii) geometric with parameter $p=0.51$,
(iv) binomial with parameters $n=35,\ p=0.75$.

10. Two dice are thrown and their total score r recorded. Calculate the probabilities $p(r)\ (2\le r\le 12)$ and sketch the corresponding probability distibution. Repeat when r is the numerical difference between their scores.

Solutions:
$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}$,
$\frac{3}{18}, \frac{5}{18}, \frac{4}{18}, \frac{3}{18}, \frac{2}{18}, \frac{1}{18}$.

11. A marksman has an 80% chance of scoring a bullseye. When he fires five shots at a target calculate the probabilities of his scoring 0, 1, 2, 3, 4, 5 bullseyes.

Solutions:
0.0003, 0.0064, 0.0512, 0.2050, 0.4096, 0.3277.

12. *What is the expected value of*

i. the score when a die is thrown,

ii. the total score when two dice are thrown?

Solutions:
3.5, 7.

---

13. If two dice are thrown, what is the expected value of the higher score (or the score on each, if both are the same)?

Solution: $4\frac{17}{36}$.

---

14. A metre rule is marked at intervals of one decimetre, including its ends. If two of these eleven points are chosen at random, what is the expected distance between them?

Solution: $3\frac{7}{11}$.

---

15. Jones and Smith each cut a pack of cards. If both their cards are of the same suit Smith pays Jones $5 unless they are both hearts, when he pays $10. If, however, the suits are different Jones pays Smith $2. Whom does this favour, and by how much?

Solution: Jones, by $6\frac{1}{4}p$.

---

16. Two players stake $1 each in a game in which eight counters are tossed on the ground. The counters are painted black on one side and white on the other. If the number of blacks showing is odd the thrower wins the other's stake, if all counters are of the same colour he wins a double stake, and otherwise he loses his stake. Calculate the expected gain of the thrower and state what assumptions you make.

Solution: $ $\frac{3}{128}$.

---

17. Calculate:

i. the variance of the score obtained on throwing a single die,

ii. the variance of the total score obtained on throwing two dice,

iii. the variance of the average score on two dice.

How are your answers to (i) and (ii) and to (ii) and (iii) related?

Solution: (i) 2.92, (ii) 5.83, (iii) 1.46.

---

18. A computer is made to produce randomly the numbers 0, 1, 2, $\cdots$, 9. What is the expected value and variance of numbers so produced?

Solution: 4.5, 8.25.

---

19. What are the mean and variance of the number of different factors (other than 1 and the number itself) of an integer chosen at random in the range 1 to 30?

Solution: 1.73, 3.13.

---

20. A box contains 100 tokens which differ in mass, but are otherwise identical. 20 of the tokens have a mass of 4.8g each, 35 a mass of 5.2g, 25 a mass of 5.7g, 15 a mass of 6.5g, and the remaining 5 have a mass of 8.0g each. Calculate (to 2 decimal places) the expected value and variance of the mass of a token chosen at random from the box.

Solution: 5.58,0.60.

---

21. An examination questions consists of two parts, A and B, and the probability of a pupil getting part A correct is $\frac{2}{3}$. If he gets A correct, the probability of getting B correct is $\frac{3}{4}$; otherwise it is $\frac{1}{6}$. There are three marks for a correct solution to part A, two marks for part B, and a bonus mark if both parts are correct. Calculate the expected value and variance of the pupil's total mark for the question.

Solution: 3.61, 6.69.

---

22. Calculate the expected value and standard deviation of the number of heads obtained when an unbiased coin is tossed (i) 4, (ii) 36, (iii) 100 times.

Solutions: (i) 2, 1, (ii) 18, 3, (iii) 50, 5.

---

23. Six players take it in turns to cut a pack of cards (excluding jokers), and each time the cards are replaced before the next player cuts. Calculate the expected value and standard deviation of the total numbers of spades turned up when each player has cut.

Solution: $\frac{3}{2}$, $\frac{3\sqrt{2}}{4}$,

---

24. Prove that for a binomial probability distributions arising from n trials, the maximum possible value of the variance is $\frac{n}{4}$.

---

25. An electronics firm packs the resistors that it produces in boxes of 800. On average one component in 100 is faulty. Calculate the expected number of defective resistors in a box, and the variance of this number. How could you quickly obtain a good approximation to the variance in this case? What would the percentage error (of the actual value) be if this approximation were used?

Solution: 8, 7.92; approximation np; 1.01%.

---

26. A box contains a large number of screws. The screws are very similar in appearance, but are in fact of 3 different types, A, B and C, which are present in equal numbers. For a given job only screws of type A are suitable. If 4 screws are chosen at random, find the probability that:

i. exactly two are suitable,

ii. at least two are suitable.

If 20 screws are chosen at random, find the expected value and varince of the number of suitable screws.

Solutions: (i) $\frac{8}{27}$, (ii) $\frac{11}{27}$, $\frac{20}{3}$, $\frac{40}{9}$.

---

27. A mathematical model for the fraction X of the sky covered with cloud $(0 < X < 1)$ assigns to this a density function

$$f(x) = \frac{k}{\sqrt{x - x^2}}.$$

Calculate:

i. the value of k.

ii. the expected fraction covered by cloud.

iii. the probability that not more than $\frac{1}{4}$ of the sky is covered.

Solutions: (i) $k = \frac{1}{\pi}$, (ii) $\frac{1}{2}$, (iii) $\frac{1}{3}$.

---

28. The probability density function of a distribution is given by

$$f(x) = \frac{e^{-x} x^{\lambda - 1}}{(\lambda - 1)!} \qquad (x \geq 0, \lambda \in \mathbb{N}).$$

Find the expected value and variance.

Solution: λ, λ.

---

29. The probability density function is given by

$$f(x) = \begin{cases} A(1 - x^2), & \text{for} -1 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the value of A, and then calculate:

$$(i) \quad P(-\frac{1}{2} \leq X \leq \frac{1}{2}), \qquad (ii) \quad P(X > \frac{1}{3}).$$

Obtain an expression for the distribution function.

Solutions:

$A = \frac{3}{4}$, (i) $\frac{11}{36}$, (ii) $\frac{7}{27}$,

$$F(x) = \begin{cases} 0, \text{ if } x \le -1, \\ \frac{1}{4}(2+3x-x^3), \text{ if } -1 < x \le 1, \\ 1, \text{ if } 1 < x. \end{cases}$$

---

30. The probability density function of a distribution is given by

$$f(x) = \begin{cases} A \sin \pi x, \text{ for } 0 \le x \le 1, \\ 0, \text{ elsewhere.} \end{cases}$$

Find the value of A, and obtain an expression for the distribution function. Calculate:

$$(i) \quad P(X < \frac{1}{3}), \qquad (ii) \quad P(\frac{1}{2} < X < \frac{2}{3}).$$

Solutions:

$A = \frac{\pi}{2}$,

(i) $\frac{1}{4}$, (ii) $\frac{1}{4}$,

$$F(x) = \begin{cases} 0, \text{ if } x \le 0, \\ \frac{1}{2}(1 - \cos \pi x), \text{ if } 0 < x \le 1, \\ 1, \text{ if } 1 < x. \end{cases}$$

---

31. Find the value of A, if the following functions are density functions:

i. $f(x) = \begin{cases} 0, \text{ if } x \le 0, \\ Ax^2 e^{-2x}, \text{ if } x > 0. \end{cases}$

ii. $f(x) = \begin{cases} 0, \text{ if } x \le 0, \\ Axe^{-2x^2}, \text{ if } x > 0. \end{cases}$

iii. $f(x) = \frac{A}{x^2+4}, \qquad x \in \mathbf{R}$

iv. $f(x) = \begin{cases} 0, \text{ elsewhere,} \\ Ax(1-x)^2, \text{ if } 0 < x \le 1. \end{cases}$

Solutions:

(i) 4, (ii) 4, (iii) $\frac{2}{\pi}$, (iv) 12.

32. The length of time (in minutes) that a certain young lady on the telephone is a random variable, with density function given by

$$f(x) = \begin{cases} Ae^{-x/5}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

i. Find the value of A.

ii. Graph the probability density function.

iii. What is the probability that the number of minutes that the young lady will talk on the telephone is (a) more than 10 minutes, (b) less than 5 minutes, (c) between 5 and 10 minutes?

Solutions:

(i) $\frac{1}{5}$, (iii) (a) 0.1353, (b) 0.6321, (c) 0.3226.

33. The number of times that a certain piece of equipment (say, a light switch) operates before having to be discarded is found to be a random phenomenon given by

$$p(x) = \begin{cases} A(\frac{1}{3})^x, & \text{for } x = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

i. Find the value of A which makes p a probability mass function.

ii. Sketch the probability mass function.

iii. What is the probability that the number of times the eqiupment will operate before having to be discarded is (a) greater than 5, (b) an even number (regard 0 as even), (c) an odd number?

Solutions:

(i) $\frac{2}{3}$, (iii) (a) $\left(\frac{1}{3}\right)^6$, (b) $\frac{3}{4}$, (c) $\frac{1}{4}$.

34. Pareto's distribution with parameters r and A, in which r and A are positive, is defined by the probability function

$$f(x) = \begin{cases} rA^r \frac{1}{x^{r+1}}, & \text{for } x \geq A, \\ 0, & \text{for } x < A. \end{cases}$$

Show that Pareto's distribution possesses a finite n th moment if and only if $n < r$. Find the mean and variance of Pareto's distribution in the cases in which they exist.

Solution:

If $r \geq 3$, $\frac{Ar}{r-1}$, $\frac{A^2 r}{(r-2)(r-1)^2}$.

35. Show that for a continuous probability distibution

$$\int_0^{+\infty}(1-F(x))dx=\int_0^{+\infty}yf(y)dy,$$

$$-\int_{-\infty}^{0}F(x)dx=-\int_0^{-\infty}yf(y)dy,$$

Consequently the expectation m of the probability distribution may be written

$$m=\int_{-\infty}^{+\infty}yf(y)dy=\int_0^{+\infty}(1-F(x))dx-\int_{-\infty}^{0}F(x)dx.$$

36. Compute the expectation and variance of the following probability distributions. (Recall that $[x]$ denotes the largest integer less than or equal to x.)

i. $F(x)=\begin{cases}0, \text{ for } x<0,\\ 1-\frac{1}{3}e^{-\frac{x}{3}}-\frac{2}{3}e^{-[\frac{x}{3}]}, \text{ for } x\geq 0.\end{cases}$

ii. $F(x)=\begin{cases}0, \text{ for } x<0,\\ 8\int_0^x ye^{-4y}dy+\frac{e^{-2}}{2}\sum_{k=0}^{[x]}\frac{2^k}{k!}, \text{ for } x\geq 0.\end{cases}$

iii. $F(x)=\begin{cases}0, \text{ for } x<1,\\ 1-\frac{1}{2x^2}-\frac{1}{2^{[x]}}, \text{ for } x\geq 1.\end{cases}$

iv. $F(x)=\begin{cases}0, \text{ for } x<1,\\ 1-\frac{2}{3x}-\frac{1}{3^{[x]}}, \text{ for } x\geq 1.\end{cases}$

37. Consider a radar set of a type whose failure law is exponential. If radar sets of this type have a failure rate $\lambda=1set/1000hours,$ find a length T of time such that the probability is 0.99 that a set will operate satisfactorily for a time greater than T.

Solution: $T=10$ hours.

38. Describe the probability distribution of the following random phenomenon: the number N of times a fair die is tossed until an even number appears

(i) for the first time,

(ii) for the second time,

(iv) for the third time.

Solutions:

$N-r$ obeys a negative binomial probability law with parameters $p=\frac{1}{2}$ and

(i) $r=1,$ (ii) $r=2,$ (iii) $r=3.$

39. Find $P(1 \leq X \leq 2)$ for the random variable X if X has

i. normal distribution with parameters $m = 1$ and $\sigma = 1$,

ii. binomial distribution with parameters $n = 10$ and $p = 0.1$,

iii. geometric distribution with parameter $p = \frac{1}{3}$,

iv. uniform distribution over the interval $\frac{1}{2}$ to $\frac{3}{2}$.

Solutions:

(i) 0.3413, (ii) 0.5811, (iii) $\frac{5}{9}$, (iv) $\frac{1}{2}$.