FEGYVERNEKI SÁNDOR,

PROBABILITY THEORY AND MATHEMATICAL STATISTICS

6

Új Magyarország FEJLESZTÉSI TERV

A Műszaki Földtudományi Alapszak tananyagainak kifejlesztése a TÁMOP 4.1.2-08/1/A-2009-0033 pályázat keretében valósult meg.

VI. Miscellaneous exercises

Solutions: ⦿ visible ◯ invisible

1. Let X and Y be independent random variables. Let $Z = Y - X$. Let $A = \{|Z| \le 1\}$. Find $P(A \mid X = 1)$, $F_{Z|X}(0 \mid 1)$, $f_{Z|X}(0,1)$, $P(Z \le 0 \mid A)$. If X and Y are each

i. uniformly distributed over the interval 0 to 2.

ii. normally distributed with parameters $\mu = 0$ and $\sigma = 2$.

iii. exponentially distributed with parameter $\lambda = 1$.

Solutions:

(i) 1, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, (iii) 0.865, 0.632, 0.368, 0.5.

2. Let X and Y be independent random variables. Let $U = X + Y$ and $V = Y - X$. Let $A = \{|V| \le 1\}$. Find $P(A \mid U = 1)$, $F_{V|U}(0 \mid 1)$, $f_{V|U}(0,1)$, $P(U \ge 0 \mid A)$, $f_{V|U}(u \mid v)$. If X and Y are each

i. uniformly distributed over the interval 0 to 2.

ii. normally distributed with parameters $\mu = 0$ and $\sigma = 2$.

iii. exponentially distributed with parameter $\lambda = 1$.

Solutions:

(ii) 0.276, 0.5, 0.2, 0.5, $\frac{1}{2}\phi\left(\frac{v}{2}\right)$.

3. A continuous random variable X is said to have a gamma distribution $\alpha > 0$ and $\lambda > 0$ if its density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}, & \text{if } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Here Γ is the celebrated gamma function defined by $\Gamma(t) = \int_0^\infty x^{t-1}e^{-x}dx, \qquad t > 0.$

Prove that

i. $\Gamma(n+1) = n!, \qquad n = 0, 1, 2, \ldots$

ii. $\Gamma(t+1) = t\Gamma(t)$ for all $t > 0$.

Determine the characteristic function, expectation and variance. Prove that if Y is independent of X and has a gamma distribution with parameters β and λ then $Z = X + Y$ has a gamma distribution with parameters $\alpha + \beta$ and λ

Solutions:
$\left(\frac{\lambda}{\lambda - iu}\right)^{\alpha}$, $\frac{\alpha}{\lambda}$, $\frac{\alpha}{\lambda^2}$.

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4. "Student's" $t-$ distribution with parameter $a > 0$ is defined as the continuous probability distribution specified by the probability density function

$$f(x) = \frac{1}{\sqrt{a\pi}} \frac{\Gamma(\frac{a+1}{2})}{\Gamma(\frac{a}{2})} (1 + \frac{x^2}{a})^{-\frac{a+1}{2}}.$$

Note that $t-$ distribution with parameter $a = 1$ coincides with the Cauchy distribution. Show that for $t-$ distribution with parameter a

i. the n th moment exists only for $n < a$,

ii. if $n < a$ and n is odd, then $E(X^n) = 0$,

iii. if $n < a$ and n is even, then

$$E(X^n) = a^{\frac{n}{2}} \frac{\Gamma(\frac{n+1}{2})\Gamma(\frac{a-n}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{a}{2})}.$$

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5. Four points are chosen independently and at random on a circle. Find the probability that the chords X_1X_2 and X_3X_4 intersect: (a) without calculation using a symmetry argument, (b) from the definition by an integral.

Solution:
$2\int_0^1 x(1-x)dx = \frac{1}{3}$. Two out of six permutations produce an intersection.

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6. Prove that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} = \frac{1}{2}.$$

Hint : Use the central limit theorem.

Hint:
Use Poisson distribution.

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7. Suppose that X is a discrete, nonnegative, integer-valued random variable. Prove that

$$E(X) = \sum_{k=0}^{\infty} P(X > n).$$

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8. Let a, b, c be random variables with distribution $U(0,1)$. Find the probability that the equation $ax^2 + bx + c = 0$ has real roots.

Solution:
$\frac{5+3\ln 4}{36}$.

9. Suppose that X has the distribution function

$$F(x)=1-e^{-\frac{x^m}{c}}, \quad if \quad x>0.$$

It is called Weibull's distribution. What is the meaning of c in the probability theory?

Solution: $c=E(X^m)$.

10. Along a straight road, L miles long, are n distinguishable persons, distributed at random. Show that the probability that no two persons will be less than a distance d miles apart is equal to, for d such that $(n-1)d \le L$,

$$(1-(n-1)\frac{d}{L})^n.$$

11. Let X_1, X_2 and X_3 be independent normally distributed random variables, each with mean 1 and variance 3. Find $P(X_1+X_2+X_3>0)$.

Solution: 0.8413.

12. A fair coin is tossed n times. Let T_n be the number of times in the n tosses that a tail is followed by a head. Show that

$$E(T_n)=\frac{n-1}{4} \qquad \text{and} \qquad E(T_n^2)=\frac{n-1}{4}+\frac{(n-2)(n-3)}{16}.$$

13. Machine components have lengths which are normally distributed with mean 57.2 mm. It is found that 13% of the components have lengths greater than 57.4 mm.

i. What is the standard deviation of the component lengths?
ii. What length will be exceeded by 20% of the components?

Solutions:
(i) 0.18, (ii) 57.35.

14. In a certain book the frequency function for the number of words per page may be taken as approximately normal with mean 800 and standard deviation 50. If I choose three pages at random, what is the probability that none of them has between 830 and 845 words?

Solutions: 0.753.

15. If X is normally distributed with mean m and standard deviation σ, calculate the value of c such that

i. $P(|X-m|<c\sigma)=0.5$,

ii. $P(|X-m|<c\sigma)=0.9$.

> **Solutions:**
> (i) 0.675, (ii) 1.65.

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16. The envelope of a narrow-band noise is sampled periodically, the samples being sufficiently far apart to assure independence. In this way n independent random variables $X_1, X_2, \ldots, X_n$ are observed, each of which is Rayleigh distributed with parameter σ. Find the probability density function of the largest value in the sample.

> **Solution:**
> $\dfrac{4(4t)^9 e^{-4t}}{9!}, \quad \dfrac{5}{8}, \quad \dfrac{\lambda(\lambda t)^{k-1} e^{-\lambda t}}{(k-1)!}.$

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17. A certain device, used for measuring Earth tremors in a seismological station, can only maintain the required degree of sensitivity for a total of ten tremors, and after this is disposed of. If such tremors occur randomly, but averaging 4 per day, find the probability density function for the useful lifetime (in days) of such device, and use this to show that the mean lifetime is $2\frac{1}{2}$ days. What is the variance of the lifetime?

Find the probability density function if the device can be used for k tremors and tremors average λ per day.

> **Solutions:**
> (i) $\dfrac{(1-k)(1+t)}{1+(1-k)t}$, (ii) $\dfrac{(1-k)(1+2t)}{1+(2-k)t}$.

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18. For a certain type of bacterium, the time X from birth to death is a random variable with probability density function $(1+x)^{-2}, \quad (x \geq 0)$. A culture of such bacteria is routinely inspected at regular intervals of time.

One bacterium, inspected at time t after birth, is found to be dead already. Find the probability that this bacterium has been dead for at least time $kt \quad (0<k<1)$.

Another bacterium of the same type is alive at time t after birth, but is found to be dead by the end of the further time t. Find the probability that this bacterium had been dead for at least time $kt \quad (0<k<1)$.

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19. Let X be an exponential random variable with parameter $\lambda > 0$. Prove that

i. $E(X^k) = \dfrac{k!}{\lambda^k}, \quad k \in \mathbf{R}$.

ii. $P(X > t+h \mid X > t) = P(X > h), \quad t > 0, h > 0$.

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20. Let X be a geometric random variable. Prove that

$$P(X = k+m \mid X \geq m) = P(X = k), \quad k = 0,1,2,\ldots, m = 0,1,2,\ldots.$$

> **Solution:**
> $\dfrac{(2+p)q}{p^2}$.

21. Let X be a geometric random variable. Determine $E(X^2-1)$.

Solution:
$\dfrac{1-(1-p)^{n+1}}{(n+1)p}$.

22. Let X be a binomial random variable. Determine $E(\dfrac{1}{1+X})$.

Solution:
$\dfrac{1-e^{-\lambda}}{\lambda}$.

23. Let X be a Poisson random variable. Determine $E(\dfrac{1}{1+X})$.

Solution:
$P(2X+1=2k+1)=\dfrac{\lambda^k e^{-\lambda}}{k!}$, $(k=0,1,2,\ldots)$, $2\lambda+1$, 4λ.

24. Let X be a Poisson random variable. Determine the probability mass function of $2X+1$, and calculate $E(2X+1)$ and $Var(2X+1)$.

25. Suppose the number of arrivals, Y, of the same entity per unit of time is described by a Poisson random variable with parameter λ Prove the time T between any two successive arrivals (interarrival time) is independent of the interarrival time of any other successive arrivals and has an exponential distribution with parameter λ

Solutions:
(i) $p_n=\dfrac{1}{2}$, (ii) $p_n=\dfrac{1}{2}\left(1+\left(\dfrac{a-b}{a+b}\right)^{n-1}\right)$,

26. Two jars, one white and the other black, contain $a+b$ balls each; in the white jar there are a white and b black balls and in the black jar b white and a black. Single draws are made as follows: at the r th draw a ball is drawn from the white or black jar according to whether the $(r-1)$ st ball drawn was white or black, the colour of the ball noted, and then returned. If p_n is the probability that the n th ball is white, show that

$$(a+b)p_n=b+(a-b)p_{n-1}.$$

Determine p_n, when the jar from which the first draw is made is (i) chosen at random, (ii) white.

27. Let $X_1, X_2, \ldots$ be non-negative integer valued, independent, identically distributed random variables with their common generating function $G(z)$ and let N be a non-negative integer valued random vaiable. Let

$$R = \sum_{n=1}^{N} X_n.$$

Prove that $G_R(z) = G_N(G(z))$ and $E(R) = E(N)E(X_1)$. Calculate $Var(R)$.

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28. Let $X_1, X_2, \dots, X_n$ be independent random variables with their common distribution $U(0,1)$. Define $Y_1 = \min\{X_1, \dots, X_n\}$, and if Y_i is defined, let $Y_{i+1} = \min\{X_k > Y_i : 1 \le k \le n\}$. Then $Y_1 < Y_2 < \dots < Y_n$ are random variables, called order statistics from the distribution $U(0,1)$.

i. Determine the density function of Y_k.

ii. Determine the joint density function of (Y_k, Y_l).

Prove that

i. $E(Y_1 \mid Y_n = x) = \dfrac{x}{n}$,

ii. $E(Y_k \mid Y_l = x) = \dfrac{k}{l} x$,

iii. $E(Y_k) = \dfrac{k}{n+1}$,

iv. $E(\dfrac{Y_k}{Y_{k+1}}) = \dfrac{k}{k+1}$.

v. Determine $\lim_{n\to\infty} P(Y_k < \dfrac{x}{n})$.

Solutions:

(i) $\binom{n}{k} k x^{k-1}(1-x)^{n-k} \quad (0 \le x \le 1)$.

(ii) If $0 \le x \le y \le 1$ then

$$f(x,y) = \binom{n}{k-1}(n-k+1)\binom{n-k}{l-k+1}(n-l+1)\, x^{k-1}(y-x)^{l-k-1}(1-y)^{n-l}.$$

(vi) $E\left(\dfrac{Y_k}{Y_{k+1}} \mid Y_{k+1} = t\right) = \dfrac{1}{t} E(Y_k \mid Y_{k+1} = t) = \dfrac{k}{k+1}$,

(vii) $\displaystyle\int_0^x \frac{t^{k-1} e^{-t}}{(k-1)!} dt.$

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29. Let $Z_1, Z_2, \dots, Z_n$ be independent identically distributed random variables on (Ω, Σ, P) with their common distribution F on R continuous and strictly increasing. If $X_i = F(Z_i) \quad (1 \le i \le n)$, show that $X_1, \dots, X_n$ are random variables satisfying the hypothesis of Exercise 28. Deduce from the above that if Z_i^* is the i th-order statistic of $Z_1, \dots, Z_n$, then

$$P(Z_i^* < x) = \frac{n!}{(i-1)!(n-i)!} \int_0^{F(x)} y^{i-1}(1-y)^{n-i} dy.$$

30. The random variable Z is defined as the unique index such that

$$X_1 \geq X_2 \geq \ldots \geq X_{Z-1} < X_Z.$$

If the $X_1, \ldots, X_Z$ have a common continuous distribution F prove that

$$P(Z=n) = \frac{n-1}{n!} \quad \text{and} \quad E(Z) = e.$$

Let F be $U(0,1)$. Prove that

$$P(X_1 \leq x, Z=n) = \frac{x^{n-1}}{(n-1)!} - \frac{x^n}{n!},$$

whence

$$P(X_1 \leq x, Z \; even) = 1 - e^{-x}.$$

Define Y as follows: A "trial" is the sequence $X_1, X_2, \ldots X_Z;$ it is a "failure" if Z is odd. We repeat independent trials as long as necessary to produce a "success." Let Y equal the number of failures plus the first variable in the successful trial. Prove that

$$P(Y < x) = 1 - e^{-x}.$$

31. Let the random variable X have a Poisson distribution with expectation λ Show that

$$P(X \leq c) = 1 - P(Y < 2\lambda),$$

where the random variable Y has χ^2 distribution with $2(c+1)$ degrees of freedom.

Hint: Use repeated integration by parts.

Solution:

$E(X_n) = 0, \quad Var(X_n) = \sigma^2 \dfrac{\alpha^{2n} - 1}{\alpha^2 - 1}.$

32. Consider a sequence of independent random variables $Y_1, Y_2, \ldots,$ each with mean zero and variance σ^2. A new set of random variables $X_1, X_2, \ldots$ is defined from the relationship

$$X_k = \alpha X_{k-1} + Y_k, \qquad k = 1, 2, \ldots$$

and X_0 is defined to be zero.

Find $E(X_n)$ and $Var(X_n)$.