## CHAPTER 3

## ELECTRODYNAMICS

IN THIS chapter we shall develop equations which govern the behaviour of magnetic fields and electric currents in superconductors.

### 3.1. Consequence of Zero Resistance

In a superconducting metal the superelectrons encounter no resistance to their motion, so, if a constant electric field **E** is maintained in the material, the electrons accelerate steadily under the action of this field:

$$m\dot{\mathbf{v}}_s = e\mathbf{E} \tag{3.1}$$

where $\mathbf{v}_s$ is the velocity of the superelectrons and $m$ and $e$ are their mass and charge.† If there are $n_s$ superelectrons per unit volume moving with velocity $\mathbf{v}_s$, there is a supercurrent density

$$\mathbf{J}_s = n_s e \mathbf{v}_s.$$

Substituting this into (3.1) we see that an electric field produces a continuously increasing current with a rate of increase given by

$$\dot{\mathbf{J}}_s = \frac{n_s e^2}{m}\mathbf{E}. \tag{3.2}$$

To obtain an equation describing magnetic fields we remember that a magnetic field is related to an electric field and current by Maxwell's equations

$$\dot{\mathbf{B}} = -\operatorname{curl}\mathbf{E} \tag{3.3}$$

and

$$\operatorname{curl}\mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}. \tag{3.4}$$

In this chapter we wish to develop equations which explicitly relate the fields in superconductors to the currents flowing in them, and we shall

† In this book $e$ is used for the charge on the electron, and this symbol includes the negative sign, $e = -1{\cdot}602 \times 10^{-19}$ coulomb.

adopt the viewpoint, discussed in § 2.2.2, that the metal of a superconducting body is, like other non-ferromagnetic metals, non-magnetic, i.e. $\mu_r = 1$, so that any flux density in the metal must be due to the currents. We also take the view (see Appendix A) that while currents in the metal affect **B**, they do not affect **H**, and within the superconductor we therefore replace (3.4) by

$$\text{curl } \mathbf{B} = \mu_0(\mathbf{J}_s + \dot{\mathbf{D}}),$$

where $\mathbf{J}_s$ is the current density within the metal. Furthermore, unless the fields are varying very rapidly in time, the displacement current $\dot{\mathbf{D}}$ is negligible in comparison with $\mathbf{J}_s$. Hence inside a superconductor we can write Maxwell's equations in the form

$$\dot{\mathbf{B}} = -\text{curl } \mathbf{E}, \tag{3.3a}$$

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J}_s. \tag{3.4a}$$

Substituting (3.2) into (3.3a) gives

$$\dot{\mathbf{B}} = -\frac{m}{n_s e^2} \text{curl } \dot{\mathbf{J}}_s, \tag{3.5}$$

and we can eliminate $\mathbf{J}_s$ by means of (3.4a):

$$\dot{\mathbf{B}} = -\frac{m}{\mu_0 n_s e^2} \text{curl curl } \dot{\mathbf{B}}.$$

For brevity, let us use a single symbol $\alpha$ for the constant $m/\mu_0 n_s e^2$, so the last equation becomes

$$\dot{\mathbf{B}} = -\alpha \text{ curl curl } \dot{\mathbf{B}}. \tag{3.6}$$

Now

$$\text{curl curl } \dot{\mathbf{B}} = \text{grad div } \dot{\mathbf{B}} - \nabla^2 \dot{\mathbf{B}},$$

but from Maxwell's equations div $\mathbf{B} = 0$, so (3.6) becomes

$$\dot{\mathbf{B}} = \alpha \nabla^2 \dot{\mathbf{B}}$$

or

$$\nabla^2 \dot{\mathbf{B}} = \frac{1}{\alpha} \dot{\mathbf{B}}. \tag{3.7}$$

This is a differential equation which **B** must satisfy. To see what this implies, consider the plane boundary of a superconductor with a uniform magnetic field applied parallel to this boundary (Fig. 3.1). Suppose that the flux density outside the metal is $\mathbf{B}_a$, and let the direction normal to the boundary be called the $x$-direction. Because the applied field is uniform, $\mathbf{B}$ will have the same direction everywhere, so we may regard (3.7) as a scalar equation; also there will be no gradients of the field parallel to the boundary, so in this case (3.7) reduces to

$$\frac{\partial^2 \dot{B}}{\partial x^2} = \frac{1}{\alpha}\dot{B}. \tag{3.8}$$

The solution of this is†

$$\dot{B}(x) = \dot{B}_a \exp\left(\frac{-x}{\sqrt{\alpha}}\right), \tag{3.9}$$

where $B(x)$ is the flux density at a distance $x$ inside the metal and $\dot{B}_a$ is the value of $\dot{B}$ outside the metal. This means that $\dot{B}$ dies away exponent-

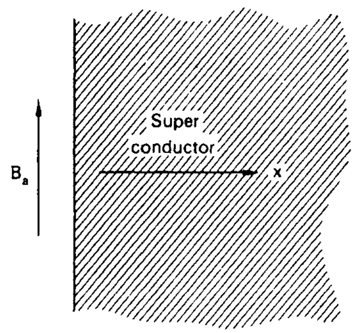


FIG. 3.1. Magnetic field applied parallel to boundary of superconductor.

ially as we penetrate into the superconductor. In other words, changes in flux density do not penetrate far below the surface, so at a sufficient distance inside the metal the flux density has a constant value which does not change with time, irrespective of what is happening to the applied field.

## 3.2. The London Theory

We have deduced the above behaviour by applying the usual laws of electrodynamics to a conductor with zero resistance; however, though (3.9) completely describes the magnetic properties of a perfect conduct-

† There is an alternative solution, $B_a \exp(+x/\sqrt{\alpha})$, but this approaches infinity as $x$ increases.

or, it does not adequately describe the behaviour of a superconductor. The Meissner effect shows that inside a superconductor the flux density is not only constant but the value of this constant is always zero; so not only $\dot{B}$ but $B$ itself must die away rapidly below the surface. F. and H. London† suggested that the magnetic behaviour of a superconducting metal might be correctly described if (3.7) applied not only to $\dot{\mathbf{B}}$ but to $\mathbf{B}$ itself,

$$\nabla^2\mathbf{B} = \frac{1}{\alpha}\mathbf{B}. \tag{3.10}$$

If this were so, the magnetic flux density $B$ would die away within the metal in a manner similar to the behaviour of $\dot{B}$, as described by (3.9),

i.e. $$B(x) = B_a \exp(-x/\sqrt{\alpha}).$$

Examination of the argument by which we derived (3.7) shows that we could have derived (3.10) if everywhere we had replaced $\dot{\mathbf{B}}$ by $\mathbf{B}$. If we retrace the argument, we arrive at (3.5) which would now have the more restrictive form

$$\mathbf{B} = \frac{-m}{n_s e^2}\,\mathrm{curl}\,\mathbf{J}_s. \tag{3.11}$$

This equation and (3.2), namely

$$\dot{\mathbf{J}}_s = \frac{n_s e^2}{m}\mathbf{E}, \tag{3.2}$$

which together describe the electrodynamics of the supercurrent, are known as the *London equations*. Equation (3.2) describes the resistanceless property of a superconductor, there being no electric field in the metal unless the current is changing; and (3.11) describes the diamagnetism.

Note that these equations are not deduced from fundamental properties and do not "explain" the occurrence of superconductivity. The London equations are restrictions on the ordinary equations of electromagnetism, and are introduced so that the behaviour deduced from these laws agrees with that observed experimentally.

The London equations lead us to replace (3.7) by (3.10). Let us now use this latter equation to determine the distribution of magnetic flux inside a superconductor when it is in a uniform magnetic field of flux den-

† F. London and H. London, *Proc. Roy. Soc.* (*London*) **A155**, 71 (1935).

sity $B_a$ applied parallel to its surface (Fig. 3.2). In this case we can use the one-dimensional form of (3.10),

$$\frac{\partial^2 B(x)}{\partial x^2} = \frac{1}{\alpha} B(x),$$

where $B(x)$ is the flux density at distance $x$ inside the metal. This equation has the solution

$$B(x) = B_a \exp\left(\frac{-x}{\sqrt{\alpha}}\right), \tag{3.12}$$

where $B_a$ is the flux density of the applied field at the surface. Equation (3.12) shows that the flux density dies away exponentially inside a superconductor, falling to $1/e$ of its value at the surface at a distance $x = \sqrt{\alpha}$. This distance is called the *London penetration depth,* $\lambda_L$. Now $\alpha = m/\mu_0 n_s e^2$, so the London penetration depth is given by

$$\lambda_L = \sqrt{(m/\mu_0 n_s e^2)}. \tag{3.13}$$

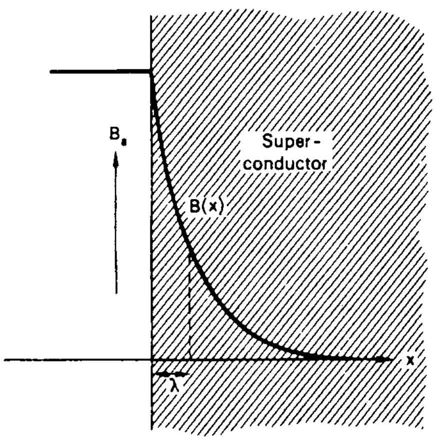


FIG. 3.2. Variation of flux density at boundary of a superconductor.

If we substitute for $m$ and $e$ the usual values of the electron's mass and charge, and take $n_s$ to be about $4 \times 10^{28}$ m$^{-3}$ (i.e. the usual concentration in metals; about one conduction electron per atom), the London penetration depth turns out to be about $10^{-6}$ cm.

We can now write (3.12) in the form

$$B(x) = B_a \exp(-x/\lambda_L). \tag{3.14}$$

The London equations predict, therefore, a very rapid exponential decay of the flux density at the surface of a superconductor. In Chapter 2 we defined, by means of (2.2), a general penetration depth which was independent of the particular form of the decrease of the flux density (i.e. whether exponential or otherwise). Substitution of (3.14) into (2.2) shows that the London penetration depth $\lambda_L$, defined by (3.13) and leading to an exponential penetration law of the form (3.14), satisfies this general definition of penetration depth.

In the previous chapter it was pointed out that any current in a superconductor must flow near the surface. For a uniform magnetic field applied parallel to the surface (Fig. 3.1) in the $z$-direction, (3.4a) reduces to $-\dfrac{\partial B}{\partial x} = \mu_0 J_y$. From (3.14) $\partial B/\partial x$ equals $-\dfrac{B_a}{\lambda_L}\exp(-x/\lambda_L)$, so we have

$$J_y = \frac{B_a}{\mu_0 \lambda_L}\exp(-x/\lambda_L)$$

which we may write as

$$J_y = J_a \exp(-x/\lambda_L).$$

We see, therefore, that any current flows close to the surface within the penetration depth.

We have mentioned that according to the two-fluid model of superconductivity (§ 1.4), the concentration of superelectrons $n_s$ decreases as the temperature is raised, falling to zero at the transition temperature. The London equations give a penetration depth which is inversely proportional to $n_s^{\frac{1}{2}}$ (3.13) and so predict that the penetration depth should increase with increasing temperature, rising to infinity as the temperature approaches the transition temperature. As we saw in § 2.4.1, this is what is observed experimentally.

We can now write the London equations (3.11) and (3.2) as

$$\operatorname{curl} \mathbf{J}_s = -\frac{1}{\mu_0 {\lambda_L}^2}\mathbf{B}$$

$$\dot{\mathbf{J}}_s = \frac{1}{\mu_0 {\lambda_L}^2}\mathbf{E}.$$

It is important to realize that the London equations do not replace Maxwell's equations, which, of course, still apply to all currents and the fields they produce. The London equations are additional conditions obeyed by the supercurrents.

In the most general case the total current $\mathbf{J}$ is the sum of a normal current and a supercurrent:

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s.$$

The normal current need only obey Maxwell's equations and Ohm's law,

$$\mathbf{J}_n = \sigma'\mathbf{E},$$

where $\sigma'$ is a conductivity associated with the normal electrons. We can now bring together the special equations which apply to a superconducting metal:

$$\mathbf{J} = \mathbf{J}_n + \mathbf{J}_s \tag{3.15}$$

$$\mathbf{J}_n = \sigma'\mathbf{E} \tag{3.16}$$

$$\operatorname{curl} \mathbf{J}_s = -\frac{1}{\mu_0 \lambda_L^2}\mathbf{B} \tag{3.17}$$

$$\dot{\mathbf{J}}_s = \frac{1}{\mu_0 \lambda_L^2}\mathbf{E}. \tag{3.18}$$

From these equations we can in principle work out the distribution of fields and currents in superconducting bodies under various conditions. In the steady state, when fields and currents are not changing with time, the only current is the supercurrent, i.e. $\mathbf{J}_n = 0$, and we need only employ the London equations (3.17) and (3.18). These lead to

$$\nabla^2\mathbf{B} = \frac{1}{\lambda_L^2}\mathbf{B}. \tag{3.19}$$

It should be pointed out that the form of (3.10) introduced to describe the distribution of magnetic flux density inside a superconductor was only a guess, based on the known properties of superconductors, and we must not expect the London equations inferred from it to be exactly correct. In fact, these equations are only approximations, though for many purposes they are sufficiently accurate. For example, the London equations predict a small penetration depth; a small penetration depth is indeed observed experimentally, but the value is greater than the London prediction by a factor of two or more. A discussion of the limitations of the London theory and a description of the more refined Ginzburg–Landau theory are given in Chapter 8.

### 3.2.1. An application of the London theory

We can in principle use (3.19) to find the distribution of flux density within any superconducting body by applying to the solution of this equation boundary conditions imposed by the shape of the body and the form of the applied magnetic field. As an example, we now calculate the distribution of flux density inside a superconducting slab, of finite thickness, with plane parallel faces, when it is in a uniform magnetic field applied parallel to these faces (Fig. 3.3). This is a configuration we shall

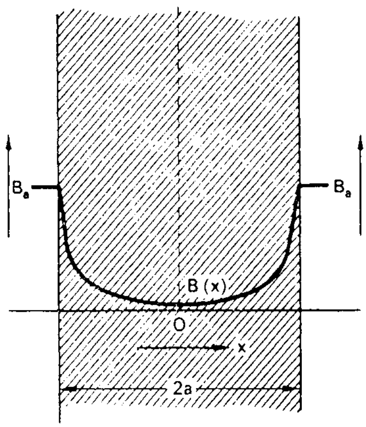


FIG. 3.3. Superconducting slab in parallel applied magnetic field.

need to make use of in a later chapter (Chapter 8). Let the thickness of the slab be $2a$ and let the $x$-coordinate be normal to the faces, with the origin at the mid-plane. Because the applied field is uniform,

$$\frac{\partial B}{\partial y} = 0 = \frac{\partial B}{\partial z}$$

and we can write (3.19) as

$$\frac{\partial^2 B(x)}{\partial x^2} = \frac{1}{\lambda_L^2} B(x).$$

The solution of this equation is

$$B(x) = B_1\, e^{+x/\lambda_L} + B_2\, e^{-x/\lambda_L}. \tag{3.20}$$

The values of the constants $B_1$ and $B_2$ are obtained by noting that, from symmetry, the variation of $B$ in both the $+x$ and $-x$ directions must be the same. Therefore $B_1 = B_2$. Further, when $x = \pm a$, the flux density equals that of the applied field, $B(\pm a) = B_a$. So (3.20) becomes

$$B(x) = \frac{B_a}{\cosh(a/\lambda_L)} \cosh(x/\lambda_L). \tag{3.21}$$

The flux density therefore varies as $\cosh(x/\lambda_L)$. The width of most specimens will be much greater than the penetration depth $\lambda_L$ and then the flux density will be very small except close to the faces where $|x| \to |a|$.