CHAPTER 8

THE SUPERCONDUCTING PROPERTIES OF SMALL SPECIMENS

IT WAS pointed out in Chapter 2 that the penetration depth λ is very small, and that most superconducting specimens have dimensions which are very much greater than λ. Sometimes, however, a situation arises, as for example with thin films or fine wires, in which one or more of the dimensions of the specimen is comparable with λ. We shall see in this chapter that the superconducting properties of such specimens are in some ways significantly different from those of large specimens.

8.1. The Effect of Penetration on the Critical Magnetic Field

We saw in Chapter 4 that if a specimen of superconducting metal is in a uniform applied magnetic field H_a, the superconductor is driven into the normal state when H_a is increased above a critical value H_c. From a thermodynamic point of view, this is because the Gibbs free energy of a superconducting specimen is changed by an amount $-\int_0^{H_a} \mu_0 M dH_a$ in an applied field H_a, where M is the induced magnetic moment.

In the superconducting state M is negative, so the free energy is increased, and if this increase is sufficient to make the free energy in the superconducting state exceed that in the normal state, the specimen becomes normal. The magnetic moment M is equal to $\int I dV$, where V is the volume of the specimen and I is the intensity of magnetization given by

$$B = \mu_0 H + \mu_0 I. \tag{8.1}$$

In Chapter 4 it is assumed that $B = 0$ everywhere inside the superconductor, so that $I = -H$ and $M = -HV$; in other words, it was assumed that the magnetic moment per unit volume is independent of the shape and size of the specimen. It follows from this that the critical magnetic field is given by

$$\tfrac{1}{2}\mu_0 H_c^2 = g_n - g_s \quad \text{[see (4.3)]}, \tag{8.2}$$

where g_n and g_s are the free energies per unit volume of the normal and superconducting phases in zero magnetic field. H_c should therefore be independent of the size of the specimen.

This argument needs modifying when penetration of the field is taken into account. We saw in Chapter 2 that B does not fall abruptly to zero at the surface of the specimen, but decreases with distance (x) into the interior of the specimen approximately as $e^{-x/\lambda}$, where λ, the "penetration depth", is about 5×10^{-6} cm in most superconductors. It follows that just inside the surface B is not zero, and that in this region the value of I given by (8.1) is no longer equal to $-H$. Instead, the magnitude of I increases from zero at the surface to the value H in the interior of the specimen, and as a result the magnitude of the magnetic moment M is less than it would be if λ were zero. Hence, for a given value of H_a, the magnetic contribution to the free energy, $-\int_0^{H_a} \mu_0 M dH_a$ or $-\frac{1}{2}\mu_0 M H_a$, is less than it would be if penetration did not occur, and the applied magnetic field has to be increased beyond the value given by (8.2) before the transition to the normal state can take place. In other words, the critical magnetic field is increased as a result of the penetration of the magnetic flux. The magnitude of the increase depends on the reduction in the magnetic moment M, which in turn depends on the dimensions of the specimen relative to the penetration depth λ. The effect is only noticeable if the volume contained within a distance λ of the surface is comparable with the total volume of the specimen.

8.2. The Critical Field of a Parallel-sided Plate

The effect of penetration on the critical magnetic field of a specimen is most easily illustrated by reference to the case of a parallel-sided plate with a magnetic field H_a applied parallel to the surfaces of the plate. The length of the plate in the direction of the field and its width are both supposed much larger than its thickness (Fig. 8.1). This particular geometry is chosen partly for its practical importance and partly because the flux distribution within the plate can be easily calculated with the aid of the London equations.

If the direction normal to the surfaces of the plate is chosen as the

x-direction, the variation of the flux density with x as given by the London equations (§ 3.2.1) is

$$B(x) = \frac{\cosh(x/\lambda_L)}{\cosh(a/\lambda_L)} \mu_0 H_a, \tag{8.3}$$

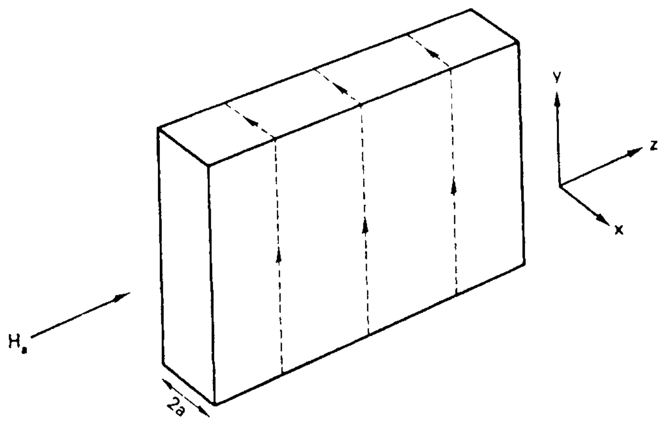

FIG. 8.1. Superconducting plate of thickness $2a$ with a magnetic field parallel to its surfaces. Its length and height are assumed much greater than $2a$. The broken lines show the direction of the screening currents.

where x is measured from the mid-plane of the film and the thickness of the film in $2a$. We saw in § 3.2 that, for large specimens, the variation of B with x predicted by the London equations is such that λ_L satisfies the general definition of λ given by (2.2). We shall therefore neglect the distinction between λ_L and λ in what follows, and write the solutions of the London equations in terms of λ. The dependence of B on x is illustrated graphically in Fig. 8.2, for the case where the thickness of the film is much greater than λ. Since the value of the magnetic field strength is H_a throughout the film,† the intensity of magnetization is equal to $(B/\mu_0) - H_a$, and the magnetic moment M (given by $\int I dV$) and the magnetic contribution to the free energy (given by $-\int_0^{H_a} \mu_0 M dH_a$ or $-\frac{1}{2}\mu_0 M H_a$) are both proportional to the cross-hatched area. For the case of Fig. 8.2 this is only marginally different from the value it would have if λ were zero. However, if $a \sim \lambda$, the situation is as shown in Fig. 8.3, and it is clear

† See Appendix A.

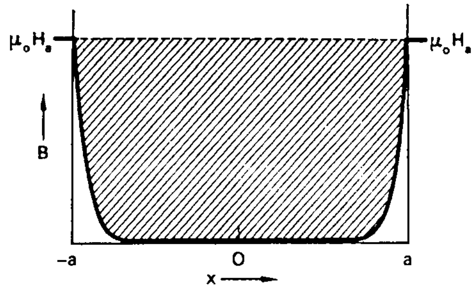

FIG. 8.2. Variation of B with distance normal to the surface for a plate of thickness $2a$ ($2a \gg \lambda$). The cross-hatched area is proportional to the magnetic moment and to the magnetic free energy.

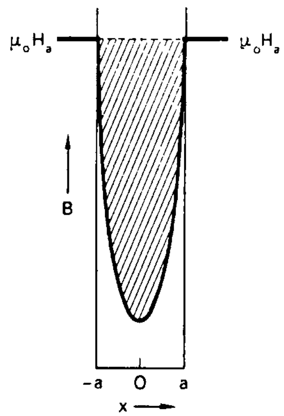

FIG. 8.3. Variation of B with distance normal to the surface for a plate of thickness $2a$ ($a \sim \lambda$).

that the reduction in the cross-hatched area relative to the value it would have if λ were zero is now very considerable. At every point the magnetization is $I(x) = \dfrac{B(x)}{\mu_0} - H_a$, so the value of M, the magnetic moment per unit area of the plate, is given by

$$M = \int_{-a}^{a} \left[\frac{B(x)}{\mu_0} - H_a\right] dx$$

$$= 2H_a \int_{0}^{a} \left[\frac{\cosh(x/\lambda)}{\cosh(a/\lambda)} - 1\right] dx$$

$$= -2aH_a \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right] \tag{8.3a}$$

It is convenient to write $M = -2akH_a$ so that

$$k = \left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right] \tag{8.4}$$

Due to the penetration of the flux, the effective susceptibility of the plate is $-k$ instead of -1. Note that k is positive, and that $k = 1$ if $\lambda = 0$. The magnetic contribution to the Gibbs free energy per unit area of the plate is

$$-\tfrac{1}{2}\mu_0 M H_a = \mu_0 a k H_a^2$$

so that the critical magnetic field is given by

$$\mu_0 a k H_c^2 = G_n - G_s = 2a(g_n - g_s)$$

where G_n and G_s refer to unit area of the plate, and g_n and g_s to unit volume as before. If we use H_c' to denote the critical field of a plate of thickness $2a$ for a penetration depth λ and H_c the critical field if λ were zero, then H_c' is given by

$$\tfrac{1}{2}\mu_0 k H_c'^2 = g_n - g_s \tag{8.5}$$

compared with $\frac{1}{2}\mu_0 H_c^2 = g_n - g_s$ given by (8.2).

Hence

$$H_c' = k^{-\frac{1}{2}} H_c = H_c\left[1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda}\right]^{-\frac{1}{2}} \tag{8.6}$$

i.e. the critical field is increased due to the penetration of the flux. We have referred to H_c as the critical magnetic field for the case of zero penetration, but since (8.6) involves only the ratio a/λ, we could equally well regard H_c as the critical magnetic field for an infinitely large specimen. For this reason H_c is usually referred to as "the bulk critical field".

Equation (8.6) can be simplified for the cases of $a \gg \lambda$ or $a \ll \lambda$. If $a \gg \lambda$,

$$1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda} \simeq 1 - \frac{\lambda}{a}$$

and
$$H_c' \simeq H_c\left(1 - \frac{\lambda}{a}\right)^{-\frac{1}{2}} \simeq H_c\left(1 + \frac{\lambda}{2a}\right), \tag{8.7}$$

which has the following simple physical interpretation. If $a \gg \lambda$ (8.4) takes the form $k \simeq 1 - (\lambda/a)$, so that $M \simeq -2(a - \lambda)H_a$. This is as if the intensity of magnetization I had remained equal to $-H_a$ throughout the plate, but the thickness of the plate had shrunk to $2(a - \lambda)$, i.e. as if each surface of the plate had receded inwards a distance λ. This is in accordance with the phenomenological definition of λ given by (2.2). For the other extreme case of a $a \ll \lambda$,

$$1 - \frac{\lambda}{a}\tanh\frac{a}{\lambda} \simeq \frac{a^2}{3\lambda^2}$$

so that

$$H_c' \simeq \sqrt{(3)}\frac{\lambda}{a}H_c. \tag{8.8}$$

It is important to get some idea of the order of magnitude of the increase in the critical field due to penetration of the flux. We have seen (§ 2.4.1) that the penetration depth obeys the relationship

$$\lambda = \lambda_0\left\{1 - \left(\frac{T}{T_c}\right)^4\right\}^{-\frac{1}{2}},$$

where λ_0 is about 500 Å (i.e. 500×10^{-10} m) for most pure metals. Equation (8.7) now shows that, for temperatures not too near the transition temperature, the increase in H_c will only be significant (say 10% or more) if the thickness of the plate is about 5000 Å or less, i.e. if we are dealing with a thin film. For the case of a *very* thin film, of thickness 100 Å or so (8.8) shows that H_c' may exceed H_c by an order of magnitude, especially if the temperature is close to the critical temperature.

8.3. More Complicated Geometries

Although the basic physics is the same as for the rectangular plate, the effect of penetration on the magnetic properties of a cylinder or sphere is much more difficult to calculate, the case of the cylinder involving Bessel functions. When the dimensions of the specimen are much greater than λ, we may, however, make use of the simple argument discussed immediately after (8.7), according to which the effect of penetration is as if the magnetization remained equal to $-H_a$ throughout the specimen but the surface of the cylinder were to recede inwards by a distance λ. Using this argument it may easily be shown that the critical magnetic field of a long cylinder of a radius a with the field applied parallel to its axis is given by

$$H_c' = H_c\left(1 + \frac{\lambda}{a}\right).$$

For the case of $a \ll \lambda$, it can be shown† that $H_c' = \sqrt{(8)}(\lambda/a)H_c$.

† For example, D. Shoenberg, *Superconductivity*, C.U.P., 1962, p. 234.

8.4. Limitations of the London Theory

It is clear from (8.6) to (8.8) that, since λ is of the order of 10^{-5} cm, the effect of penetration will be too small to produce any appreciable effect on the critical magnetic field unless one of the dimensions of the sample at right angles to the field is about 5000 Å or less, as occurs in the case of a thin film.

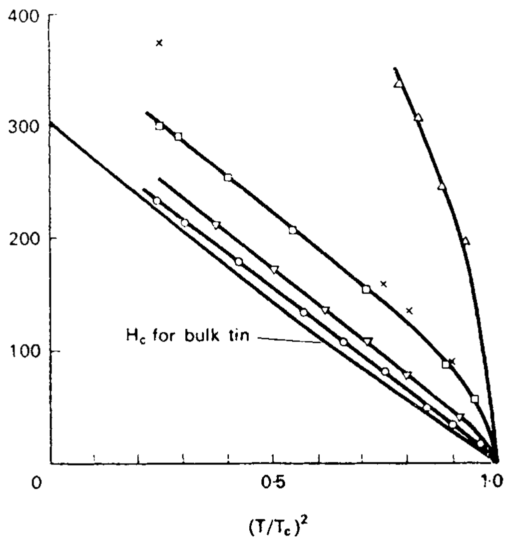

FIG. 8.4. Variation of parallel critical field with reduced temperature for tin films of various thicknesses (△, 1000 Å; □, 2000 Å; ▽, 5000 Å; ○, 10000 Å). Also shown are points calculated from an effective penetration depth as prescribed by Ittner (×, 2000 Å). (After Rhoderick.)

The dependence of the critical magnetic field of thin films on their thickness is quite pronounced, as can be seen from Fig. 8.4. The critical fields of these films were determined by observing the restoration of resistance by a magnetic field parallel to the surface of the film. In the case of a film 1000 Å thick, the critical magnetic field close to the critical temperature is over an order of magnitude greater than that for the bulk metal. The increase relative to the bulk field is most marked near the critical temperature because, as was pointed out in Chapter 2, the penetration depth is found experimentally to vary with temperature approximately as $[1 - (T/T_c)^4]^{-\frac{1}{2}}$ and becomes infinite as T approaches T_c.

A comparison of the experimental results with the predictions of the London theory as expressed by (8.6) shows good qualitative agreement, but quantitative comparison is not easy. One could, of course, assume the truth of (8.6) and use this equation to calculate an effective value of λ from the measured values of H_c, H_c' and a. The important point, however, is whether the value of λ obtained in this way agrees with the values obtained from the magnetization of bulk specimens and shows the same temperature variation. Only in this way is it possible to check the validity of the London theory. It should be remembered that the measurements of λ obtained from the magnetization of bulk specimens, discussed in Chapter 2, are independent of any particular penetration law and depend only on the definition of λ given by

$$\lambda = \frac{1}{B(0)} \int_0^\infty B(x)\,dx,$$

where x is measured from the surface. As we pointed out on page 97, (8.7) is consistent with this definition.

Equation (8.6), however, is derived from (8.3), which is the penetration law predicted by the London theory. Whether or not (8.6) correctly predicts the critical fields for films of *any* thickness, using the value of λ determined from bulk measurements is therefore equivalent to verifying the truth of the penetration law expressed by (8.3).

Attempts to correlate theory and experiment in this way have been only partially successful. Figure 8.4 shows points calculated for a film of thickness 2000 Å using for λ_0 the value 520 Å obtained by magnetization measurements of bulk specimens and assuming a variation of λ with temperature given by

$$\lambda(T) = \lambda_0\left[1 - \left(\frac{T}{T_c}\right)^4\right]^{-\frac{1}{2}}.$$

It is seen that the theoretical points lie consistently above the experimental ones.

To account for the discrepancy we must recall how the London theory originated. It is essentially a *phenomenological* theory, that is to say, it was introduced because it gives a fairly good description of the Meissner effect. It is important that we should not accord the London theory the status of, say, Maxwell's equations, which are believed to be exact expressions of inviolable physical laws. One obvious limitation is that it is essentially a classic:l theory which treats the electrons as classical particles, although we should expect quantum effects to be significant. Two important assumptions made in the London theory are that the penetration depth λ_L is independent of the strength of the applied magnetic field and also of the dimensions of the specimen. It should not really surprise us if the first of these assumptions turns out not to be strictly true. As can be seen from (3.13), it is equivalent to assuming that the effective number of superconducting electrons is independent of the applied field. However, the application of a magnetic field is known to modify the behaviour of electrons in a profound way, so we might expect it to change the degree of order, i.e. the effective number of superelectrons. It has been shown experimentally by Pippard that the penetration depth does in fact increase with applied magnetic field, although in bulk superconductors the effect is not large except near the transition temperature. The London theory is therefore essentially a *weak field* theory. The effect of such a dependence of λ on the magnetic field is that the magnetic moment M of a film is no longer linearly proportional to H_a, as (8.3a) would predict if λ were constant, but increases less rapidly. The graph of M against H_a is therefore non-linear, as shown in Fig. 8.5 and

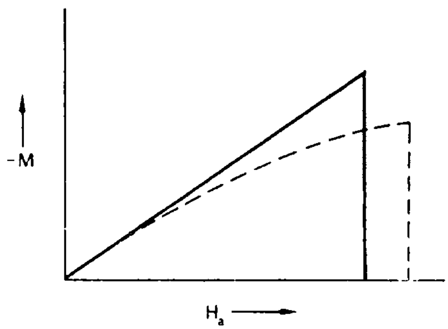

FIG. 8.5. Magnetization curve of a thin superconductor.———— assuming λ independent of H_a. — — — assuming λ increases with H_a.

H_a has to be increased further before the free energy of the superconducting state becomes equal to that of the normal state; consequently H_c is increased.

The second assumption mentioned above, that the penetration depth is independent of the dimensions of the specimen, seems less open to objection, and it is difficult to explain in simple terms why it should not be correct. Suffice it to say that, as was seen in Chapter 6, superconducting electrons do not behave completely independently of each other, but exhibit "long range order" extending for a distance known as the coherence length ξ, which in a bulk superconductor is about 10^{-4} cm. If the dimensions of the specimen are less than this bulk coherence range, the value of ξ is reduced, and various properties of the superconductor (among them the penetration depth) are modified.

Various attempts have been made, by Ittner, Tinkham and others, to "patch up" the London theory in so far as its predictions regarding the critical fields of thin films are concerned by incorporating in it the field-dependence, temperature-dependence and size-dependence of the penetration depth which result from more recent theories, such as the microscopic theory of Bardeen, Cooper and Schrieffer (see Chapter 9). The success of these attempts is only moderate, and since they incorporate into the London theory new features which are in fact foreign to it, the procedure is not entirely satisfactory. It seems far preferable to look for a theory, such as the Ginzburg–Landau theory, in which the necessary non-linearity of M with respect to H and size-dependence of λ are inherent.

8.5. The Ginzburg–Landau Theory

The Ginzburg–Landau theory† is an alternative to the London theory. To a certain extent it is a phenomenological theory also, in the sense that it makes certain *ad hoc* assumptions whose justification is that they correctly describe the phase transition in zero field, but unlike the London theory, which is purely classical, it uses quantum mechanics to predict the effect of a magnetic field. The Ginzburg–Landau theory involves a good deal of algebra and a complete description of it would take us far beyond the scope of this book. However, we will try to give a brief description of what it is about, and of some of its more important predictions.‡

The first assumption of the Ginzburg–Landau theory is that the behaviour of the superconducting electrons may be described by an "effective wave function" Ψ, which has the significance that $|\Psi|^2$ is equal to the density of superconducting electrons. It is then assumed that the free energy of the superconducting state differs from that of the normal state by an amount which can be written as a power series in $|\Psi|^2$. Near the critical temperature it is sufficient to retain only the first two terms in this expansion. Ginzburg and Landau then point out that if, for any reason, the wave function Ψ is not constant in space but has a gradient, this gives rise to kinetic energy whose origin is the same as that of the

† V. L. Ginzburg and L. D. Landau, *J.E.T.P.* **20**, 1064 (1950).

‡ An introduction to the Ginzburg–Landau theory is given by A. D. C. Grassie, *The Superconducting State*, Sussex University Press, 1975.

kinetic energy term $(\hbar^2/2m)\nabla^2\Psi$ which appears in Schrödinger's equation for a particle of mass m. To take account of this, an additional term, proportional to the square of the gradient of Ψ, is added to the expression for the free energy of the superconducting phase. The effect of a magnetic field is introduced by resorting to a theorem in classical mechanics which states that the effect of the Lorentz force ($q\mathbf{v} \times \mathbf{B}$) on the motion of a charged particle in a magnetic field $\mathbf{B}$ may be completely accounted for† by replacing the momentum $\mathbf{p}$, wherever it occurs in the expression for the kinetic energy, by the more complicated expression $\mathbf{p} - q\mathbf{A}$. Here $\mathbf{A}$ is the magnetic vector potential defined by $\mathbf{B} = \text{curl}\,\mathbf{A}$. To make the transition to quantum mechanics, $\mathbf{p}$ is replaced by the operator $-i\hbar$ grad. The total magnetic contribution to the free energy of the superconducting state is therefore given by

$$\frac{\hbar^2}{2m}\int \Psi^* \left[i \,\text{grad} + \frac{eA}{\hbar} \right]^2 \Psi \, dV,$$

the integral being taken over the whole volume V of the specimen.

The central problem of the Ginzburg–Landau approach is now to find functions $\Psi(x, y, z)$ and $\mathbf{A}(x, y, z)$ which make the total free energy of the specimen a minimum subject to appropriate boundary conditions. For weak magnetic fields the problem is easily soluble and reduces to the same form as the London equations. In a strong magnetic field the equations are only soluble by numerical means. In the case of an infinitely thick plate with the applied magnetic field parallel to the surface, the solution predicts that $|\Psi|^2$ is constant in the interior of the plate but falls off towards the surface by an amount which increases with the applied

† This can be seen by the following simple argument. Suppose that a particle of charge q is moving in a field-free region with velocity $\mathbf{v}_1$ and that a magnetic field is applied at time $t = 0$. The field can only build up at a finite rate, and while it is changing there will be an induced electric field which satisfies Maxwell's relation curl $\mathbf{E} = -\dot{\mathbf{B}}$. If $\mathbf{A}$ is the vector potential, curl $\mathbf{E} = -(d/dt)$ (curl $\mathbf{A}$), and integration with respect to the space coordinates gives $\mathbf{E} = -(d\mathbf{A}/dt)$ apart from a constant of integration which does not concern us. Hence the momentum at time t is given by

$$m\mathbf{v}_2 = m\mathbf{v}_1 + \int_0^t q\mathbf{E}\,dt = m\mathbf{v}_1 - q\int_0^t \frac{d\mathbf{A}}{dt}\,dt$$

$$= m\mathbf{v}_1 - q\mathbf{A},$$

so that $m\mathbf{v}_2 + q\mathbf{A} = m\mathbf{v}_1$. Hence the vector $\mathbf{p} = m\mathbf{v} + q\mathbf{A}$ is conserved during the application of the field and must be regarded as the effective momentum when a magnetic field is present. However, the kinetic energy ε depends only on $m\mathbf{v}$, and if $\varepsilon = f(m\mathbf{v})$ before the field is applied, we must write $\varepsilon = f(\mathbf{p} - q\mathbf{A})$ in the presence of the field.

magnetic field. Since the penetration depth depends on the number of superconducting electrons at the surface, as in the London theory, and hence on $|\Psi|^2$, we at once get a field-dependent penetration depth. In the case of a thin film, because of the boundary conditions, the variation of Ψ with x depends on the thickness of the film, and since λ again depends on $|\Psi|^2$ the penetration depth is a function of the thickness of the film. Thus the two missing elements in the London theory are automatically provided by Ginzburg and Landau. The critical magnetic field can be calculated by the usual method of equating the free energy of the film in the superconducting state with that in the normal state. The general expression for H_c' is complicated, but simplifies in two special cases:

(i) $a \gg \lambda$

In this case

$$H_c' = H_c\left(1 + \frac{\alpha\lambda}{2a}\right),$$

where $2a$ is the thickness of the film, λ is the penetration depth in a *weak* magnetic field, and α is a coefficient very close to unity. This is essentially the same as the London result.

(ii) $a < \lambda$

In this case Ψ is approximately constant throughout the film, and $H_c' = \left[\frac{\sqrt{(6)}\lambda}{a}\right] H_c$. This differs from the London result (8.8) by a factor of $\sqrt{2}$. A surprising consequence of the Ginzburg–Landau theory is that in this case Ψ falls *gradually* to zero as H approaches H_c' so that the transition is a second-order one. (It should be remembered that in bulk super-conductors the transition is second order in the absence of a magnetic field but first order in the presence of a field.) The transition is first- or second-order according as $a \gtrless \frac{\sqrt{(5)}\lambda}{2}$. This prediction of the theory has been amply confirmed by Douglass using the tunnelling technique (see Chapter 10).

The Ginzburg–Landau theory predicts magnitudes of the critical magnetic fields which are not very different from those of the London theory in the limiting cases of very thick and very thin films. In the intermediate case, where solutions have to be obtained numerically, the

Ginzburg–Landau theory does not give a significantly better fit with experiment than the London theory if values of λ obtained from measurements on large specimens are used. The great success of the Ginzburg–Landau theory is that it correctly predicts the change from first- to second-order transitions with decreasing thickness, which the London theory does not.

Various refinements have been incorporated into the Ginzburg–Landau theory to improve its quantitative predictions. The general situation with respect to the theoretical interpretation of critical fields of thin films is still, however, not really satisfactory.

8.6. Edge Effects

One important consequence of the dependence of the critical magnetic field of a film on its thickness is that the sharpness of the magnetic transition for a thin film depends very much on the nature of its edges. As a rule films are prepared in the form of a strip by evaporating the superconducting metal on to an insulating base (or "substrate") through a mask which behaves like a stencil. Because the mask is never in precise contact with the substrate, and also because the metal atoms are able to wander about on the substrate before they finally come to rest, the edges of the film are never completely sharp but always tend to be tapered, as shown in Fig. 8.6. The tapered edges, being thinner than the rest of the

FIG. 8.6. Typical cross-section of evaporated film, showing tapered edges. The right-hand edge has been trimmed to produce a well defined rectangular geometry.

film, have a higher critical magnetic field and if the film is tested for superconductivity by passing a current through it and seeing whether any voltage difference appears across it ends, the edges will remain superconducting and give rise to zero resistance even when the rest of the film is normal. This has two consequences. First, the magnetic field strength at which a voltage difference appears may be considerably greater than the true critical magnetic field of the film; and second, because the edges are unlikely to be perfectly uniform along the length of the strip, the transition from zero to full normal resistance may extend over a considerable range of values of the magnetic field. To obtain sharp resistance transitions at the true value of magnetic field, the edges of a film are usually trimmed as shown on the right-hand side of Fig. 8.6. The effect of trimming a film is shown in Fig. 8.7.

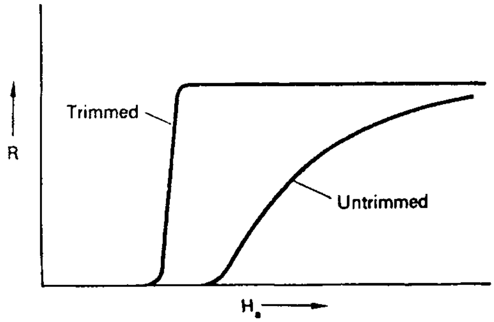

FIG. 8.7. Effect of trimming the edges on resistance transition of thin film.

8.7. Transitions in Perpendicular Magnetic Fields

So far we have limited the discussion to the case where the magnetic field is applied parallel to the surface of the film. Since films are always thin compared with their other two dimensions, a film is essentially a "long, thin specimen" in the sense used in Chapter 6, and demagnetizing effects are quite negligible. However, if a magnetic field is applied perpendicular to the surface of the film, demagnetizing effects become very important. If the film takes the form of a relatively narrow strip whose width w is much less than its length but much greater than its thickness d, then we may approximate the strip by an elliptical cylinder whose cross-section has axes w and d. The demagnetizing coefficient for such a geometry is given by $n \simeq 1 - (d/w)$. According to Chapter 6, we expect the film to enter the intermediate state if exposed to a perpendicular field of magnitude $H_c(1 - n) \simeq H_c(d/w)$. Typical values encountered in practice are $w \sim 10^{-1}$ cm and $d \simeq 10^{-5}$ cm, so that $(d/w) \sim 10^{-4}$. We should therefore expect the film to be driven normal by an extremely small perpendicular field—in fact, the earth's field should be more than sufficient. In practice this is not so, the reason being that the theory of the intermediate state outlined in Chapter 6 is no longer valid when the thickness of the film is comparable with the penetration depth, mainly because the concept of surface energy discussed in Chapter 6 now requires considerable modification. A complete discussion of the subject would take us far beyond the scope of this book. According to Tinkham, the intermediate state in thin films resembles the mixed state in type-II superconductors (see Chapter 12) and can be described in terms of current vortices.

Experimentally it is found that the transition to the normal state of thin films in perpendicular magnetic fields does occur at a significantly lower field strength than in the case of parallel fields, but not as low as the previous paragraph would indicate. For this reason, in carrying out studies of transitions in parallel magnetic fields, great care has to be taken to ensure that the film is accurately parallel to the applied field so that there is no component of the field perpendicular to the surface. In the case of resistive transitions this is easily accomplished by setting the magnetic field so that about half of the normal resistance is restored, and then rotating the magnet (or the specimen) until the resistance is a minimum, which means that the critical magnetic field has its maximum value.

8.8. Critical Currents of Thin Specimens

For large specimens we have already seen that the critical current can be calculated from the critical magnetic field by making use of Silsbee's criterion. This states that in the absence of an applied magnetic field the critical current is that current which produces at the surface of the specimen a magnetic field equal to the critical field H_c. Unfortunately this simple rule does not hold for specimens which have one or more dimensions comparable with λ, the most obvious reason being that even if some sort of modified Silsbee's rule were to hold we should not know whether to insert the bulk critical field H_c or the actual critical field for the small specimen H_c'. The problem is a complicated one and depends, among other things, on the current distribution in the film. We therefore need some way of calculating the current distribution, and for simplicity we shall follow the London theory, notwithstanding the limitations of this theory which we have already mentioned. An entirely adequate theory of critical currents in thin specimens has yet to be formulated, but the predictions of the London theory are sufficiently correct to give some qualitative indication of the effects that are to be expected.

A further difficulty is that we cannot adopt the simple thermodynamic approach of equating the free energies of the superconducting and normal phases because, in the presence of a transport current supplied by an external source, the transition is irreversible due to the fact that energy is continuously dissipated in the normal state. Fortunately a way round this difficulty was found by H. London, who pointed out that the existence of a critical magnetic field H_c for a bulk superconductor can be regarded as resulting from the existence of a critical current density J_c, as we saw in Chapter 4. To illustrate this principle, consider the special case of a superconducting plate of thickness $2a$ with a magnetic field applied parallel to its surfaces (see Fig. 8.1). Suppose that the thickness of the plate lies along the x-direction, that the applied field is in the z-direction, and that the dimension of the plate in the y-direction is much greater than $2a$. Then to exclude flux from the interior of the plate the shielding currents must flow parallel to the y-axis, as shown in the figure. As we have already seen, the solution of the London equations gives

$$B(x) = \frac{\cosh(x/\lambda)}{\cosh(a/\lambda)} \mu_0 H_a,$$

where x is measured from the midplane of the plate. The current density J_y can be found from Maxwell's equation† curl $\mathbf{B} = \mu_0 \mathbf{J}$, which for the geometry shown in Fig. 8.1 simplifies to

$$J_y = -\frac{1}{\mu_0}\frac{\partial B}{\partial x} = -\frac{H_a}{\lambda}\frac{\sinh(x/\lambda)}{\cosh(a/\lambda)};$$

B and J are plotted as functions of x in Fig. 8.8.

Notice that although B has the same direction throughout the plate, the current density has opposite directions in the two halves. The current density is greatest at the surfaces of the plate, where it has the magnitude $(H_a/\lambda)\tanh(a/\lambda)$. According to London's postulate, the plate is driven into the normal state when the current density at the surface reaches a critical value J_c. However, we know that for thick specimens ($a \gg \lambda$) the plate is driven normal when $H_a = H_c$, and in this case $\tanh(a/\lambda) \to 1$, so the relationship between the critical current density J_c and the critical magnetic field is

$$J_c = \frac{H_c}{\lambda}. \tag{8.9}$$

Now suppose that instead of applying an external field H_a to the plate we pass in the y-direction a current which has the magnitude $\mathscr{I}$ per unit width of the plate in the z-direction. Associated with this transport current there will be a flux density B both inside and outside the plate. Clearly, from the symmetry of the situation, the current density

† Note that we have written this equation in terms of **B** rather than the usual **H**. This is because **J** is not the "free" current density (see Appendix A) but the induced magnetization current.

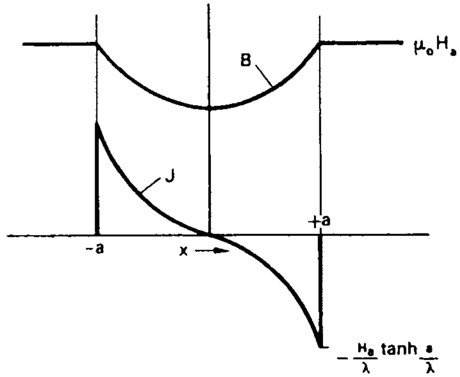

FIG. 8.8. Variation of B and $\mathcal{J}$ with x for plate of thickness $2a$ in uniform applied field H_a.

associated with the transport current must have the same direction in each half of the plate, but B will have opposite directions. We therefore need a solution of the London equations which has the form

$$B(x) = -B(-x) \qquad \text{and} \qquad \mathcal{J}(x) = \mathcal{J}(-x).$$

Such a solution is

$$\mathcal{J}(x) = \mathcal{J}(a)\frac{\cosh(x/\lambda)}{\cosh(a/\lambda)}$$

and

$$B = -\mu_0\lambda^2 \operatorname{curl}\mathbf{J} = -\mu_0\lambda^2 \frac{\partial \mathcal{J}_y}{\partial x}$$

$$= -\mu_0\lambda\mathcal{J}(a)\frac{\sinh(x/\lambda)}{\cosh(a/\lambda)}; \tag{8.10}$$

B and $\mathcal{J}$ are plotted as functions of x in Fig. 8.9. $\mathcal{J}(a)$ is related to the current per unit width $\mathscr{I}$ by

$$\mathscr{I} = \int_{-a}^{a} \mathcal{J}(x)\,dx = 2\lambda\mathcal{J}(a)\tanh(a/\lambda)$$

The current density is again a maximum at the surface and if we assume that the plate will begin to go normal when $\mathcal{J}(a)$ has the value $\mathcal{J}_c = H_c/\lambda$, as before, then the critical current per unit width of the plate $\mathscr{I}'_c$, is given by

$$\mathscr{I}'_c = 2\lambda\mathcal{J}_c \tanh(a/\lambda)$$

$$= 2H_c \tanh(a/\lambda). \tag{8.11}$$

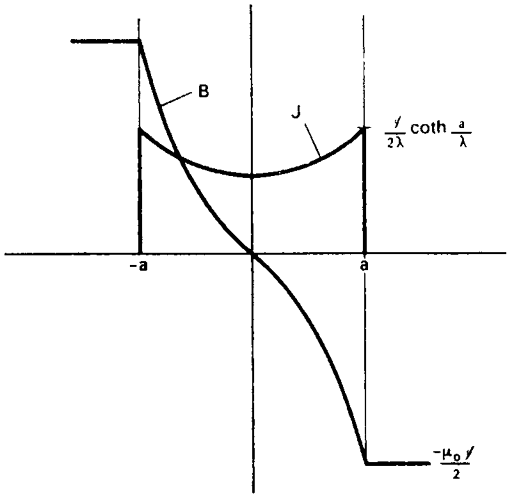

FIG. 8.9. Variation of B and J with x for plate of thickness $2a$ carrying current $\mathcal{I}$ per unit width.

If $a \gg \lambda$, tanh $(a/\lambda) \to 1$, and the critical current per unit width becomes $\mathcal{I}_c = 2H_c$. In this case, from (8.10) the magnitude of the flux density at the surface is $\mu_0 H_c$, i.e. the magnetic field strength is H_c, which is in accordance with Silsbee's rule. In other words, ***for plates which are much thicker than λ, regarding the destruction of superconductivity as determined by a critical current density J_c is in all respects equivalent to thinking of it as associated with a critical magnetic field H_c.*** Note that the critical current per unit width of the plate is independent of its thickness, which is to be expected because all the current is concentrated within a penetration depth of the two surfaces.

In the case when a is comparable with λ, the critical current per unit width $\mathcal{I}_c'$ is, according to (8.11), given by

$$\mathcal{I}_c' = 2H_c \tanh (a/\lambda) = \mathcal{I}_c \tanh (a/\lambda), \qquad (8.12)$$

so that the critical current is ***reduced*** by the factor tanh (a/λ), in contrast with the fact that the critical magnetic field is *increased.* If $a \ll \lambda$, tanh $(a/\lambda) \simeq (a/\lambda)$ and $\mathcal{I}_c'$ is proportional to the thickness of the film, as might be expected from the fact that in this case the current is almost uniformly distributed throughout the cross-section of the plate.

It is interesting to calculate the magnetic field at the surface of the film when superconductivity is destroyed by a current. From (8.10) the field strength is given by $|H(a)| = \lambda \mathcal{J}(a) \tanh (a/\lambda)$, and if $\mathcal{J}(a)$ is given by $\mathcal{J}(a) = \mathcal{J}_c = H_c/\lambda$, then

$$H(a) = H_c \tanh (a/\lambda).$$

Since $\tanh (a/\lambda) < 1$, we see that the magnetic field strength at the surface when superconductivity is quenched by a current, which we denote by H_i, is not only less than the magnitude of the externally applied field necessary to drive the film normal in the absence of a current H_c', but is even less than the bulk critical field H_c, so that Silsbee's rule is not obeyed in any form. If $a \ll \lambda$, $H_i \simeq H_c a/\lambda$, and we have already seen from (8.8) that for this case $H_c' = \sqrt{(3)} H_c \lambda/a$, so that $H_i H_c' = \sqrt{(3)} H_c^2$. The Ginzburg–Landau theory gives essentially the same relationship between H_i, H_c' and H_c, but with a factor 4/3 instead of $\sqrt{3}$.

8.9. Measurements of Critical Currents

Reliable measurements of the critical currents of thin films are hard to make for two reasons—first, because the attainment of a suitable geometry is difficult and, second, because it is not easy to prevent effects due to Joule heating confusing the issue.

The discussion of § 8.2 assumed that the film was infinitely wide in the z-direction, so that the current density was independent of z. For a film of finite width this is no longer true. If the thickness of the film were large compared with λ, the distribution of current over its surface would be exactly the same as occurs in an analogous situation in electrostatics, namely the distribution of charge on a conductor of the same geometry (see § 2.3). It is well known that charge tends to be concentrated where the radius of curvature is greatest, so it follows that in the case of a superconducting plate of rectangular geometry the current density will be greatest at the edges. If the thickness of the plate is comparable with λ, the analogy with electrostatics is no longer complete, but the current density is still greatest at the edges of the plate (or film). The critical current will therefore be very sensitive to the geometrical perfection of these edges and, as we saw in § 8.6, it is difficult to obtain perfectly sharp edges unless the films are trimmed.

A further complication which often arises when one tries to measure the critical currents of thin films is that if there happens to be a "weak spot", such a small region which is thinner or narrower than the rest of the film, or has slightly different metallurgical properties, then this weak spot will be driven normal before the true critical current of the film is reached, and the normal region may then spread by thermal propagation as discussed in § 7.2. This may be avoided by using current in the form of short pulses (say $1/10\,\mu$sec or even less) which do not allow time for any appreciable thermal propagation to take place, or by taking great care to see that the film is evaporated on to a substrate with a high thermal conductivity in good thermal contact with the helium bath.

The most careful experiments on critical currents of thin films to date seem to be those of Glover and Coffey, who calculated the current distribution from the London equations, and assumed the film to be driven normal when the current density at the edges of the strip reached J_c. They found that their results were consistent with the existence of a critical current density and that the value of J_c for tin at 0°K was about 2×10^7 A cm^{-2}.