APPENDIX A

THE SIGNIFICANCE OF THE MAGNETIC FLUX DENSITY *B* AND THE MAGNETIC FIELD STRENGTH *H*

IT IS important for a proper appreciation of the magnetic properties of superconductors that the reader should have a sound understanding of the significance of B and H. This is especially so in view of the fact that the MKS system involves a rather different approach to magnetism than the mixed e.s.u.–e.m.u. system.

A.1. Definition of B

The modern approach is to abandon the concept of free magnetic poles and instead to discuss magnetism entirely in terms of the interaction between currents. From this point of view, the magnetic flux density vector **B** is the basic magnetic quantity, in the sense that in electrostatics the fundamental quantity is the electric field vector **E**. If a conductor carrying a current i is placed in a magnetic field a force will act on the conductor due to the presence of the field. The magnetic flux density **B** of the field is *defined* by the relationship:

$$d\mathbf{F} = i d\mathbf{l} \times \mathbf{B}, \tag{A.1}$$

where $d\mathbf{F}$ is the force on a current element $i d\mathbf{l}$. This is analogous to the definition of **E** as the force on unit charge, and establishes **B** as the fundamental property.

Magnetic fields are generated by currents, and the flux density resulting from a given geometrical arrangement of currents in free space can be calculated from the Biot–Savart law:

$$d\mathbf{B} = \frac{\mu_0 i \, d\mathbf{l} \times \hat{\mathbf{r}}}{4\pi r^2}, \tag{A.2}$$

where $d\mathbf{B}$ is the contribution to the flux density at a point P due to a current element $i d\mathbf{l}$, r is the distance of P from the element $d\mathbf{l}$, $\hat{\mathbf{r}}$ the unit vector in the direction of r, and μ_0 the permeability of free space. This expression for **B** corresponds to the expression

$$d\mathbf{E} = \frac{dq\,\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} \tag{A.3}$$

for the electric field in free space due to a charge dq, ε_0 being the permittivity of free space. It can be shown from the Biot–Savart law that *in free space* **B** satisfies the Ampère circuital law

$$\oint \mathbf{B} \, . \, d\mathbf{l} = \mu_0 \mathscr{I}, \tag{A.4}$$

where the line integral $\oint \mathbf{B} \, . \, d\mathbf{l}$ is taken round *any* closed path and $\mathscr{I}$ is the net current linking that path.

It can easily be shown by applying the Ampère circuital law to an infinitely long solenoid that the flux density inside it is uniform and given by

$$B = \mu_0 m i, \tag{A.5}$$

where m is the number of turns per unit length and i the current through each of them. Furthermore, the flux density outside the solenoid is zero.

A.2. The Effect of Magnetic Material

All the foregoing equations with the exception of (A.1) apply *only in free space.* To introduce the effect of magnetic material, consider a long cylinder of paramagnetic material inside an infinitely long solenoid, as shown in Fig. A.1. Paramagnetism arises because the material contains

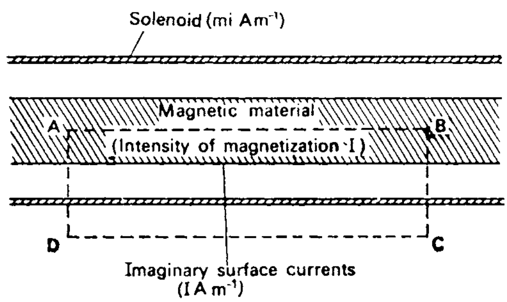
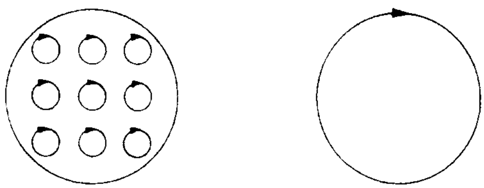


FIG. A.1. Rod of magnetic material in infinite solenoid. $ABCD$ ---- path of integration (A.7).

within it elementary atomic dipoles, and these dipoles tend to be aligned by the field of the solenoid, so that they point predominantly in the direction of the field. Remembering that the atomic dipoles are due not to free magnetic poles, but to small circulating currents which arise either from electron spin or from the orbital motion of electrons within the atoms, it will be seen that these circulating currents are as shown in Fig. A.2a when viewed parallel to the axis of the solenoid. Because of the aligning influence of the field, all these currents circulate in the same sense. The degree of magnetization of the material can be described by specifying its *intensity of magnetization* (usually called simply its "magnetization") **I**, which is a vector pointing in the direction of magnetization and having a magnitude equal to the resultant magnetic dipole moment per unit volume.

The total flux density within the cylinder is now the resultant of the flux density due to the solenoid and that due to the atomic currents.



(a) (b)

FIG. A.2. Equivalence of aligned current dipoles and surface current (viewed in direction of field). The atomic current loops which generate the magnetic dipoles all circulate in the same direction due to the aligning action of the field, as in (a). The average flux density produced by these current loops within the material is the same as would be produced by imaginary surface currents of density I A m^{-1}, as in (b).

Clearly **B** will not be uniform within the magnetic material but will fluctuate from point to point with the periodicity of the atomic lattice. But there will be an average value of **B**, and it can be shown† that if the magnetic material has a magnetization **I** this average value is exactly the same as would be produced by fictitious currents flowing around the periphery of the cylinder in planes perpendicular to the axis, as in Fig. A.2b, and having a surface density of I A m^{-1}. The magnetization of the paramagnetic material is therefore equivalent to an imaginary solenoid carrying a current of I A m^{-1} and the additional flux density produced by this solenoid is, in accordance with (A.5), given by

$$\mathbf{B}_{\mathrm{m}} = \mu_0 \mathbf{I}.$$

† See, for example, A. F. Kip, *Fundamentals of Electricity and Magnetism*, McGraw-Hill, 1962.

The flux density due to this imaginary solenoid simply adds to the flux density produced by the real solenoid, so the magnitude of the total flux density within the material is given by

$$B = \mu_0 mi + \mu_0 I. \tag{A.6}$$

There are two conventions in the literature about the dimensions of I. Most books on electromagnetic theory give I the dimensions of amperes per metre, as we have done here. Books dealing with the magnetic properties of solids, however, often describe their magnetism in terms of a magnetic *polarization* I', a magnetic *flux density* which the sample adds onto the flux density of the applied field. So I' has the same dimensions as B, and (A.6) becomes

$$B = \mu_0 mi + I'. \tag{A.6a}$$

There are arguments in favour of each convention, but the definition of I embodied in (A.6) as a magnetic *field strength* contributed by the sample seems to be more in keeping with the parallelism between magnetism and electrostatics, and we have adopted it in this book.

A.3. The Magnetic Field Strength

If we take the line-integral of **B** around the path $ABCD$ in Fig. A.1, where $AB = CD = x$, we find from (A.6), remembering that $B = 0$ outside the solenoid,

$$\oint_{ABCD} \mathbf{B} \,.\, d\mathbf{l} = (\mu_0 mi + \mu_0 I)x. \tag{A.7}$$

We can write this as

$$\oint \mathbf{B} \,.\, d\mathbf{l} = \mu_0(\mathscr{I}_f + \mathscr{I}_m), \tag{A.8}$$

where $\mathscr{I}_f = xmi$ is the total current through the turns of the solenoid linking $ABCD$, and $\mathscr{I}_m = Ix$ is the total effective surface current which is equivalent to the magnetization I. $\mathscr{I}_f$ is often referred to as the "free" current. Hence (A.4) only remains valid if we identify $\mathscr{I}$ with $\mathscr{I}_f + \mathscr{I}_m$. This is not a useful relationship, however, because although we know $\mathscr{I}_f$ we do not in general know $\mathscr{I}_m$. It is therefore convenient to introduce a new vector, called the *magnetic field strength* **H**, which is defined by

$$\mathbf{B} = \mu_0 \mathbf{H} + \mu_0 \mathbf{I} \tag{A.9}$$

so that (A.6) gives

$$H = mi \tag{A.10}$$

and (A.7) becomes
$$\oint \mathbf{H} \,.\, d\mathbf{l} = xmi = \mathscr{I}_f. \tag{A.11}$$

Hence Ampère's circuital law for **H** involves only the *real* or "free" current $\mathscr{I}_f$ and is independent of the presence of the magnetic material. It is this important result which makes **H** a useful quantity. It is an unfortunate feature of the MKS system that the fundamental distinction between B and H, as exemplified by (A.8) and (A.11), tends to be obscured by the presence of a dimensional factor μ_0 in (A.8) which is absent in (A.11).

For many materials (but not iron) it is found that the magnetization I is proportional to the magnetic field intensity *within the specimen*, so that inside the material $\mathbf{I} = \chi\mathbf{H}$ where χ is the susceptibility. Hence

$$\begin{aligned}\mathbf{B} &= \mu_0(\mathbf{H} + \mathbf{I}) = (1 + \chi)\mu_0\mathbf{H} \\ &= \mu_r\mu_0\mathbf{H},\end{aligned}$$

where $\mu_r = 1 + \chi$ is the relative permeability, which is a pure number. For ferromagnetic or paramagnetic materials χ is positive and $\mu_r > 1$, but for diamagnetic materials χ is negative and $\mu_r < 1$.

To sum up, for the case of a long thin rod, the flux density **B** within the rod includes a contribution from the magnetization of the material, while the magnetic field strength **H** does not. In this case the magnetic field strength inside the rod is the same as if the rod were not there.

A.4. The Case of a Superconductor

The foregoing discussion applies also to the case of a type-I superconductor (or a type-II superconductor below H_{c1}), but with an important distinction. The flux density within the superconductor is zero if we ignore penetration effects, and this perfect diamagnetism is brought about by *real* currents which circulate around the periphery of the superconductor; they are in fact the screening currents discussed in Chapter 2.

When it comes to the concept of B and H in a superconductor, there are two ways in which we can proceed. We can focus attention on the screening currents and regard them as real currents not different in nature from the current in the windings of the solenoid. On this view, which regards the bulk of the superconductor as non-magnetic, it is appropriate to rewrite (A.6) in the form

$$B = \mu_0(mi + j_s), \tag{A.12}$$

where j_s is the surface density of the screening currents per unit length parallel to the axis. The vanishing of B within a superconductor is due to the fact that the terms in brackets in (A.12) are equal and opposite. In this case we have for any closed path

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0(\mathscr{I}_f + \mathscr{I}_s), \tag{A.13}$$

where $\mathscr{I}_f$ is the total current through the solenoid turns linking the path, and $\mathscr{I}_s$ the total diamagnetic screening current linking the same path. However, although the screening currents are certainly real, we cannot measure them with an ammeter, and we shall continue to define the magnetic field strength $\mathbf{H}$ so that $\oint \mathbf{H} \cdot d\mathbf{l}$ is always equal to the "free" or *measurable* current $\mathscr{I}_f$. This means that we retain (A.10), which states that

$$H = mi, \tag{A.10}$$

just as in the case of a paramagnetic material, and *the screening currents affect B but not H inside the material.*

Alternatively, it is possible to invert the argument summarized in § A.2 and regard the real currents flowing on the surface of the superconductor as equivalent to fictitious dipoles uniformly distributed throughout the body of the superconductor. (Since the superconductor is diamagnetic, these imaginary dipoles would point the opposite way to the real dipoles in a ferromagnetic or paramagnetic specimen.) From this point of view, we can talk about the intensity of magnetization I of a superconductor and regard this either as the magnetic moment per unit volume of the equivalent dipoles or the surface density of the screening currents in amps per metre. A practical definition of I is that it is the total magnetic moment of the specimen arising from the surface currents divided by the volume of the specimen. I is equal to and has the same dimensions as the quantity j_s occurring in (A.12). In terms of I, we may write, in accordance with (A.6),

$$B = \mu_0(mi + I), \tag{A.6}$$

where the term $\mu_0 mi$ represents the flux density due to the solenoid and $\mu_0 I$ is the flux density resulting from the fictitious equivalent dipoles. As before, we write

$$\mathbf{B} = \mu_0\mathbf{H} + \mu_0\mathbf{I}, \tag{A.9}$$

and the vanishing of B implies that within the superconductor the field strength H must be equal to $-I$, so that we have $\chi = -1$ or $\mu_r = 0$, i.e. the bulk of the superconductor is perfectly diamagnetic.

Whichever way we choose to look at it, we have

$$\oint \mathbf{H} \, . \, d\mathbf{l} = \mathscr{I}_f \neq \frac{1}{\mu_0} \oint \mathbf{B} \, . \, d\mathbf{l}, \qquad \text{(A.11a)}$$

and the value of H inside a long thin superconductor is the same as if the superconductor were not there. (H is in fact the quantity we have called the applied field H_a). Whichever view we adopt, we say that deep within a superconductor $B = 0$, but in the presence of an applied field H does not vanish.†

For most purposes the second approach is more convenient, since it allows us to apply to a superconductor concepts such as energy of magnetization and demagnetizing field which were first developed for the case of ordinary magnetic materials. As an example, the treatment of the intermediate state given in Chapter 6 would be very difficult if we had to take the screening currents into account explicitly. There are, however, cases where we are particularly interested in the spatial distribution of flux density (and of screening currents) near to the surface of a superconductor. An example is the development of the London equations in Chapter 3. In this case, it is advantageous to adopt the first approach and recognize the existence of the real surface currents which circulate around the specimen whose bulk consists of non-magnetic material. The distinction between B and H is now revealed by (A.11a) and (A.13), which in point form become

$$\text{curl } \mathbf{H} = \mathbf{J}_f \qquad \text{and} \qquad \text{curl } \mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_s).$$

Within the superconductor, $\mathbf{J}_f = 0$, so that curl $\mathbf{H} = 0$ and curl $\mathbf{B} = \mu_0 \mathbf{J}_s$.

Both approaches are encountered in the literature, and the reader should familiarize himself with each of them.

A.5. Demagnetizing Effects

So far we have limited our discussion to the case of a long thin specimen in which end effects are unimportant. We now show that the

† This standpoint is not invariably taken in the literature. It is quite common to find authors who write $B = \mu_0 H$ everywhere within the superconductor. In this case the vanishing of B implies the vanishing of H also. But there is really no point in introducing B and H if they are everywhere simply related by a universal constant of proportionality. This way of looking at it causes difficulties in the explanation of demagnetizing effects, and also means that the usual boundary condition involving the continuity of the tangential component of $\mathbf{H}$ at the boundary between two media, which we have made use of in Chapter 6, is no longer valid at the interface between the superconducting and normal phases.

relationship $B = \mu_0(H + I)$ has to be interpreted somewhat differently if the specimen is not long in comparison with its width. To be specific, consider a sphere of paramagnetic material which is placed in a uniform applied field, say within a long solenoid. It is possible to show that the magnetization **I**, defined as the magnetic moment per unit volume, is uniform within the sphere and is parallel to the axis of the solenoid. It can also be shown† that the flux density due to the magnetization is the same as would be produced by surface currents having a constant surface density *I per unit length measured parallel to the axis*. The flux density produced by such a distribution of surface currents around the surface of the sphere is equal to $\frac{2}{3}\mu_0 I$, so within the sphere the flux density is uniform and given by

$$B_i = \mu_0 mi + \tfrac{2}{3}\mu_0 I. \qquad \text{(A.13)}$$

Comparing this with (A.6) and (A.9), which state that for a long thin specimen

$$B = \mu_0 mi + \mu_0 I = \mu_0 H + \mu_0 I,$$

we might be tempted to define the magnetic field strength by

$$B = \mu_0 H + \alpha\mu_0 I,$$

where α is a numerical factor depending on the geometry of the specimen, equal to unity for a long thin specimen and to 2/3 for a sphere. If we were to do this, the field strength inside the body would be equal to mi both for a long thin specimen and for a sphere. However, unlike the case of a long thin rod, in the case of the sphere the surface currents alter the flux density (and therefore the field strength, since $B = \mu_0 H$) *outside* the sphere, and an argument identical with that given on page 65 (Chapter 6), shows that we should no longer have $\oint \mathbf{H} \cdot d\mathbf{l} = \mathcal{I}_f$. It can be shown that if we wish to retain $\oint \mathbf{H} \cdot d\mathbf{l} = \mathcal{I}_f$ for any closed curve, where $\mathcal{I}_f$ is the total current in those turns of the solenoid which link the curve, then we must also retain (A.9) as the definition of the magnetic field strength $\mathbf{H}_i$ inside the sphere, i.e.

$$\mathbf{B}_i = \mu_0 \mathbf{H}_i + \mu_0 \mathbf{I}. \qquad \text{(A.9')}$$

But, according to (A.13),

$$B_i = \mu_0 mi + \tfrac{2}{3}\mu_0 I. \qquad \text{(A.13)}$$

† See, for example, A. F. Kip, *Fundamentals of Electricity and Magnetism*, McGraw-Hill, 2nd ed., 1969, p. 370.

These equations can both be satisfied only if

$$H_i = mi - \tfrac{1}{3}I,$$

i.e. the magnetic field strength inside the sphere is different from the value mi which it would have if the sphere were absent and which we have called the applied field H_a. Hence for a sphere

$$H_i = H_a - \tfrac{1}{3}I,$$

and for a body of arbitrary shape

$$\mathbf{H}_i = \mathbf{H}_a - n\mathbf{I}, \tag{A.14}$$

where n is the "demagnetizing factor", equal to 1/3 for a sphere. For a superconductor, if we adopt the approach which ignores the screening currents but regards the bulk of the superconductor as perfectly diamagnetic, then

$$B_i = \mu_0 H_i + \mu_0 I = 0,$$

so that

$$I = -H_i$$

and from (A.14) $H_i = H_a + nH_i$, i.e. the field strength inside the specimen is *increased* to $H_i = H_a/(1 - n)$. For the general case we may write

$$\mathbf{B}_i = \mu_0 \Big(\underbrace{\underset{\substack{\downarrow \\ \text{applied} \\ \text{field}}}{\mathbf{H}_a} - \underset{\substack{\downarrow \\ \text{demagnetizing} \\ \text{field}}}{n\mathbf{I}}}_{\text{internal field, } \mathbf{H}_i} + \underset{\substack{\downarrow \\ \text{magnetization}}}{\mathbf{I}} \Big)$$