

## GUIDANCE AND CONTROL

### OPTIMALIZING TECHNIQUES FOR INJECTION GUIDANCE

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#### ABSTRACT

The class of problems being considered here is the determination of optimizing guidance laws for systems where the trajectories described by the state variables are well predicted. For an example problem of injection guidance, it is assumed that the actual trajectories are adequately described by perturbation or linear equations of motion about the reference trajectory. Within the assumptions of linear prediction of the state variables, a guidance law was determined which is optimal in the sense that the least amount of fuel is used. A numerical solution was obtained for the case where the position and velocity state variables satisfy a specified constraint at the time of thrust cutoff. The method used to obtain the optimal guidance laws is based on Pontryagin's maximum principle. This is a method of solving a problem in the calculus of variations wherein the set of admissible extremals is closed. In particular, it applies to the problem considered here where the magnitude of the control is bounded.

#### INTRODUCTION

The design of the booster stage of a space vehicle is the result of extensive optimization studies. However, the design must of necessity be based on an approximate model of the system, that is, the vehicle and the environment through which the vehicle moves. Thus, sufficient allowance or overdesign in terms of fuel and structures must be made for the expected variation from the design model. This variation from a reference model essentially determines the control and the guidance computation requirements. If the penalty for overdesign is severe, as it is for many aerospace missions, there is justification for increasing the complexity of the control and guidance system in order to reduce the expected variability in critical performance parameters such as structural loading and the fuel used for control. However, the forces involved are so complex that optimum solutions to the nonlinear equations of motion are difficult to obtain, and the resulting guidance laws are

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## GUIDANCE AND CONTROL

difficult to mechanize without greatly simplifying the mathematical model of the system. By obtaining a reference solution of the approximate nonlinear equations of motion, a more exact model can be developed based on linear approximations about the reference solution. Also, the linear approximations greatly simplify the mechanization problem, since the property of superposition of solutions then holds.

Self-contained guidance systems for space vehicle boosters are generally based on an open loop thrust attitude vs. time program during the early portion of flight and then on perturbation guidance, where deviations from a reference path are multiplied by precomputed error sensitivities to develop a control command. Perturbation guidance has a great advantage over explicit guidance, or whole value computation, because the error sensitivities can be computed beforehand and then stored in the guidance computer. The main disadvantage is that a reference trajectory must be followed closely enough for the linear perturbation equations of motion to be valid.

### DEVELOPMENT OF SYSTEM EQUATIONS

It is assumed that the nonlinear equations of motion of the vehicle have been solved to give a reference or nominal solution. This solution is a description of the trajectories of the state variables in phase space. The particular variables chosen as state variables are in principle arbitrary, but actually they are restricted to a class that adequately describes the system and is relatively stable. The major assumption is then made that the motion of the system will be sufficiently close to the reference solution that the linear differential equations of perturbed motion will apply. This set of linear equations is considered to be the set of differential equations of motion of the system. The solution to this set is the linear prediction model to be used in the study.

The reference solution is obtained on the basis of a nominal set of values for the predicted thrust level, air density, winds, launch times, etc. During the actual mission, measurements of the state variables and of the environmental parameters will yield new values. The problem posed is, how should the guidance law based on nominal values be modified in accordance with the new set of values? In particular, what is an optimum guidance law based on the measured values of the state variables and the new a priori description of forces? In the solution offered below, the criterion of optimality is the least amount of fuel used while satisfying end constraints on certain of the state variables. The techniques used to find optimum solutions are based on the maximum principle of Pontryagin.

## GUIDANCE AND CONTROL

The nonlinear equations of motion for a rocket must be integrated numerically to find a reference solution or trajectory. Implicit in these equations is a set of constraints which represent a particular guidance law, that is, the direction and magnitude of the thrust as functions of one or more parameters of the system. The following simplifying assumptions are made:

- 1) Motion is in a plane over a flat, nonrotating Earth (X is horizontal, Z is vertical).
- 2) Aerodynamic moments are exactly cancelled by inner loop flight control.
- 3) Thrust is entirely due to momentum thrust ( $T_h = \dot{M}V_e$ ).
- 4) Gravitational force is in the vertical direction.
- 5) The ARDC 1959 standard atmosphere exists.
- 6) A guidance law has been specified for the reference trajectory.

The equations of motion for the reference trajectory are the following first-order set:

$$\begin{aligned}
 dX/dt &= V_x & dz/dt &= V_z & & [1a-b] \\
 dV_x/dt &= 1/M \left[ -D \cos \left[ -L \sin \left[ + U_2 V_e \cos(B+U_1(t)) \right] \right] \right] & & & [1c] \\
 dV_z/dt &= 1/M \left[ -D \sin \left[ +L \cos \left[ -G + U_2 V_e \sin(B+U_1(t)) \right] \right] \right] & & & [1d] \\
 dM/dt &= -U_2(t)
 \end{aligned}$$

The geometry is defined in Fig. 1. For prescribed functions of  $U_1(t)$  and  $U_2(t)$ , a reference solution is obtained. This solution is shown in Figs. 2 and 3.

The perturbation equations are obtained from the first terms in a Taylor's series expansion of Eqs. 1.

Rewriting Eqs. 1 in vector form as  $d\bar{X}/dt = F(\bar{X}, \bar{U}, t)$  [2]  
the perturbation equations are

$$\frac{d\Delta X_i}{dt} = \frac{\partial f_i(\bar{X}, \bar{U}, t)}{\partial X_j} \Delta X_j + \frac{\partial f_i(\bar{X}, \bar{U}, t)}{\partial U_k} \Delta U_k \quad \begin{matrix} i, j=1, 2, \dots, 5 \\ k=1, 2 \end{matrix} \quad [3]$$

where  $\partial f_i(\bar{X}, \bar{U}, t)/\partial X_j$  and  $\partial f_i(\bar{X}, \bar{U}, t)/\partial U_k$  are evaluated on the reference solution.

Substituting  $x$  for  $\Delta X$  and  $u$  for  $\Delta U$  and putting Eq. 3 in matrix form gives

$$d\vec{x}/dt = A(t)\vec{x} + B(t)\vec{u} \quad [4]$$

## GUIDANCE AND CONTROL

The elements of  $A(t)$  are listed in Appendix A. It is assumed that the variations in air density are a known function of altitude and that the lift and drag coefficients are only functions of angle of attack. The effects of variations in temperature, viscosity, and Mach number on the lift and drag coefficients could be included similarly, as was done by Tsien (Ref. 1, pp. 195-197). From now on it is assumed that  $U_2 = \dot{M}$  and  $U_1 = 0$ .

It is obvious that the selection of perturbation state variables is arbitrary and depends to a large extent on the particular interest in mind. However, the system must be relatively stable or the solutions will not stay within the region of first-order validity. Since this region is not known a priori, it can only be determined by comparing the linear solutions with neighboring solutions of the nonlinear differential equations. For the present problem, the free system would be extremely unstable if the aerodynamic moments were not assumed to be cancelled by an attitude control loop.

The normalized fundamental solution matrix  $\Phi(t)$  for the differential equation set, Eqs. 4a-4f, is obtained by integrating the differential equations from initial conditions of unity. However, since it is the end or burnout conditions that are of interest, the solution will be obtained for terminal conditions of unity. This is accomplished simply by integrating backward in time. The inverse of the fundamental solution matrix is also required and is obtained by solving the adjoint differential equations corresponding to Eqs. 4a-4f. That is

$$\frac{d\bar{\Phi}}{dt} = -\bar{\Phi}A(t) \quad [5]$$

These equations are listed in Appendix A. Eq. 5 and the procedure for converting to negative time are described in Appendix B.

The solution matrices,  $\Phi(t)$  and  $\Phi^{-1}(t)$ , represent the linear prediction model. The perturbation vector  $x(t)$  can then be determined for any initial conditions and disturbance function  $G(t)$

$$\bar{x}(t) = \Phi(t) \bar{x}(t_0) + \Phi(t) \int_{t_0}^t \Phi^{-1}(\tau) \bar{G}(\tau) d\tau \quad [6]$$

With an arbitrary disturbance vector  $g(t)$ , the system of differential equations is of the form

$$\dot{x}_i = a_{ij} x_j + b_{ik} u_k + g_i(t) \quad [7]$$

## GUIDANCE AND CONTROL

The particular equations that have a control variable are

$$\begin{aligned} \dot{x}_3 &= a_{3j} x_j + g_3(t) + b_{31} u_1 + b_{32} u_2 & \left[ \begin{array}{l} 8a \\ 8b \\ 8c \end{array} \right] \\ \dot{x}_4 &= a_{4j} x_j + g_4(t) + b_{41} u_1 + b_{42} u_2 & j=1, \dots, 5 \\ \dot{x}_5 &= a_{5j} x_j + g_5(t) + b_{51} u_1 + b_{52} u_2 \end{aligned}$$

where

$$\begin{aligned} b_{31} &= \frac{-V_e \dot{M}}{M} \sin B & b_{41} &= \frac{V_e \dot{M}}{M} \cos B & b_{52} &= -1 \\ b_{32} &= \frac{V_e}{M} \cos B & b_{42} &= \frac{V_e}{M} \sin B \end{aligned}$$

Physically,  $u_1(t)$  is an increment in the thrust vector deflection and  $u_2(t)$  a variation in the mass flow rate. Both of these functions will be assumed to be bounded. It is assumed that

$$\begin{aligned} |u_1(t)| &\leq 0.02 \equiv K_1 \\ |u_2(t)| &\leq 0.01 \dot{M} \equiv K_2 \dot{M} \end{aligned}$$

The choice of the value of  $K_2$  is arbitrary. Limiting the control deflection variation  $u_2$  to a small value is necessary for the assumption that the control is described by linear non-coupled terms with no control dynamics.

The problem to be solved is to determine  $u_1(t)$  and  $u_2(t)$  for a set of initial conditions  $X(t_0)$  and disturbance functions  $g_i(t)$ , so that the resulting motion of the system is optimum. The method used in finding the optimum  $u$ 's is based on the maximum principle of Pontryagin. Since this method is relatively new to engineering literature in this country it will be described briefly prior to a precise statement of the problem and the solution.

### MAXIMUM PRINCIPLE

The theory used in this paper, called the "maximum principle", was developed by three Russian mathematicians, Pontryagin, Boltyanskii, and Gamkrelidze. This principle was conjectured by Pontryagin in 1956 and completely proved by Boltyanskii, with certain theorems concerning uniqueness and existence established by Gamkrelidze, by the end of 1960. The<sup>3</sup> maximum principle is described in some detail by Rozonoer(2).

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<sup>3</sup>Numbers in parentheses indicate References at end of paper.

## GUIDANCE AND CONTROL

This reference is the primary source for the following description. A general description of the theory and the application to a nonlinear system is given in the preceding article by Dahlard Lukes.

A system is considered whose state variables are subject to  $n$  ordinary differential equations of constraint

$$\dot{x}_i = f_i(x_1, \dots, x_n; u_1, \dots, u_r; t) \quad i=1, \dots, n \quad [9]$$

where  $x(t) = (x_1, \dots, x_n)$  is a vector in the phase space whose motion is referred to as the trajectory, and  $u(t) = (u_1, \dots, u_r)$  is called the control vector. It is required that  $u(t)$  belong to the set  $U$  where  $U$  is some closed subset of  $r$ -dimensional Euclidean space  $E^r$ , and furthermore the components  $u_k(t)$  of  $U$  are required to be piecewise continuous in  $U$ . A control  $u(t)$  that satisfies these two conditions is called admissible and will be the only type of control vector referred to hereafter.

The state of the system is described by a point in  $E^n$ , and when a control  $u(t)$  is specified along with an initial condition of the system

$$x^0 = (x_1^0, \dots, x_n^0)$$

then the behavior of the system in the form of a trajectory  $x(t)$  in the phase space  $E^n$  is uniquely determined from Eqs. 9.

The basic problem in the theory of optimum systems is the selection of that control  $u(t) \in U$  for which the behavior of the system is optimum in some predesignated sense. These optimum criteria assume many forms, but only those will be considered which can be reduced to the case of requiring a linear combination of the final values of the phase coordinates to be maximized or minimized, that is, of requiring that the quantity

$$S = \sum_{k=1}^n c_k x_k$$

be minimized, where the  $c_k$  are given constants. Furthermore, these optimum criteria usually restrict the final position of the trajectory in the phase space in some way. In general, this restriction can be formulated as a transfer of the trajectory in the phase space from the fixed point  $x^0$  to some fixed closed set  $G$  of  $E^n$ .

Using the above terminology, the optimum problem can be stated as follows. From the set  $U$  of controls<sup>4</sup> that transfer

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<sup>4</sup>It must be remembered that they must be admissible by the earlier convention.

## GUIDANCE AND CONTROL

the system described by Eq. 9 from the point  $x^0 \in E^n$  to the fixed closed set  $G \in E^n$ , a control  $u(t)$  is selected so that the sum

$$S = \sum_{i=1}^n c_i x_i(T)$$

takes on the minimal value (compared to the values it would assume for other controls in  $U$ ). It should be noted that the problem formulated is one in which the time of control  $T - t_0$  is fixed, that is,  $t_0$  and  $T$  are fixed. The case where the time of control is not fixed can be treated in an entirely analogous way.

The control  $u(t) \in U$  causing the minimum (maximum) of a functional  $S$  is called the min-optimum (max-optimum) control relative to  $S$ . The results necessary to obtain this  $u(t)$  are presented below.

Introduce the set of  $n$  time functions  $p_1(t), \dots, p_n(t)$ , which are components of the vector  $p(t) \in E^n$ . The variables  $p_i(t)$  are required to satisfy the set of differential equations

$$p_i(t) = - \sum_{s=1}^n p_s \frac{\partial f_s(x, u, t)}{\partial x_i} \quad i=1, \dots, n \quad [10]$$

where the  $f_s(x, u, t)$  are from Eqs. 9. Eqs. 10 determine  $p(t)$  uniquely when the boundary conditions for  $p(t)$  are given. Now the following function is introduced

$$H(x, p, u, t) = \sum_{i=1}^n p_i f_i(x, u, t)$$

Then Eqs. 9 and 10 can be described in the form

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = - \frac{\partial H}{\partial x_i} \quad x_i(t_0) = x_i^0 \quad i=1, \dots, n \quad [11]$$

Eqs. 11 coincide in form to the canonical equations of Hamilton in analytical mechanics. The function  $H$  is analogous to the Hamiltonian and the vector  $p(t)$  to the momentum vector. The connection between the equations of the theory of optimum systems and the equations of mechanics is discussed in Ref. 2.

Now  $u(t)$  will be some admissible control, and  $x^u(t)$  will denote its corresponding trajectory; also  $p(t)$  will be one of

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<sup>5</sup>The problem being investigated, except for the requirement that  $U$  be closed, is the classical problem of Mayer in the variational calculus.

## GUIDANCE AND CONTROL

the vector functions satisfying Eq. 10. Upon substituting the values  $x_i^u(t)$  and  $p_i(t)$  into  $H(x, p, u, t)$ , the following function is obtained:

$$K(u_1, \dots, u_r; t) = [H x^u(t), p(t), u, t]$$

which is, for fixed  $p(t)$ , a function of  $u(t) \in U$ . A control  $u(t)$  is said to satisfy the condition of the maximum (minimum) relative to the vector  $p(t)$  if for all  $t' \in [t_0, T]$ , the function  $K(u, t')$  reaches an absolute maximum (minimum) on  $U$  for the values  $u_k = u_k(t')$   $k = 1, \dots, r$ .

$$(x) = \sum_{i=1}^n c_i x_i \quad [12]$$

is defined as a function of the point  $x = (x_1, \dots, x_n)$  of the phase space. Next the set of points in  $G$  for which  $\Psi(x)$  takes on a minimum value, compared with all other points of  $G$ , will be called the set  $G^*$ . It is clear that  $S$  cannot be smaller than  $\Psi^*$ , where  $\Psi^*$  is defined to be this minimum value of  $\Psi$  on the set  $G^*$ . Then according to the previous terminology, any control  $u(t)$  that transfers the trajectory from  $x^0$  to the set  $G^*$  during the time  $T-t_0$  is min-optimum, because it makes  $S$  its smallest possible value, namely  $\Psi^*$ . If it is possible to transfer the trajectory to the set  $G^*$  in the interval  $T-t_0$ , the corresponding problem is called degenerate. The degenerate problem is not considered here.

A basic theorem exhibits the maximum principle. If in a nondegenerate problem, the control  $u(t)$  is min-optimum (max-optimum) relative to  $S = \sum_{i=1}^n c_i x_i (T)$ , then there exists a vector function  $p^u(t)$  such that  $u^u(t)$  satisfies the maximum (minimum) condition relative to  $p^u(t)$ .

In essence, the theorem states that if a function  $u(t)$  is thought to be min-optimum, it is necessary that  $H$  be maximized with respect to  $u$  and relative to the vector  $p(t)$ .

Next, the method for obtaining boundary conditions for the vector  $p(t)$  is considered. Since the motivation is too cumbersome to be included here, the results are stated for a special class of problems, namely those in which the set  $G$  is a convex set described by the inequality  $F(x_1, \dots, x_n) \leq 0$ . Then if the point  $x(T)$  is on the boundary of  $G$ ,  $[F x(T)] = 0$ . Since  $G$  is convex, a supporting hyperplane to  $G$  can be found at each point on its boundary. If  $F(x)$  is differentiable, the coefficients of the tangent plane are  $b_i$ , where



## GUIDANCE AND CONTROL

$$b_i = \left. \frac{F(x)}{x_i} \right|_{x=x(T)} \quad i=1, \dots, n$$

It should be recalled that the constants  $c_i$  are the coefficients of  $x_i(T)$  in  $S$ , and  $\lambda$  and  $\mu$  will be two nonnegative real parameters; then the following boundary conditions on  $p(T)$  result

$$p_i(T) = -\lambda c_i - \mu b_i [x_1(T), \dots, x_n(T)]$$

where it is permissible to let one of the parameters,  $\lambda$  or  $\mu$ , be equal to one (since only their ratio turns out to be important), and where the remaining one must then be chosen so that  $S(T)$  ends up on the surface of  $G$ , that is, so that  $F[x_1(T), \dots, x_n(T)] = 0$ .

The maximum principle is used the following way. Obtain the maximum of  $H$  with respect to the explicit occurrence of  $u$  as a function of  $x(t)$  and  $p(t)$ . Substitute these expressions for  $u(t)$  back into the systems of differential equations to obtain a set of  $2n$  differential equations in  $2n$  unknowns subject to  $2n$  boundary conditions and involving one parameter ( $\lambda$  or  $\mu$ ) and one relation (namely,  $F(x_1(T), \dots, x_n(T)) = 0$ ). The solution of this differential algebraic system constitutes a solution to the problem.

### STATEMENT OF PROBLEM

The problem is to find an optimum solution in terms of  $u_i(t)$ . The equations involving  $u_i(t)$  are Eqs. 8.

The  $u_i(t)$  are to be found from the class of piecewise continuous functions belonging to the set  $U$  such that  $|u_1(t)| \leq K_1$ ,  $|u_2(t)| \leq K_2 \dot{M}$  where the  $K_i$  are known constants.

The optimization criterion is that the vehicle should consume a minimum amount of fuel in achieving the desired end conditions. This constraint on fuel is posed in the form of an integral as

$$J[t_0, t_b, u(t)] = \int_{t_0}^{t_b} \left[ \dot{M}(t) + u_2(t) + \frac{1}{2} u_1^2(t) \dot{M}(t) \right] dt$$

where  $\dot{M}(t)$  is a known function of time. <sup>6</sup>  $\frac{1}{2} u_1^2(t) \dot{M}(t)$  represents the additional mass rate due to  $u_1(t)$   $\dot{u}_2(t)$  can be

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<sup>6</sup>  $\dot{M}(t)$  is a positive number.

## GUIDANCE AND CONTROL

positive or negative, indicating either an increase or decrease from the reference value.

The initial conditions on the  $x_i$  are  $x_i(t_0) = x_i^0$ . The terminal constraint is

$$2.5 x_1(t_b) + 2.5 x_2(t_b) + 10^3 x_3(t_b) - 0.5 \times 10^3 x_4(t_b) = 0$$

This last condition arises from extrapolating the desired conditions at a future point (on Earth for the present example) to the time of burnout,  $t_b$ . For each additional condition a corresponding constraint equation would be written.

In order to apply Pontryagin's maximum principle, the problem must be stated as one of optimizing a linear functional of the final values of the coordinates. Thus a new state variable is defined

$$x_6(t_b) = \int_{t_0}^{t_b} \left[ \dot{M}(t) + u_2(t) + \frac{1}{2} u_1^2(t) \dot{M}(t) \right] dt \quad [13]$$

Then

$$\dot{x}_6 = \dot{M}(t) \left[ 1 + \frac{1}{2} u_1^2(t) + u_2(t) \right] \quad [14]$$

with  $x_6(t_0) = x_6^0 = 0$ . This leads to a new system of equations,  $\dot{x}_i = f_i(x, u, t)$  ( $i=1, \dots, 6$ ), subject to the desire that  $S = x_6(t_b)$  should be minimum.

### SOLUTION OF PROBLEM

The vector function  $p(t)$  is introduced where  $p(t)$  satisfies

$$p_i = - \sum_{s=1}^6 p_s \frac{\partial f_s(x, u, t)}{\partial x_i} \quad i=1, \dots, 6 \quad [15]$$

The function  $H$  is introduced

$$H(x, p, u, t) = \sum_{i=1}^6 p_i f_i(x, u, t) \quad [16]$$

so that  $x_i$  and  $p_i$  can be written as

$$\dot{x}_i = \frac{\partial H}{\partial p_i} \quad \dot{p}_i = - \frac{\partial H}{\partial x_i} \quad x_i(t_0) = x_i^0 \quad i=1, \dots, 6$$

Next it must be determined if the problem is nondegenerate. For the present problem,  $G^*$  is the hyperplane  $x_6 = 0$ , but this implies that no fuel is consumed; thus  $G^*$  is not attainable,

## GUIDANCE AND CONTROL

and the problem is nondegenerate. The terminal constraint leads to the following conditions on  $p(t)$  at  $t = t_b$

$$p(t_b) = -\lambda c_i - \mu b_i$$

where

$$\begin{array}{lll} c_i = 0 & & i=1, \dots, 5 \\ c_6 = 1 & b_1 = 2.5 & b_3 = 10^3 \\ & b_2 = 2.5 & b_4 = -0.5 \times 10^3 \end{array}$$

The  $b_i$  are coefficients of the supporting hyperplane B, and the signs are to be selected so that G will be on that side of B where

$$\sum_1^4 b_i [x_i - x_i(t_b)] \leq 0$$

Since G coincides with B, it is impossible to make this selection, and the possibility must be allowed that  $-b_i$  is the set of coefficients to B which are desired. In the phase space, this simply means that at  $t$  near  $t_b$ , it is not certain which side of G the trajectory will be on. For simplicity in the computations,  $\mu = 1$  is chosen and  $\lambda$  is left to be determined.

From theorem 4 of Ref. 2, if  $u(t)$  satisfies the maximum condition relative to the vector  $p(t)$ , the coordinates of which at  $t = t_b$ , are

$$p_i(t_b) = -\lambda c_i - \mu b_i \quad i=1, \dots, 6 \quad [17]$$

where  $\mu \geq 0$ ,  $\lambda > 0$ , then the control is min-optimum regarding  $S = x_6(t_b)$ .

To carry out the maximum condition, H is considered as given by Eq. 16 in conjunction with Eqs. 8 and 14. By separating the terms containing  $u_i(t)$  from those containing just  $x$ , H is written as

$$\begin{aligned} H = R(x) + u_1(t) & \left( -\frac{V \dot{M}}{M} \sin B \right) p_3 + u_1(t) \left( \frac{V \dot{M}}{M} \cos B \right) p_4 \\ & - \frac{1}{2} u_1^2(t) \dot{M} \lambda + u_2(t) \left[ -\lambda + \left( \frac{V \dot{e}}{M} \cos B \right) p_3 + \left( \frac{V \dot{e}}{M} \sin B \right) p_4 - p_5 \right] \end{aligned} \quad [18]$$

Taking partial derivatives with respect to the  $u_i(t)$  and setting them equal to zero gives

$$\frac{\partial H}{\partial u_1} = 0 = -\frac{V \dot{M}}{M} p_3 \sin B + \frac{V \dot{M}}{M} p_4 \cos B - u_1 \dot{M} \lambda \quad [19a]$$

GUIDANCE AND CONTROL

$$\frac{\partial H}{\partial u_2} = 0 = \left[ -\lambda + \left( \frac{V_e}{M} \cos B \right) p_3 + \left( \frac{V_e}{M} \sin B \right) p_4 - p_5 \right] \quad [19b]$$

Hence, the stationary point is

$$u_1 = \frac{-V_e}{\lambda M} (p_3 \sin B - p_4 \cos B) \quad [20]$$

The only values of  $u_1$  and  $u_2$  which will make  $H$  attain its maximum value on the set  $U$  are  $u_2$  on the boundary of  $U$  (that is,  $u_2 = \pm K_2$ ) and either  $u_1$  on the boundary of  $U$  (that is,  $u_1 = \pm K_1 \dot{M}$ ) or equal to its stationary value (Eq. 20).

For each fixed  $t \in [t_0, T]$ ,  $u_1(t)$  and  $u_2(t)$  must be chosen so that  $H$  is a maximum on  $U$  at that moment with respect to any other choices of  $u(t)$  on  $U$ . For convenience

$$P(t) \equiv -p_5 + \left( \frac{V_e}{M} \cos B \right) p_3 + \left( \frac{V_e}{M} \sin B \right) p_4 - \lambda$$

Clearly  $H$  can be made maximum with respect to  $u_2(t)$  by choosing

$$u_2(t) = K_2 \dot{M} \operatorname{sgn} P(t)$$

because then the term

$$u_2(t) P(t) = P(t) K_2 \dot{M} \operatorname{sgn} P(t) = K_2 \dot{M} |P(t)|$$

is the largest possible. However, the choice is not so simple with regard to  $u_1(t)$  because the function  $H$  might be maximized with respect to  $u_1(t)$  when  $u_1(t)$  is on the boundary of  $U$  or when  $u_1(t)$  assumes its stationary value. Furthermore, it is possible that  $H$  would be maximized at different times by  $u_1(t) = K_1$ ,  $-K_1$  or by  $u_1(t) =$  stationary value.

To prove that the stationary value is the correct choice, the values of  $H$  should be compared for the various choices of  $u_1(t)$ .

Considering  $H$  to be  $H(u_1, u_2)$

$$H^+ \equiv H(K_1, u_2)$$

$$H^- \equiv H(-K_1, u_2)$$

$$H^c \equiv H \left[ \frac{-V_e}{\lambda M} (p_3 \sin B - p_4 \cos B), u_2 \right]$$

## GUIDANCE AND CONTROL

Then  $H^c - H^+$  and  $H^c - H^-$  should be examined. For convenience,  $H^+$  and  $H^-$  will be lumped together as  $\tilde{H}$  to give

$$H^c - \tilde{H} = \frac{V_e^2}{M^2} M(p_3 \sin B - p_4 \cos B)^2 + 2K_1 \frac{V_e}{M} M(p_3 \sin B - p_4 \cos B)\lambda + K_1^2 M\lambda^2$$


---


$$2\lambda$$

As a quadratic equation in  $\lambda$ , it is seen that the discriminant is identically zero. Hence, except for the choice of  $\lambda$ , which is a double root of the numerator, the expression  $H^c - \tilde{H}$  is of one sign. Since  $\lambda = 0$  makes  $H^c - \tilde{H} > 0$ , it is concluded that  $H^c - \tilde{H} \geq 0$  or  $H^c \geq \tilde{H}$  when  $\lambda$  is a positive number. This means that by choosing  $u_1(t)$  as the critical point,  $H$  is maximized at every instant of time; and by the maximum principle,  $u_1(t)$  is then the min-optimum control function. Because the system of equations being used is linear with respect to the coordinates, theorem 4 of Ref. 2 guarantees that the  $u_1(t)$  so chosen is sufficient for the minimization of  $S = x_6(t)$ . Hence, the optimum control vector  $u(t)$  exists, and  $u_2(t) = K_2 \dot{M} \operatorname{sgn} P(t)$  and  $u_1(t) = -(V_e/\lambda M)(p_3 \sin B - p_4 \cos B)$  are its components.

### SYNTHESIS OF CONTROL FUNCTION

The problem of control synthesis is one of calculating the optimum control in the form of a function of time for prescribed initial conditions and disturbance functions. The maximum principle essentially solves this problem, inasmuch as the calculation of the optimum control in the form of a function of time consists of the integration of a certain system of differential equations with corresponding boundary conditions.

In the present case, the values for  $u_1(t)$  and  $u_2(t)$  of the optimum control vector are substituted into the  $x_1$  equations to give the following system of differential equations

$$\dot{x}_1 = a_{1j}(t) x_j + g_1(t) \quad [21a]$$

$$\dot{x}_2 = a_{2j}(t) x_j + g_2(t) \quad [21b]$$

$$\dot{x}_3 = a_{3j}(t) x_j + g_3(t) + \frac{V_e^2 \dot{M}}{\lambda M^2} \sin B(p_3 \sin B - p_4 \cos B) + \frac{V_e}{M} \cos B(K_2 \dot{M} \operatorname{sgn} P(t)) \quad [21c]$$

GUIDANCE AND CONTROL

$$\dot{x}_4 = a_{4j}(t) x_j + g_4(t) - \frac{V_e^2 \dot{M}}{\lambda M^2} \cos B(p_3 \sin B - p_4 \cos B) + \frac{V_e}{M} \sin B(K_2 \dot{M} \operatorname{sgn} P(t)) \quad [21d]$$

$$\dot{x}_5 = a_{5j}(t) x_j + g_5(t) - K_2 \dot{M} \operatorname{sgn} P(t) \quad [21e]$$

$$\dot{x}_6 = \dot{M} + K_2 \dot{M} \operatorname{sgn} P(t) + \dot{M} \left[ (V_e^2 / \lambda^2 M^2) (p_3 \sin B - p_4 \cos B)^2 \right] \quad [21f]$$

where the conditions that  $p_6(t_b) = -\lambda$  and  $\dot{p}_6(t) = 0$  have been used to solve for  $p_6(t)$ . The solution to the Eqs. 21 is taken from Appendix B.

$$x(t_b) = \Phi^{-1}(t_0) x(t_0) + \int_{t_0}^{t_b} \Phi^{-1}(\tau) G(\tau) d\tau$$

where the inverse fundamental solution matrix is normalized to the identity matrix I at  $t_b$ . The elements of  $\Phi^{-1}(t)$  are displayed as follows

$$\Phi^{-1}(t) = \begin{bmatrix} 1 & \Phi_{12}^{-1} & \Phi_{13}^{-1} & \Phi_{14}^{-1} & \Phi_{15}^{-1} & 0 \\ 0 & \Phi_{22}^{-1} & \Phi_{23}^{-1} & \Phi_{24}^{-1} & \Phi_{25}^{-1} & 0 \\ 0 & \Phi_{32}^{-1} & \Phi_{33}^{-1} & \Phi_{34}^{-1} & \Phi_{35}^{-1} & 0 \\ 0 & \Phi_{42}^{-1} & \Phi_{43}^{-1} & \Phi_{44}^{-1} & \Phi_{45}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The control terms and the disturbance forcing functions are contained in  $G(\tau)$ . To keep the problem manageable for illustration, the forcing function will be assumed as a wind shear in the horizontal direction over the time interval from 50 to 60 sec. A resulting horizontal acceleration of 1 ft/sec<sup>2</sup> is assumed and will be written as  $1; t \in (50, 60)$  for a forcing function to the  $\dot{x}_3$  equation. All other  $g_i(t)$  are assumed zero. The expression for  $G(\tau)$  containing both the control and disturbance terms is written below.  $\tilde{P}(t)$  has been replaced by  $\tilde{P}-\lambda$  to display the parameter  $\lambda$ . Thus

$$\tilde{P} = \left[ \frac{V_e}{M} p_3 \cos B + \frac{V_e}{M} p_4 \sin B - p_5 \right]$$

## GUIDANCE AND CONTROL

$$G(\tau) = \begin{bmatrix} 0 \\ 0 \\ -\frac{V_e \dot{M}}{M} u_1 \sin B + \frac{V_e}{M} u_2 \cos B + 1; t \in (50, 50) \\ \frac{V_e \dot{M}}{M} u_1 \cos B + \frac{V_e}{M} u_2 \sin B \\ -u_2 \\ \dot{M} + \frac{1}{2} \dot{M} u_1^2 + u_2 \end{bmatrix}$$

The end constraint equation becomes

$$0 = \int_{t_0}^{t_b} \left\{ (2.5 I_3^{-1}) \left( -\frac{V_e M}{M} u_1 \sin B + \frac{V_e}{M} u_2 \cos B + 1; t (50, 60) \right) \right. \\ \left. + (2.5 I_4^{-1}) \left( \frac{V_e}{M} \dot{M} u_1 \cos B + \frac{V_e}{M} u_2 \sin B \right) + (2.5 I_5^{-1}) u_2 \right\} dt \quad 22$$

$$\text{where } \begin{aligned} I_3^{-1} &= I_{13}^{-1} + I_{23}^{-1} + 400 I_{33}^{-1} - 200 I_{43}^{-1} \\ I_4^{-1} &= I_{14}^{-1} + I_{24}^{-1} + 400 I_{34}^{-1} - 200 I_{44}^{-1} \\ I_5^{-1} &= I_{15}^{-1} + I_{25}^{-1} + 400 I_{35}^{-1} - 200 I_{45}^{-1} \end{aligned}$$

The curves representing the  $I_{ij}^{-1}$  are shown in Figs. 4-6. Furthermore, by letting  $t_0$  correspond to the current time  $t$ , a value for  $\lambda$  and hence a unique control function can be determined for any point along the trajectory. Thus a closed loop guidance law (one which is a function of the present state of the system) has been obtained.

Carrying through the numerical integration for the case where  $K_2 = 0$  gives  $\lambda = 2.4 \times 10^7$ . Thus  $u_1(t)$  becomes  $u_1(t) = (-V_e / 2.4 \times 10^7 M) (-2.5 I_3^{-1} \sin B + 2.5 I_4^{-1} \cos B)$ .

Integrating the differential equations with this value for  $u_1(t)$  then gives the disturbed trajectory that uses the least fuel in satisfying the end condition constraints.

### CONCLUSION

The problem solved was that of obtaining an optimizing guidance law for injection guidance whereby the least amount of fuel is used while end point constraints are satisfied. The method of solution was based on linearization of the equations

## GUIDANCE AND CONTROL

of motion and on the application of Pontryagin's maximum principle. The generality of the results and the ease of application of the maximum principle recommend the methods used for similar optimization problems, that is, optimization of linear time-varying systems with specified boundary conditions.

### APPENDIX A: PERTURBATION EQUATIONS

Rewriting Eq. 3

$$\frac{dx}{dt} = \frac{\partial f_i}{\partial X_j} dx_j + \frac{\partial f_i}{\partial U_k} du_k \quad \begin{matrix} i, j=1,2,\dots,5 \\ k=1,2 \end{matrix}$$

The values for  $\partial f_i / \partial X_j$  and  $\partial f_i / \partial U_k$  are

$$\frac{\partial f_3}{\partial U_1} = \frac{-V_e \dot{M}}{M} \sin B \quad \frac{\partial f_3}{\partial U_2} = \frac{V_e}{M} \cos B \quad \frac{\partial f_4}{\partial U_1} = \frac{V_e \dot{M}}{M} \cos B$$

$$\frac{\partial f_4}{\partial U_2} = \frac{V_e}{M} \sin B \quad \frac{\partial f_5}{\partial U_2} = -1$$

$$\frac{\partial f_3}{\partial Z} = \frac{qs}{m_p} \frac{dp}{dZ} (C_D \cos \Gamma + C_L \sin \Gamma)$$

$$\frac{\partial f_4}{\partial Z} = \frac{qs}{m_p} \frac{dp}{dZ} (-C_D \sin \Gamma + C_L \cos \Gamma) + \frac{2gr_e^2}{(r_e + Z)^3}$$

$$\frac{\partial f_3}{\partial M} = \frac{qs}{M^2} (C_D \cos \Gamma + C_L \sin \Gamma) - \frac{V_e \dot{M}}{M^2} \cos B$$

$$\frac{\partial f_4}{\partial M} = \frac{qs}{M^2} (C_D \sin \Gamma + C_L \cos \Gamma) - \frac{V_e \dot{M}}{M^2} \sin B$$

$$\frac{\partial f_3}{\partial V_X} = \frac{-qs}{MV} \left[ 2(C_D \cos \Gamma + C_L \sin \Gamma) \frac{V_X}{V} + \left( \frac{\partial C_D}{\partial \alpha} \cos \Gamma + \frac{\partial C_L}{\partial \alpha} \sin \Gamma \right) \sin \Gamma \right]$$

$$\frac{f_4}{V_X} = \frac{-qs}{MV} \left[ 2(C_L \sin \Gamma - C_D \cos \Gamma) \frac{V_X}{V} + \left( \frac{\partial C_D}{\partial \alpha} \sin \Gamma - \frac{\partial C_L}{\partial \alpha} \cos \Gamma \right) \sin \Gamma \right]$$



## GUIDANCE AND CONTROL

$$\frac{\partial f_3}{\partial V_Z} = \frac{-qs}{MV} \left[ 2(C_D \cos \Gamma + C_L \sin \Gamma) \frac{V_Z}{V} - \left( \frac{\partial C_D}{\partial \alpha} \cos \Gamma + \frac{\partial C_L}{\partial \alpha} \sin \Gamma \right) \cos \Gamma \right]$$

$$\frac{\partial f_4}{\partial V_Z} = \frac{-qs}{MV} \left[ 2(C_D \sin \Gamma - C_L \cos \Gamma) \frac{V_Z}{V} - \left( \frac{\partial C_D}{\partial \alpha} \sin \Gamma - \frac{\partial C_L}{\partial \alpha} \cos \Gamma \right) \cos \Gamma \right]$$

The adjoint set to Eq. 3 is written as  $d\vec{p}/dt = -\vec{p}^T A$ . The components are

$$\begin{aligned} \dot{p}_1 &= 0 \\ \dot{p}_2 &= p_3 \frac{\partial f_3}{\partial Z} + p_4 \frac{\partial f_4}{\partial Z} \\ \dot{p}_3 &= p_1 + p_3 \frac{\partial f_3}{\partial V_x} + p_4 \frac{\partial f_4}{\partial V_x} \\ \dot{p}_4 &= p_2 + p_3 \frac{\partial f_3}{\partial V_Z} + p_4 \frac{\partial f_4}{\partial V_Z} \\ \dot{p}_5 &= p_3 \frac{\partial f_3}{\partial B} + p_4 \frac{\partial f_4}{\partial B} \\ \dot{p}_6 &= 0 \end{aligned}$$

### APPENDIX B: DERIVATION OF LINEAR PREDICTION EQUATIONS

The linear differential equations describing the system are in vector and matrix form

$$d\vec{x}/dt = A(t) \vec{x} + B(t) \vec{u}(t) \quad [B-1]$$

Changing the independent variable from  $t$  to  $s = T-t$ , where  $T$  is a fixed time, Eq. B-1 becomes

$$d\vec{x}/ds = -A(s) \vec{x} - B(s) \vec{u}(s) \quad [B-2]$$

The homogeneous part of Eq. B-2 has a normalized fundamental solution matrix  $\Phi(s)$  which satisfies the equation  $d\Phi(s)/ds = -A(s)\Phi(s)$   $\Phi(0)=I$  [B-3]

The inverse of the normalized fundamental solution matrix  $\Phi^{-1}(s)$  satisfies the equation  $d\Phi^{-1}(s)/ds = \Phi^{-1}(s)A(s)$   $\Phi^{-1}(0)=I$  [B-4]

## GUIDANCE AND CONTROL

Eq. B-4 is derived by noting that

$$\Phi^{-1}(s)\Phi(s) = I$$

$$\frac{d}{ds} \Phi^{-1}(s)\Phi(s) = \frac{d\Phi^{-1}(s)}{ds} \Phi(s) + \Phi^{-1}(s) \frac{d\Phi}{ds} = 0$$

But from Eq. B-3,  $d\Phi(s)/ds = -A(s)\Phi(s)$ , so that, substituting for  $d\Phi(s)/ds$  and post-multiplying by  $\Phi^{-1}(s)$

$$d\Phi^{-1}/ds = \Phi^{-1}(s)A(s)$$

which is Eq. B-4. The vector equation associated with Eq. B-4 is

$$d\vec{p}/ds = \vec{p} A(s) \quad [B-5]$$

where  $\vec{p}$  is the adjoint set to  $\vec{x}$  and is a row vector.

To include the forcing function vector  $\vec{u}(s)$ , post-multiply Eq. B-4 by  $\vec{x}$  and pre-multiply Eq. B-2 by  $\Phi^{-1}(s)$ , giving

$$d\Phi^{-1}(s)/ds \vec{x} = \Phi^{-1}(s)A(s) \vec{x}$$

$$\Phi^{-1}(s) d\vec{x}/ds = -\Phi^{-1}(s)A(s)\vec{x} - \Phi^{-1}(s)B(s)\vec{u}(s)$$

Adding these two equations gives

$$d/ds (\Phi^{-1}(s)\vec{x}) = -\Phi^{-1}(s)B(s)\vec{u}(s)$$

Integrating

$$\Phi^{-1}(s)\vec{x}(s) = \Phi^{-1}(0)\vec{x}(0) - \int_0^s \Phi^{-1}(s')B(s')\vec{u}(s')ds'$$

but  $\Phi^{-1}(0) = I$ , and

$$\vec{x}(0) = \Phi^{-1}(s)\vec{x}(s) + \int_0^s \Phi^{-1}(s')B(s')\vec{u}(s')ds'$$

$$\text{or } \vec{x}(T) = \Phi^{-1}(t)\vec{x}(t) + \int_t^T \Phi^{-1}(\tau)B(\tau)\vec{u}(\tau)d\tau \quad [B-6]$$

where  $\Phi^{-1}(t)$  has been obtained by integrating back from  $t=T$ .

### ACKNOWLEDGMENT

The numerical solutions to the linear equations of motion were obtained by Michael Ward. Much of the problem formulation is due to Arnold Peske; the background on linear prediction is due to Richard Kiene, William Marshall, and Wallace Ito. The

## GUIDANCE AND CONTROL

application of the theory of optimum systems has benefited greatly from discussions with Dahlard Lukes, Wallace Ito, and many others in the Military Products Group, Research Dept.

### NOMENCLATURE

$\vec{A}$	= aerodynamic force vector
B	= Euler angles of vehicle attitude
D	= drag force along velocity vector
$E^n$	= n-dimensional Euclidian space
$\vec{G}$	= gravitational force vector
$\vec{g}(t)$	= disturbance vector
L	= lift force normal to velocity vector
M	= mass
$\dot{M}$	= mass flow rate
$\vec{N}$	= aerodynamic torque vector
$\vec{p}$	= state vector adjoint to $\vec{x}$
S	= linear combination of functionals
T	= thrust
$U_1^h(t)$	= reference thrust angle deflection
$U_1^m(t)$	= reference mass flow rate
$V^2$	= characteristic velocity
$\alpha^e$	= angle of attack
	= flight path angle in vertical plane

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2 Rozonoer, L.I., "L.S. Pontryagin's maximum principle in the theory of optimum systems", Part I, Part II, Avtomatika i Telemekhanika 20, Nov. 1959.

GUIDANCE AND CONTROL

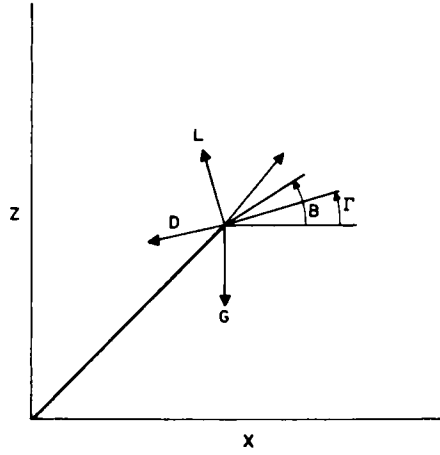


Fig. 1 Geometry of reference system

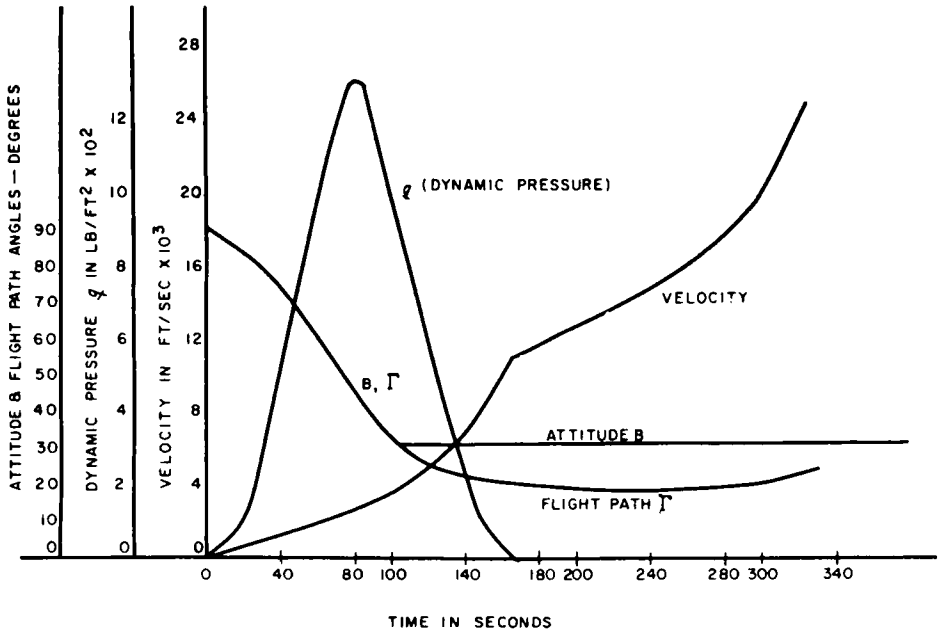


Fig. 2 Reference trajectories

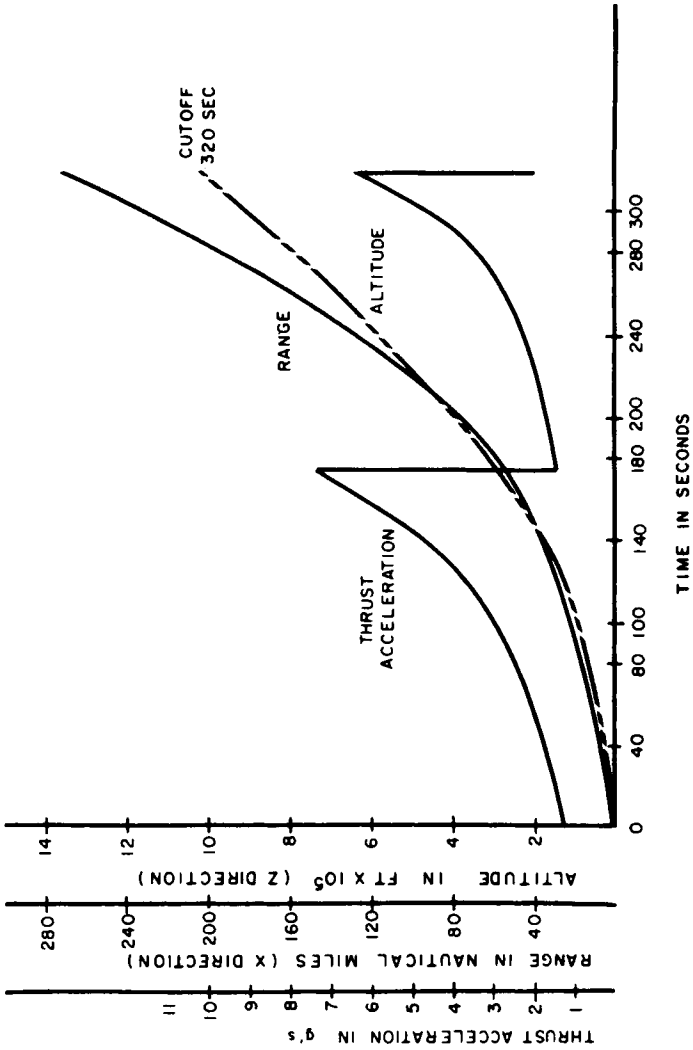


Fig. 3 Reference trajectories

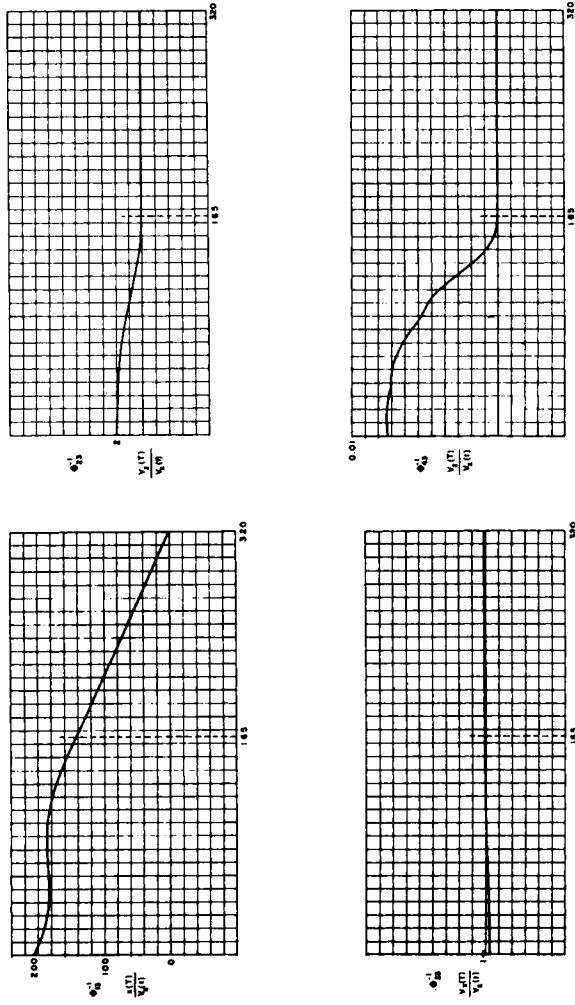


Fig. 4 Influence functions  $\Phi_3^{-1}$

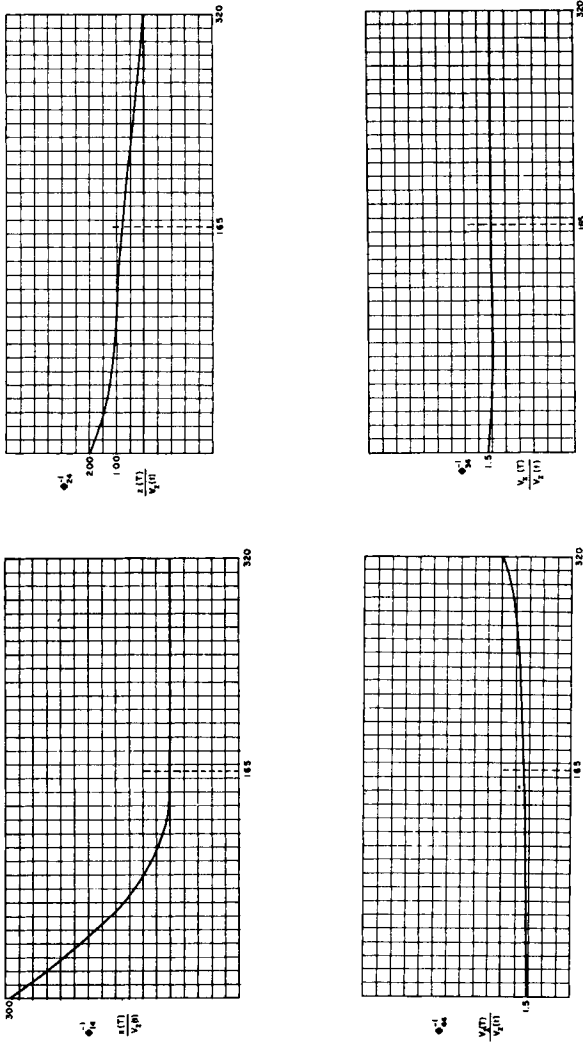


Fig. 5 Influence functions  $\Phi_4^{-1}$

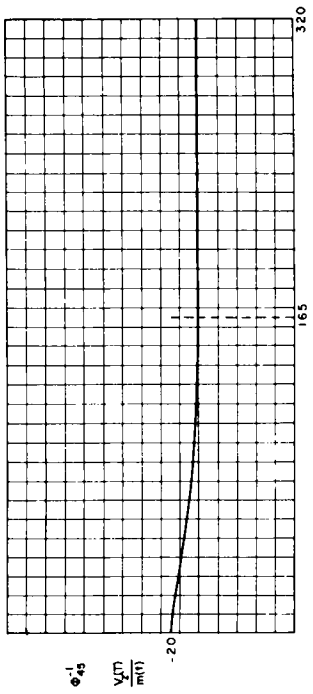
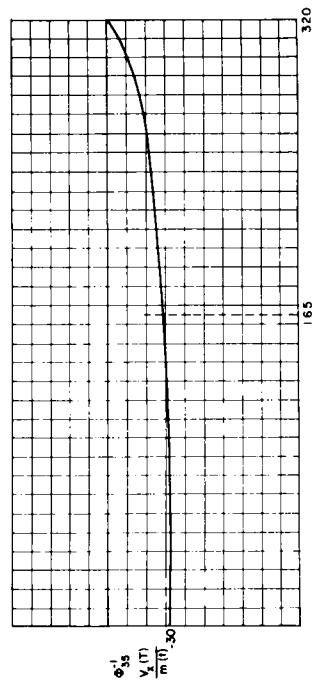
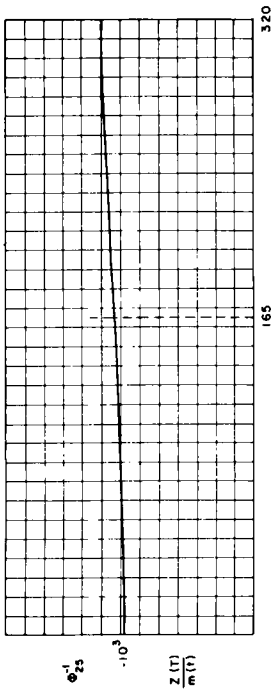
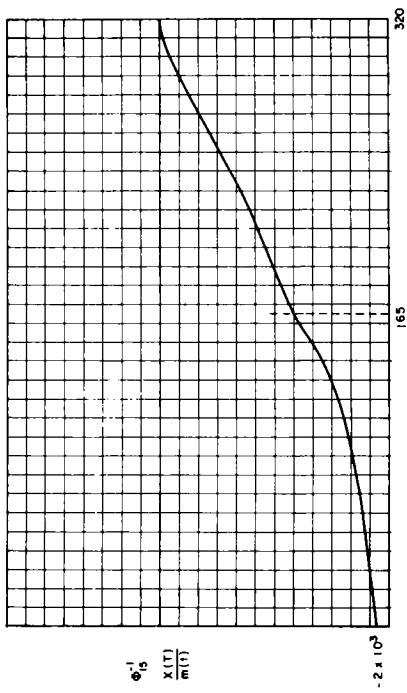


Fig. 6 Influence functions  $\Phi_y^{-1}$