

TECHNIQUES FOR ERROR
ANALYSIS OF TRAJECTORIES

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ABSTRACT

An error analysis technique is presented which can be used to predict the propagation of errors along a trajectory in terms of the error vector at the end of the trajectory. The method, called the conjugate solution method, uses perturbation theory and assumes that the equations of disturbed motion result in a solution that remains in a region of linear approximation about a given nominal trajectory. The error analysis technique can be used to predict the effect of random errors as well as non-random errors. As an example, the technique is applied to the analysis of error propagation along a typical ICBM trajectory. The application of the conjugate solution technique and conventional techniques to similar problems are evaluated qualitatively through a comparison of computer requirements and in the ease of evaluating the data.

INTRODUCTION

An investigation of the performance of a complex physical system generally involves the solution of a set of equations that describe the system mathematically. It is usually necessary to make approximations and assumptions in the mathematical statement of the system to simplify the equations. For example, in the preliminary design of a ballistic missile a relatively simple set of differential equations is derived from physical laws. These equations are set up in a suitable coordinate system, and solutions are found. Some of the assumptions are:

- 1) System parameters are known or can be measured exactly.
- 2) Certain random disturbing forces in the measurement of the system parameters, e.g., radar noise, are absent.

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3) Gravitational and atmospheric forces can be computed exactly.

4) Random disturbing forces external to the system, e.g., wind or deviations from a standard atmosphere, are absent.

It is thus necessary to evaluate the system by determining the effect of these simplifications in the form of an error analysis. From a knowledge of the errors introduced in a system, component tolerances can be specified, and a system can be designed that meets a desired performance specification.

Another factor that arises in the analysis of error propagation along trajectories, e.g., during the boost phase of a ballistic missile, is that the equations of motion have time-varying coefficients. Thus the conventional frequency transform methods are not applicable, and analysis techniques must be based on time-domain descriptions of the system and the errors.

METHODS OF ERROR ANALYSIS

There are several methods available for time-domain analysis of systems described by differential equations with time-varying coefficients. Evaluation of these systems requires solution by digital computers.

These methods are described in literature dealing with particular missile systems. The methods have general application, although they are described here with application to the boost phase of a ballistic missile. Moore (1)³ describes one technique that uses the original system equations with application to the propagation of burnout errors to impact time. Perturbation methods are described by Anderson (2) in evaluating inertial guidance systems, and by Rosenberg (3) for computing trajectories in the neighborhood of a nominal trajectory. The methods using adjoint techniques are described by Bliss (4) with application to free fall missile ballistics, by Marshall (8) with an extension to the evaluation of errors introduced by uncertainties in physical constants, and by Pfeiffer (6) with application to the guidance problem for a ballistic missile. Laning and Battin (7) describe application of adjoint techniques to the study of random inputs to linear systems. The purpose of this paper is to discuss the several methods and to evaluate them as an aid in selecting the method best suited to a particular problem.

³Numbers in parentheses indicate References at end of paper.

One method that is generally used involves the repeated solution of the nonlinear equations of motion of the system. The results are used for investigating: 1) propagation of initial condition errors and guidance sensitivities; 2) evaluation of instrument errors; and 3) evaluation of CEP.

Another method makes the assumption of linear perturbations about a nominal trajectory. A set of perturbation equations is derived from the nonlinear equations of motion, and many solutions of these equations are found. The results are used for investigation: 1) propagation of initial condition errors; 2) evaluation of instrument errors; and 3) evaluation of CEP.

The conjugate solution method emphasized in this paper uses the linear approximation of perturbation techniques and requires the solution to the adjoint equation. The result is the conjugate, or inverse, fundamental solution matrix to a set of linear differential equations with time-varying coefficients. This solution is basic in solving for: 1) propagation of initial condition errors; 2) a set of "guidance sensitivity" functions; 3) propagation of errors due to small forcing functions that affect the system; 4) evaluating a CEP for a missile system from knowledge of expected accuracy of measurement instruments; and 5) analysis of random inputs that affect the system.

The conjugate solution method has two major advantages over either of the other two methods: 1) fewer solutions of the equations of motion or the perturbation equations are needed for the error propagation and guidance sensitivity studies than in either of the other methods (reducing the cost of analysis by cutting down on computation time); and 2) the fundamental solution matrix used for the error propagation is also used in the guidance sensitivity studies (giving a smooth transition from the study of error propagation of the computation of sensitivities). There is no need for additional derivations as is necessary in the other methods.

Generalized System Equations

A system is considered which is described by a set of n first-order, ordinary differential equations

$$\frac{dx_1}{dt} = f_1(x_1, x_2, \dots, x_n, t) \quad [1a]$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, \dots, x_n, t) \quad [1b]$$

$$\frac{dx_n}{dt} = f_n(x_1, x_2, \dots, x_n, t) \quad [1c]$$

The coordinates x_i are chosen so that the rate of change of the coordinates is adequately described by functions of the coordinates and time; i.e., these coordinates are a good representation of the system parameters of interest. The equations are in general, nonlinear and must be solved on a digital computer. The coordinates x_i in the case of a ballistic missile during injection, for example, could be position coordinates, the velocity coordinates, vehicle attitude and attitude rate, mass and mass rates. The particular coordinates chosen will depend on factors such as the precision of control of the system and the feasibility of measurement.

A system of equations of order higher than first can be reduced to a first-order system by defining new coordinates. For example

$$\frac{d^2x}{dt^2} = f(x, t) \quad [2]$$

This can be reduced to a system of first order differential equations by defining

$$x_1 = x \quad [3a]$$

$$x_2 = \frac{dx_1}{dt} = f_1(x_1, x_2, t) \quad [3b]$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2, t) = f(x, t) \quad [3c]$$

It is convenient to write a system of equations such as Eqs. 1 in a more compact form, using matrix notations

$$x = [x_1 \ x_2 \ \dots \ x_n]^T \quad f = [f_1 \ f_2 \ \dots \ f_n]^T$$

Then Eqs. 1 can be written in the form

$$\frac{dx}{dt} = f(x, t) \quad [4]$$

Let Eqs. 1 or 4 represent the motion of a ballistic missile during injection. For a particular set of initial conditions on the n coordinates, a solution $\hat{x}(t)$ will be found. The trajectory $\hat{x}(t)$ may be optimized in some manner, such as requiring minimum fuel. By imposing constraints on the coordinates, e.g., programmed thrust for a particular attitude profile, the trajectory will terminate at some prescribed point

in n space $\hat{x}(T)$. This trajectory represents the performance of the missile within the accuracy limits of the mathematical description of the system.

Suppose that there exists a neighboring solution that is displaced by a small amount from the nominal trajectory $\hat{x}(t)$. This means that $x_1 - \hat{x}_1$ are bounded by some small positive constant along the trajectory in the interval of interest. Then

$$\frac{d\hat{x}}{dt} = f(\hat{x}, t)$$

$$\frac{dx}{dt} = f(x, t)$$

Conjugate Solution Method of Error Analysis

The equations of disturbed motion, or the first variational equations, about $\hat{x}(t)$ can be obtained by expanding Eq. 1 in a Taylor's series about $\hat{x}(t)$ and truncating after the first-order terms. This gives the linear vector equation

$$\frac{d\Delta x}{dt} = A(t) \Delta x(t) \quad [5]$$

where

$$x - \hat{x} = \Delta x = [\Delta x_1 \quad \Delta x_2 \dots \Delta x_n]^T$$

$$[A]_{ij} = \frac{\partial f_i}{\partial x_j} = a_{ij}(t)$$

Eqs. 5 are a set of linear, first-order differential equations with time-varying coefficients. They represent the perturbed motion, in a linear sense, of the vehicle about the nominal trajectory $x(t)$. The coefficients $a_{ij}(t)$ are evaluated from a knowledge of the parameters of the reference trajectory, such as velocity, position, attitude, and acceleration.

A set of n independent solutions to Eq. 5, $\Delta x^{(1)}(t)$, $\Delta x^{(2)}(t), \dots, \Delta x^{(n)}(t)$, and arranged in a matrix gives

$$\pi(t) = [\Delta x^{(1)}(t) \quad \Delta x^{(2)}(t) \dots \Delta x^{(n)}(t)] \quad [6]$$

For the particular set of boundary conditions $\pi(t) = I$, the $n \times n$ identity matrix $\pi(t)$ is a fundamental solution matrix to Eq. 5, and $\pi(t)$ will satisfy Eq. 5, namely

$$\frac{d\pi}{dt} = A(t) \pi(t) \quad [7]$$

and for any boundary condition $\Delta x(t=T)$

$$\Delta x(t) = \pi(t) \Delta x(T) \quad [8]$$

Here, the boundary conditions are set up at $t = T$, since a systems analysis is primarily concerned with the value of the miss vector at the end of a trajectory. If $\pi(t)$ is a nonsingular matrix, the inverse matrix $\pi^{-1}(t)$ exists. Premultiplying Eq. 8 by $\pi^{-1}(t)$

$$\pi^{-1}(t) \Delta x(t) = \Delta x(T) \quad [9]$$

since

$$\pi^{-1}(t) \pi(t) = \pi(t) \pi^{-1}(t) = I \quad [10]$$

the $n \times n$ identity matrix, by definition of the inverse matrix, for all t . Thus, the terminal miss vector $\Delta x(T)$ can be evaluated for any measured perturbation on the trajectory, by the matrix multiplication indicated by Eq. 9. However, this involves inversion of the fundamental solution matrix $\pi(t)$ for all points of interest. The inverse, or conjugate, fundamental solution matrix can be obtained for all time t in the interval $[0, T]$ by the solution of a set of linear differential equations.

Differentiating Eq. 10 with respect to time gives

$$\frac{d}{dt} [\pi(t) \pi^{-1}(t)] = \frac{d\pi}{dt} \pi^{-1}(t) + \pi(t) \frac{d\pi^{-1}}{dt} = 0 \quad [11]$$

where 0 is the $n \times n$ null matrix. Post multiply Eq. 7 by $\pi^{-1}(t)$

$$\frac{d\pi}{dt} \pi^{-1}(t) = A(t) \quad [12]$$

From Eqs. 11 and 12

$$-\pi(t) \frac{d\pi^{-1}}{dt} = A(t) \quad [13]$$

Premultiplying Eq. 2 by $-\pi^{-1}(t)$ gives

$$\frac{d\pi^{-1}}{dt} = -\pi^{-1}(t) A(t) \quad [14]$$

Eq. 14 is of the form Eq. 7, i.e., $\pi^{-1}(t)$ can be considered to be the fundamental solution matrix to a set of differential equations, as yet, unspecified. Postulating a set of linear differential equations to correspond to Eq. 14; let

$$\frac{d\lambda}{dt} = -\lambda(t)A(t) \quad [15]$$

Here, $\lambda(t) = [\lambda_1(t) \lambda_2(t) \dots \lambda_n(t)]$ is a row vector. Now arranging n independent solutions of Eq. 15 in a matrix

$$\pi^{-1}(t) = [\lambda^{(1)}(t) \lambda^{(2)}(t) \dots \lambda^{(n)}(t)]^T$$

where superscript T indicates the transposed vector it is seen that for the particular boundary conditions $\pi^{-1}(T) = I$, $\pi^{-1}(t)$ is a fundamental solution matrix to Eq. 15. Since $\pi^{-1}(t)$ is the inverse to $\pi(t)$, $\pi^{-1}(t)$ is called the inverse, or conjugate, fundamental solution matrix.

Thus it is seen that $\pi^{-1}(t)$ can be generated as a continuous function of time, by solving Eq. 15 for the n boundary conditions, such that $\pi^{-1}(T) = I$. Polynomials can be fitted to the tabulated values of $\pi^{-1}(t)$ for convenience as desired for later parts of the analysis. Then, from Eq. 9, the terminal miss vector can be evaluated by the indicated matrix multiplication. Inversion of the matrix $\pi(t)$ is not required. In fact, it is not necessary, at this point, to have the solution matrix $\pi(t)$ to Eq. 5. It is only necessary to solve for the n solutions of Eq. 15. The system of equations, Eq. 15, is the adjoint equation to Eq. 5. Eq. 9 is a basic formula for the conjugate solution technique of error analysis.

If Eq. 9 is written in component form, the usefulness of the variable $\lambda(t)$ can be seen. Let the j th component of the vector $\lambda^{(i)}(t)$ be denoted by $\lambda_{ij}(t)$. Then $[\pi^{-1}(t)]_{ij} = \lambda_{ij}$, and expanding Eq. 9

$$\sum_{j=1}^n \lambda_{ij}(t) \Delta x_j(t) = \Delta x_i(T) \quad [16]$$

That is $\Delta x(t = T)$ is a function of Δx_i and t , and if the partial derivatives of $\Delta x_i(T)$ are taken with respect to $\Delta x_j(t)$ for fixed time t , it is seen that

$$\lambda_{ij}(t) = \frac{\partial \Delta x_i(T)}{\partial \Delta x_j(t)} \quad [17]$$

Thus, the functions $\lambda_{ij}(t)$ can be regarded as sensitivity functions.

The boundary conditions are set up at time T , i.e., at the end of the trajectory, and Eq. 15 is solved backwards in time. Solution with the boundary condition, $\pi^{-1}(T) = I$, gives the result that $\lambda_{ij}(t)$ is the value of the error component $\Delta x_i(T)$ due to a unit error $\Delta x_j(t)$. Then, since Eq. 15 is linear, a measured error $\Delta x(t)$ of arbitrary magnitude is scaled according to Eq. 9 to give the resultant terminal error.

Alternative Methods of Error Analysis

For the first of these other methods, the missile is assumed to be described by the system of equations Eqs. 1. A nominal trajectory $\hat{x}(t)$ that meets some performance criterion is solved. Then using small perturbations of the initial conditions $\Delta x_i^0 = x_i(t_0) - \hat{x}_i(t_0)$, the nonlinear set of equations is again solved. This is done repeatedly for many perturbed initial conditions about the nominal trajectory. A functional form is assumed for the burnout error, for example

$$\Delta x_i(T) = \sum_{j=1}^n K_{ij} \Delta x_j^0 \quad [19]$$

The form of these equations could be expanded to include second-order terms, but for evaluating system accuracy, a linear set of equations such as Eqs. 19 is usually assumed. The data from the solutions to Eq. 1 for the various initial conditions is fitted to the Eq. 19 with conventional curve fitting techniques, e.g., using weighted least squares techniques. It should be noted that the coefficients K_{ij} only hold for the nominal trajectory, and for only one particular time. To get a set of functions equivalent to the λ_{ij} of Eqs. 19, this process would have to be repeated with new initial conditions corresponding to points along the trajectory. Enough points are taken so curves can be fit through them. Then the coefficients can be fitted with polynomials.

Alternatively, in the perturbation method described above, Eqs. 1 are solved to give the nominal trajectory $\hat{x}(t)$. Then, a set of perturbation equations are derived from the nonlinear set. This gives the system of equations, Eqs. 5. For each time of interest, n solutions to Eqs. 5 are obtained to give a set of linear predictor coefficients as in Eqs. 19. Eq. 5 is solved repeatedly to obtain the coefficients K_{ij} as a function of time along the trajectory, since for each time (t_0) only the coefficients K_{ij} for $t = t_0$ are obtained.

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The advantages of the conjugate solution matrix at this point are:

1) The error propagation coefficients, the sensitivity functions, are obtained by one set of n solutions to the adjoint set of differential equations.

2) Less data processing is necessary, because only the reference trajectory and the sensitivity functions need to be curve fitted.

EVALUATION OF ERRORS DUE TO DISTURBING FORCES ALONG THE TRAJECTORY

In general, the perturbation equations are a set of non-homogeneous equations

$$\frac{d\Delta x}{dt} = A(t) \Delta x(t) + B(t)u(t) \quad [20]$$

The disturbance function $u(t)$ is quite general and can represent many things of interest. For analysis of the control system, $u(t)$ can represent disturbing forces, such as wind shears or motor vibrations, or can represent control forces themselves. For analysis of the guidance system, $u(t)$ can represent position and velocity uncertainties of radar or optical measurements were used. For inertial measurements, $u(t)$ can represent errors due to drift in the reference platform and errors due to bias, nonlinearity, etc., in the velocity sensors. These forces may be included in the original formulation of the system equations Eqs. 1. The $\pi(t)$ matrix is then computed from the homogeneous part of the nonhomogeneous set that results from the truncated Taylor's series expansion. The matrix $B(t)$ is an $n \times n$ matrix that couples these disturbing forces or control forces to the system, as coordinate transformations. By proper choice of the coordinates of the system, Eq. 20 may be written

$$\frac{d\Delta x}{dt} = A(t) [\Delta x(t) + u(t)] \quad [21]$$

The effect of these disturbing forces, or forcing functions, can be evaluated from the equation (see Ref. 8, pp.

$$\Delta x(t) = \pi(t) \int_{t_0}^t \pi^{-1}(\tau) A(\tau) u(\tau) d\tau + \pi(t) \pi^{-1}(t_0) \Delta x(t_0) \quad [22]$$

$\pi^{-1}(t_0) \Delta x(t_0)$ represents the initial conditions for the integration of this equation. Eq. 22 is the second basic

equation for the conjugate solution method of error analysis. It is the equation that is used to solve for the terminal miss vector due to any deterministic disturbing forces that affect the missile along the trajectory. In particular, for $t = T$, noting that $\pi(T) = I$

$$\Delta x(T) = \int_{t_0}^T \pi^{-1}(\tau) A(\tau) u(\tau) d\tau + \pi^{-1}(t_0) \Delta x(t_0) \quad [23]$$

To compute the propagation of errors due to forcing functions $u(t)$ which are functions of time in the interval $[0, T]$, it is necessary to have the solution $\pi(t)$ to Eqs. 5 and to use Eq. 22. This requires n computer solutions to Eqs. 5 corresponding to the boundary conditions $\pi(T) = I$. For errors at burnout, $t = T$, no additional preliminary computations are necessary to solve Eqs. 23, since all the functions required for the equation are known from the previous analysis.

For each additional input $u(t)$, a set of equations is derived to determine how the function $u(t)$ affects the system. The errors introduced are assumed small, so that linear approximations can be made. The result is a system of linear differential equations, which are solved to determine the effect of the forcing functions. As an illustration, the effect is considered of accelerometer bias and misorientation of the inertial platform.

The basic equation is of the form

$$A_T = A_a + g \quad [24]$$

where A_T is the true acceleration acting on the system, A_a is the acceleration sensed by the accelerometers, and g is the gravitational acceleration. If the deviation from the true acceleration is denoted by ΔA_a , then

$$\Delta A_T = \Delta A_a + \Delta g \quad [25]$$

The term Δg enters because an error in sensed acceleration causes an error in velocity and position calculation, and hence, in the computation of the gravitational force. The error introduced by misorientation of the accelerometers can be denoted by the transformation $\Delta \phi X A_a$, where $\Delta \phi$ is the matrix that transforms the measured acceleration from the erroneous coordinates of the platform to the desired coordinates. The resultant equations are set of linear vector differential equations, which are of the form

$$\frac{d\Delta x}{dt} = g(\Delta x, t) \quad [26]$$

Additional terms for each of the error sources, as acceleration and nonacceleration sensitive gyro drift and accelerometer errors are added to Eqs. 26. If the error sources are considered independent, separate equations can be derived for the individual error sources. Eqs. 26 are then solved to evaluate errors due to these error sources.

Comparing this method with the conjugate solution method, it is seen that for arbitrary deterministic inputs, additional equations must be solved in both cases. In the conjugate solution method, the equation is Eq. 22. For the conventional methods, the equation is a linear differential equation of the form of Eq. 26. If the nonhomogeneous equation, in the conjugate solution method, is of the form of Eq. 20, the derivation of the transformation matrix $B(t)$ is equivalent to the derivations required in the other methods which lead to the differential equations, Eqs. 26. However, by judicious choice of the system coordinates in the original formulation of the system equations, Eqs. 1 no additional derivations are necessary in the conjugate solution method, since Eq. 21 can be used.

Here again, the conjugate solution method has advantages over the conventional method. The solution of the perturbation equations for the fundamental and inverse fundamental solution matrices leads directly to a set of vector equations, Eqs. 22, which are used to evaluate the errors due to small arbitrary deterministic inputs to the system.

EVALUATION OF THE CIRCULAR PROBABLE ERROR

When certain parameters of the system have a probability uncertainty, only a statistical knowledge of the errors caused by these parameters can be found. The CEP, circular probable error, is defined as the circle about the terminal point where the probability that the trajectory ends within this circle is 0.50. For evaluation of CEP, all of the analysis methods described above use equivalent techniques.

It is supposed that there are a number of independent error sources characterized by coefficients that are fixed for a particular source, but that are statistically distributed for an ensemble of sources. It is further supposed that the distribution of each of these sources has a zero mean value. Examples are gyro drift coefficient and accelerometer nonlinearity coefficient.

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From an analysis of the sources acting on the system, the contribution to the terminal error from each source will be found. If the sources are independent, the total average error from the sources will be the sum of the contribution of the average of the individual sources.

The Central Limit Theorem states that for a large number of independent random functions, the probability distribution of the sum of these random functions approaches a normal distribution regardless of the distribution of the individual sources. By the Central Limit Theorem, the terminal error due to all of the previously mentioned sources will be approximately normally distributed. Also, since the sources are independent, the variance of the total error will be the sum of the individual variances. It should be noted that the variance, for a random function with zero mean value, equals the mean squared value of the function. With the mean value and the mean squared value of the individual sources, the multidimensional normal distribution for the output can be determined. Thus, the CEP can be computed by integration of the multivariate normal distribution function to determine the value for which the miss probability is 0.50.

STOCHASTIC INPUTS

One of the principal advantages of the conjugate solution method is that the equations can be extended to stochastic inputs. Consideration is given to Eq. 22, for which $u(t)$ is a random function for which the statistical properties, in the form of a correlation matrix, are known. The following derivation shows how the mean squared error at time T due to random inputs can be evaluated.

The mean or expected value of a function $f[x(t)]$ is defined as

$$E [f(x)] = \int_{-\infty}^{\infty} \rho(x,t) f(x) dx \quad [27]$$

where $\rho(x,t)$ is the probability density function of $x(t)$. For a function of n variables $f(x_1, x_2, \dots, x_n)$, the expected value of the function is defined as

$$E [f(x_1, x_2, \dots, x_n)] = \underbrace{\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty}}_n (x_1, t_1; x_2, t_2; \dots; x_n, t_n) f(x_1, x_2, \dots, x_n) dx_1, dx_2, \dots, dx_n \quad [28]$$

where $\rho(x_1, t_1; x_2, t_2; \dots; x_n, t_n)$ is the joint probability density function of the variables x_i .

From Eq. 23

$$\Delta x(T) = \int_0^T \pi^{-1}(\tau) A(\tau) u(\tau) d\tau \quad [29]$$

where the initial condition notation is dropped for convenience in the derivation.

One component $\Delta x_i(T)$ is taken of the function $\Delta x(T)$

$$\Delta x_i(T) = \int_0^T \sum_{j=1}^n \sum_{k=1}^n \lambda_{ij}(\tau) a_{jk}(\tau) u_k(\tau) d\tau \quad [30]$$

Let

$$\sum_{j=1}^n \lambda_{ij}(\tau) a_{jk}(\tau) = c_{ik}(\tau) \quad [31]$$

Then

$$\Delta x_i(T) = \int_0^T \sum_{k=1}^n c_{ik}(\tau) u_k(\tau) d\tau \quad [32]$$

Squaring Eq. 32

$$[\Delta x_i(T)]^2 = \int_0^T \sum_{k=1}^n c_{ik}(\tau) u_k(\tau) d\tau \int_0^T \sum_{l=1}^n c_{il}(s) u_l(s) ds \quad [33]$$

$$= \int_0^T \int_0^T \sum_{k=1}^n \sum_{l=1}^n c_{ik}(\tau) c_{il}(s) u_k(\tau) u_l(s) d\tau ds \quad [34]$$

The expected value of $[\Delta x_i(T)]^2$ is

$$E[\Delta x_i(T)]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \rho(u_1, u_2, \dots, u_n) \left\{ \int_0^T \int_0^T \sum_{k=1}^n \sum_{l=1}^n c_{ik}(\tau) c_{il}(s) u_k(\tau) u_l(s) d\tau ds \right\} du_1 du_2 \dots du_n \quad [35]$$

It is necessary to use the following relation for joint probability density functions:

$$\int_{-\infty}^{\infty} \rho(u_1, t_1; u_2, t_2) du_2 = \rho(u_1, t_1) \quad [36]$$

which is readily expanded to the case of an n-variable distribution function.

Using Eq. 36 in Eq. 35

$$E[\Delta x_i(T)]^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(u_k, \tau; u_l, s) \left\{ \int_0^T \int_0^T \sum_{k=1}^n \sum_{l=1}^n c_{ik}(\tau) c_{il}(s) u_k(\tau) u_l(s) d\tau ds \right\} du_k du_s \quad [37]$$

Interchanging the orders of integration and using the fact that

$$\phi_{u_1 u_2}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(u_1, t_1; u_2, t_2) u_1(t_1) u_2(t_2) du_1 du_2 \quad [38]$$

is the correlation function of the variables $u_1(t_1)$, $u_2(t_2)$

$$E[\Delta x_i(T)]^2 = \sum_{k=1}^n \sum_{l=1}^n \int_0^T \int_0^T c_{ik}(\tau) c_{il}(s) \phi_{u_k u_l}(\tau, s) d\tau ds \quad [39]$$

The following cases are of interest:

1) u_i are uncorrelated white noise

$$\phi_{u_k u_l}(\tau, s) = \delta_{kl}(\tau - s) \quad [40]$$

where $\delta_{kl}(\tau - s)$ is the dirac delta function, then Eq. 40 becomes

$$E[\Delta x_i(T)]^2 = \sum_{k=1}^n \int_0^T [c_{ik}(\tau)]^2 d\tau \quad [41]$$

2) u_i are uncorrelated stationary random functions, i.e., all cross-correlation functions are zero

$$E[\Delta x_i(T)]^2 = \sum_{k=1}^n \int_0^T \int_0^T c_{ik}(\tau) c_{il}(s) \phi_{u_k u_l}(\tau - s) d\tau ds \quad [42]$$

3) u_i are random functions with known correlation functions

$$E [\Delta x_i(T)]^2 = \sum_{k=1}^n \sum_{l=1}^n \int_0^T \int_0^T c_{ik}(\tau) c_{il}(s) \phi u_k u_l(\tau, s) d\tau ds \quad [43]$$

It should be noted that if u_k are stationary

$$\phi u_k u_s(\tau, s) = \phi u_k u_s(\tau - s) \quad [44]$$

Eq. 39 or the equivalent forms, Eqs. 41-43, as required for the particular random functions of interest, is used to evaluate the mean squared output error for the case of random inputs to the system.

The conjugate solution method is readily extended to the case of inputs that have known correlation functions. Applications are calculation of the mean squared error at burn-out conditions or propagation of the mean squared error due to vibration environments, atmospheric turbulence and rocket motor noise, and random variations in the measurement instruments.

The conventional methods of error analysis do not have an analogous method for computing the mean squared error due to random functions acting along the trajectory. Instead, in the analysis of the propagation of random functions for particular systems, it has been shown that the standard deviation propagates according to some functional form. For example, for a gaussian random process, in a time-invariant system, or for a gaussian random process through a single integration, the standard deviation propagates in proportion to $t^{1/2}$. For a gaussian random process through a double integration, the standard deviation propagates in proportion to $t^{3/2}$. These functional forms for random processes, in conventional analysis, are frequently introduced as deterministic inputs, and the analysis is carried out as described previously for deterministic inputs.

APPLICATION AND REPRESENTATIVE RESULTS

The conjugate solution method of error analysis was applied to the study of the planar motion of an ICBM during injection, over a flat, nonrotating Earth. Results were obtained for error sensitivities, propagation of errors, and error analysis for particular error sources in the measurement instruments. Fig. 1 shows the reference frame used in the study, and the results are presented in Figs. 2 and 3.

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The coordinates were chosen as follows

- $x_1 = x$ range
- $x_2 = z$ altitude
- $x_3 = \beta$ attitude
- $x_4 = m$ mass
- $x_5 = v_x$ x-component of velocity
- $x_6 = v_z$ z-component of velocity
- $x_7 = \dot{\beta}$ attitude rate
- $x_8 = \dot{m}$ mass rate
- $x_9 = \tau$ system time
-[45]

System time was included to enable extension of the study to include variations in system time for synthesis of a controller. For the error analysis, the relation $\tau \equiv t$ was assumed.

Using Eq. 45, a set of equations was derived to correspond to Eqs. 1.

They are

$$\begin{aligned} \frac{dx_1}{dt} &= \frac{dx}{dt} = v_x & \frac{dx_3}{dt} &= \frac{d\beta}{dt} = \dot{\beta} \\ \frac{dx_2}{dt} &= \frac{dz}{dt} = v_z & \frac{dx_4}{dt} &= \frac{dm}{dt} = \dot{m} \\ \frac{dx_5}{dt} &= \frac{dv_x}{dt} = f_5(z, v_x, v_z, \beta, \dot{\beta}, m, \dot{m}, t) \\ \frac{dx_6}{dt} &= \frac{dv_z}{dt} = f_6(z, v_x, v_z, \beta, \dot{\beta}, m, \dot{m}, t) \\ \frac{dx_7}{dt} &= \frac{d\dot{\beta}}{dt} = f_7(z, v_x, v_z, \beta, \dot{\beta}, m, \dot{m}, t) \end{aligned}$$

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$$\frac{dx_8}{dt} = \frac{d\dot{m}}{dt} = 0$$

..... [46]

From Eqs. 46, a set of linear perturbation equations corresponding to Eqs. 5 was derived. The coefficients were (using the notation

$$a_{ij} = \left(\frac{\partial^f i}{\partial x_j} \right)$$

$$a_{15} = a_{26} = a_{27} = -a_{28} = 1$$

$$a_{52} = - \left[\frac{qs}{mp} \frac{dp}{dz} (C_d \cos \gamma + C_L \sin \gamma) \right]$$

$$a_{62} = \left[\frac{qs}{mp} \frac{dp}{dz} (-C_d \sin \gamma + C_L \cos \gamma) + \frac{2 g r_e^2}{(r_e + z)^3} \right]$$

$$a_{72} = \left(\frac{qs l C_m}{I p} \right) \left(\frac{dp}{dz} \right)$$

$$a_{53} = \left[\frac{g I_{sp} \dot{m}}{m} \sin \beta + \frac{qs}{m} \left(\frac{\partial C_d}{\partial \alpha} \cos \gamma + \frac{\partial C_L}{\partial \alpha} \sin \gamma \right) \right]$$

$$a_{73} = \left(\frac{q s l}{I} \right) \left(\frac{\partial C_m}{\partial \alpha} \right)$$

$$a_{54} = \left[\frac{qs}{m^2} (C_D \cos \gamma + C_L \sin \gamma) - \frac{g I_{sp} \dot{m}}{m^2} \cos \beta \right]$$

$$a_{64} = \left[\frac{qs}{m^2} (C_D \sin \gamma - C_L \cos \gamma) - \frac{g I_{sp} \dot{m}}{m^2} \sin \beta \right]$$

$$a_{74} = - \frac{q s l C_m}{m I}$$

$$a_{55} = - \frac{qs}{mv} \left[2(C_d \cos \gamma + C_L \sin \gamma) \left(\frac{v_x}{v} \right) + \left(\frac{\partial C_d}{\partial \alpha} \cos \gamma + \frac{\partial C_L}{\partial \alpha} \sin \gamma \right) \sin \gamma \right]$$

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$$a_{65} = \frac{-qs}{mv} \left[2(C_d \sin \gamma - C_L \cos \gamma) \left(\frac{v_x}{v} + \frac{\partial C_d}{\partial \alpha} \sin \gamma - \frac{\partial C_L}{\partial \alpha} \cos \gamma \right) \sin \gamma \right]$$

$$a_{75} = \frac{q s \ell}{I v} \left[\frac{2 C_m v_x}{v} + \frac{C_m}{\sin \gamma} \right]$$

$$a_{56} = \frac{qs}{mv} \left[2(C_d \cos \gamma + C_L \sin \gamma) \left(\frac{v_z}{v} \right) - \left(\frac{\partial C_d}{\partial \alpha} \right) \cos \gamma + \frac{\partial C_L}{\partial \alpha} \sin \gamma \right] \cos \gamma$$

$$a_{67} = -\frac{qs}{mv} \left[2(C_d \sin \gamma - C_L \cos \gamma) \left(\frac{v_z}{v} \right) + \frac{\partial C_d}{\partial \alpha} \sin \gamma + \frac{\partial C_L}{\partial \alpha} \cos \gamma \right] \cos \gamma$$

$$a_{76} = \frac{q s \ell}{I v} \left[\frac{2 C_m v_z}{v} - \frac{\partial C_m}{\partial \alpha} \cos \gamma \right]$$

$$a_{58} = \frac{g I_{sp}}{m} \cos \beta$$

$$a_{68} = \frac{g I_{sp}}{m} \sin \beta$$

All other a_{ij} not listed are zero.

NOMENCLATURE

C_d, C_L, C_m = drag, lift, and moment coefficients

I = inertia

I_{sp} = specific impulse

g = gravitational specific force

m = mass

q = dynamic pressure

r_e = radius of Earth

x, z = inertial coordinates

v = velocity

α = angle of attack

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- β = attitude
 γ = $\beta - \alpha$
 ρ = atmospheric density

Eq. 46 were solved under the constraint of a particular thrust profile to determine a reference trajectory that reached a desired set of burnout conditions. This trajectory was then taken as the nominal, or ideal, flight. The perturbed motion of the vehicle was assumed to remain in a region of linear approximation about this nominal trajectory. The motion of the perturbed flight is described by the first variational equations, Eqs. 5, which were derived from Eqs. 46. Digital computer solutions to Eqs. 5 were found to give the inverse fundamental solution matrix, $\Pi^{-1}(t)$. A representative set of solutions is shown in Fig. 2, which shows the altitude sensitivities to errors in the system coordinates. Those sensitivities not shown are zero. Since the perturbation equations are linear, the resultant error in range from a measured deviation Δx_i from the nominal trajectory is found by multiplying Δx_i by its corresponding sensitivity $\lambda_{2i}(t) =$

$$\frac{\partial \Delta x_2(T)}{\partial \Delta x_i(t)} = \frac{\partial \Delta x(T)}{\partial \Delta x_i(t)}, \text{ according to Eq. 9}$$

In addition, the error due to gyro drift rate, gyro mass unbalance torque, accelerometer bias error, and accelerometer linearity error was computed. Previous in-house studies have shown these to be the predominant error sources from the measurement instruments.

The resultant position and velocity errors are shown in Fig 3. Here, the error sources were assumed to be independent, and the propagated error is indicated as the square root of the sum of the squares of the individual errors.

CONCLUSIONS

The methods of error analysis were presented and explained. The advantages of the conjugate solution method were shown to be most significant in the guidance sensitivity and error propagation studies and in the study of random inputs. The methods are approximately equivalent in the study of deterministic inputs to the system and for evaluating the circular probable error, CEP. Computer requirements are significantly less in determining the error propagation and guidance sensitivity functions, using the conjugate solution method. The

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assumption of linearity about a nominal solution which is a basic assumption of accuracy analyses allows the computation to be done by a normalized (unit) error, so that the output errors can be scaled linearly according to the magnitude of the input errors.

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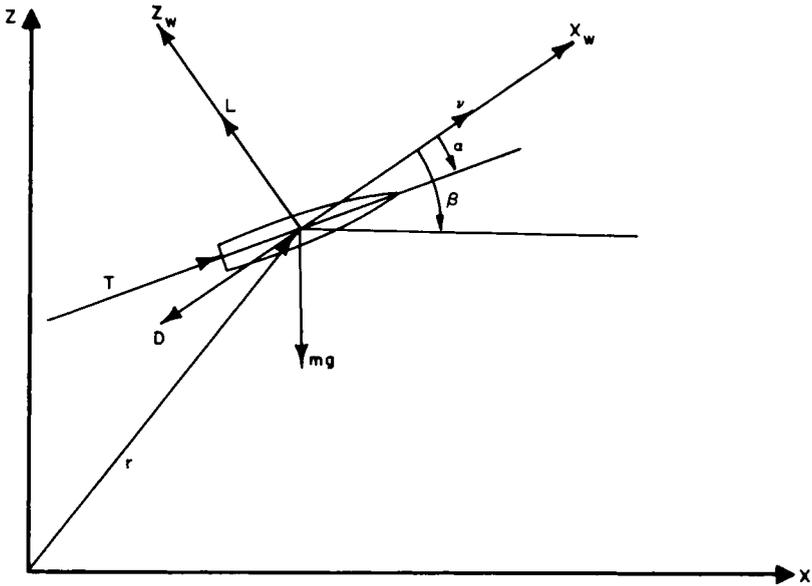


Fig. 1 Reference coordinate frame for a planar rocket flight

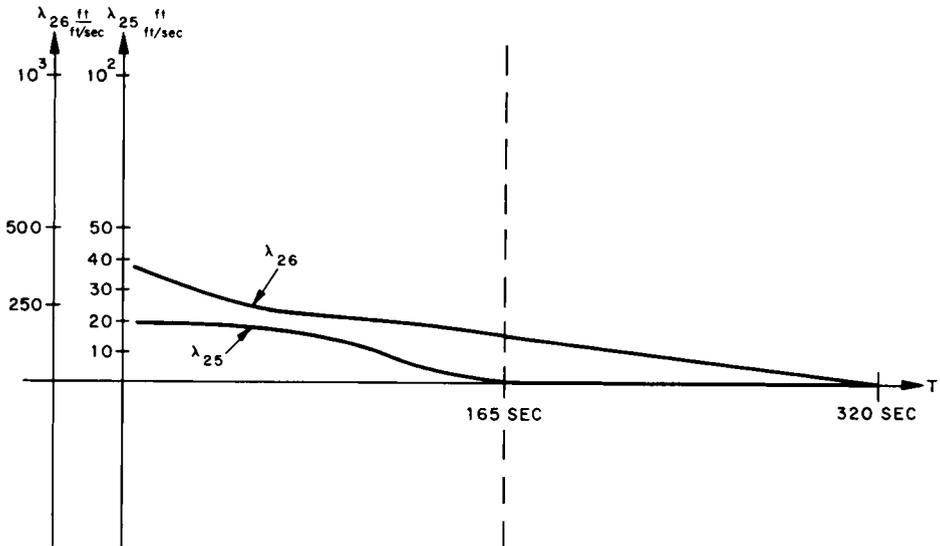


Fig. 2 Sensitivity curves for altitude coordinate errors due to errors in X- and Z-components of velocity

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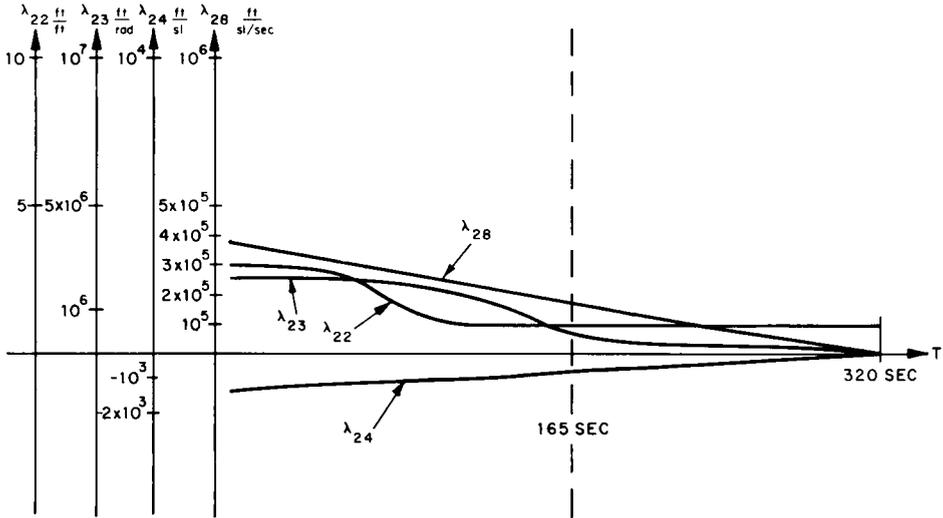


Fig. 3 Sensitivity curves for altitude coordinate errors due to errors in altitude, attitude, mass and mass rate

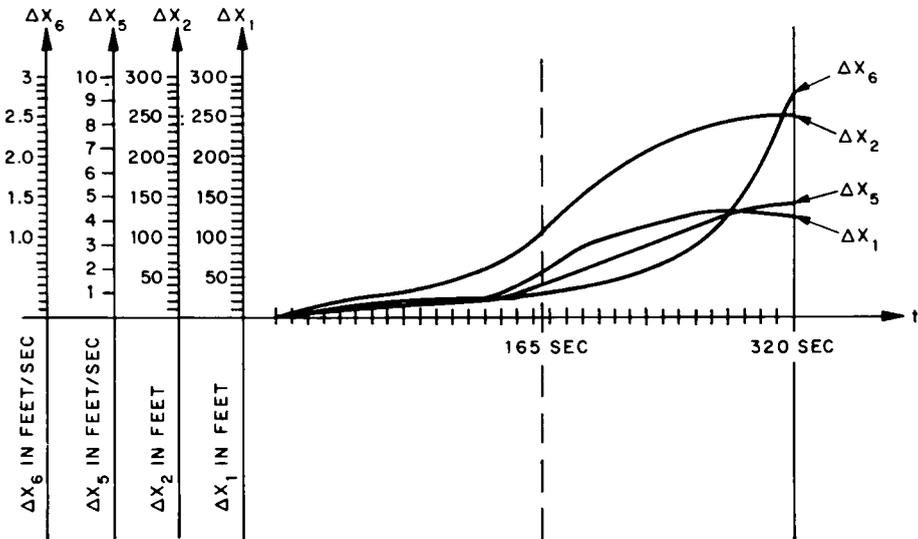


Fig. 4 Propagation of total error due to deterministic errors