CALCULUS OF PERTURBATIONS APPLIED TO LUNAR MISSION ANALYSIS

William C. Marshall

Minneapolis-Honeywell Regulator Co.
Minneapolis, Minnesota

ABSTRACT

The perturbation or linear prediction theory is applied to a realistic four-body model of the oblate Earth, triaxial moon, sun, and vehicle. The disturbed motion of the vehicle may be caused by combinations of three major error sources, namely: 1) a disturbance in the state coordinates at an initial time, e.g., an error in position and velocity at burnout or injection time; 2) small disturbing forces arising from ignored force centers or slightly inaccurate representation of non-ignored force centers, e.g., the planetary perturbations of the outer planets may be ignored, or the Earth's gravitational field may be imperfectly represented; and 3) fictitious disturbing forces which result in truncation and round-off errors propagating in time about the true (unknown) reference trajectory.

MOTIVATION

In performing a trajectory or guidance analysis of the motion of a space vehicle during a lunar mission, a precise mathematical model of all significant external forces that affect the trajectory is of fundamental interest. In many cases, the forces which act are very unpredictable and must be described statistically. Also, some forces may exist which are only known approximately or are even unknown at the present level of knowledge of the universe.


‡Research Engineer/Scientist, Military Products Group Research Dept.

‡The concept of generalized state coordinates is implied here. Thus, there is no distinction between kinematic or dynamic coordinates or other quantities such as the mass of the vehicle.
GUIDANCE AND CONTROL

Even though it may not be possible to formulate exact differential equations of motion, approximate equations of motion can be formulated. In certain problems such as the restricted two-body problem, the equations of motion of a negligible point-mass in a central inverse-squared force field have been analyzed for over 400 years. The results of these studies have been applied fairly extensively to feasibility studies of satellite, lunar, and interplanetary missions over the past five years and up to the present time.

However, with the need for extremely accurate prediction of trajectories, a precision analysis must be contemplated to determine not only error sensitivities to initial and terminal conditions (on position, velocity, and time, for example), but also to determine the effects of numerical integration errors, the effects of stochastic perturbations (such as meteorite bombardment accelerations on the vehicle or roundoff error in numerical computations), the effects of ignoring planetary perturbations of flight path (or light pressure effects), and in general any of the effects of nature that have been omitted in the approximate differential equations of motion.

The science of exterior ballistics is one field where analysts have for some time been confronted with the necessity of performing precision analysis to describe the effects of wind uncertainties, gravitational uncertainties, initial condition errors, and uncertainties in drag and lift coefficients on the flight of missiles. This area is described in Refs. 1-5.

Numerical analysts have also been concerned with the propagation of numerical integration errors through the numerical solution of the equations of motion of ballistic missile and celestial bodies as is evidenced in both the western world literature and the Russian literature, Refs. 6-9, 11-19, and 22-25.

It is the purpose of this paper to present what the author believes is the underlying theory, using the calculus of first-order perturbations or linear prediction theory, which applies to the problem areas of precision analysis mentioned previously. In particular, an analysis technique (which has been called "the adjoint method") will be derived and applied to the equations of motion of a lunar mission space vehicle. The approximate model selected is the most realistic one known to the author, embodying the oblate Earth, triaxial moon, sun, and atmospheric reaction forces.

The section on the derivation of basic relationships presents a review and derivation of the basic linear prediction
theory already referred to. An attempt is made throughout to interpret the mathematics in the language of astronautics.

The section on ignored forces describes the treatment of disturbing forces (including numerical integration error propagation). Two distinct problems are considered: 1) the inclusion of specified disturbing forces into an analysis; and 2) the determination of unknown disturbing forces through observations of the disturbed motion of the space vehicle.

The section on the Newtonian Earth-moon equations of motion contains the specific application of the linear prediction theory developed in the preceding sections to a practical set of equations of motion of a space vehicle in Earth-moon three-dimensional space.

DERIVATION OF BASIC RELATIONSHIPS

In this section, the basic relationships or calculus of first-order perturbations are derived in canonical form. Subsequently, application will be made to the particular equations of motion of a lunar mission vehicle.

Nonlinear Equations of Motion

A generalized system of first-order nonlinear ordinary differential equations are first defined as the true, or precise equations of motion of a mathematical point in configuration or phase space with the laws of motion containing m physical constants. The precise equations of motion are

$$\frac{d}{dt} \hat{X}(t_0) = \hat{X}_0$$

where the precise laws of motion are

$$\hat{f}(\hat{X}, \hat{\alpha}, t) = \begin{bmatrix} \hat{f}_1(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_m, t) \\ \vdots \\ \hat{f}_n(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n; \hat{\alpha}_1, \hat{\alpha}_2, \ldots, \hat{\alpha}_m, t) \end{bmatrix}$$

149
and the physical constants are

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix} \quad [3]$$

In a particular example, the physical constants may be true constants (whose magnitudes have a small uncertainty); or they may be dependent on time, position, velocity, attitude, or, in general, any combination of the generalized coordinates of the problem.

A solution vector $\mathbf{X}(t)$ to the precise equations of motion from precise initial conditions, $\mathbf{X}(t_0) = \mathbf{X}_0$, would describe at every instant of time the exact position, velocity, mass, attitude, etc., of a space vehicle without error. The vector $\mathbf{X}(t)$ is sometimes referred to as the state vector of the system. Between two instants of time, initial time $t_0$ and terminal time $T$, the state vector describes a trajectory or path in $(n + 1)$ dimensional $(n$ state variables and time) configuration space, as depicted in Fig. 1.

Obviously, the definition of the precise equations of motion, $\dot{\mathbf{X}}(\mathbf{x}; \mathbf{a}, t)$, and determination of precise solutions is an impossible task. Not only is the form of the equations of motion in doubt for some problems in astronautics, but, until recently, there was not general agreement as to the magnitude and error bounds on some fairly standard existant physical constants (26).

Fortunately for the analyst, in almost all problems in astronautics it is possible to formulate approximate equations of motion through the use of Newtonian (or non-Newtonian) mechanics. The approximate formulation of the equations of motion of a space vehicle are then defined by

$$\begin{align*}
\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= f(\mathbf{X}; \mathbf{a}, t) \quad [4a] \\
\mathbf{X}(t_0) &= \hat{\mathbf{X}}_0 \quad [4b]
\end{align*}$$

Numbers in parentheses indicate References at end of paper.
where the approximate laws of motion are

\[
\dot{\mathbf{x}}(\hat{\mathbf{x}}; \mathbf{a}, t) = \begin{bmatrix}
  f_1(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_m, t) \\
  f_2(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_m, t) \\
  \vdots \\
  f_n(x_1, x_2, \ldots, x_n; a_1, a_2, \ldots, a_m, t)
\end{bmatrix} \tag{5}
\]

and the approximate physical constants are

\[
\mathbf{a} = \begin{bmatrix}
  a_1 \\
  a_2 \\
  \vdots \\
  a_m
\end{bmatrix} \tag{6}
\]

where it is now tacitly assumed that as many physical constants \(m\) as exist in nature have been formulated.

The difference between the unknown laws of nature and the formulated laws are next defined as

\[
\eta(\hat{\mathbf{x}}, t) = \hat{\mathbf{f}}(\hat{\mathbf{x}}; \mathbf{a}, t) - \dot{\mathbf{x}}(\hat{\mathbf{x}}; \mathbf{a}, t) \tag{7}
\]

where \(\eta\) may be, in a particular application, as large or as small as the ineptness of the analyst in specifying the approximate laws of motion.

Subsequent to a specification of initial conditions, \(\mathbf{x}(t_0) = \hat{\mathbf{x}}_0\), Eq. 4 may be solved by analytical or numerical methods to yield a reference solution \(\hat{\mathbf{x}}(t)\). Denoting the difference between the exact solution of the (unknown) equations of motion and the reference solution to the approximate equations of motion by \(\epsilon(t)\), then

\[
\epsilon(t) = \begin{bmatrix}
  \epsilon_1(t) \\
  \epsilon_2(t) \\
  \vdots \\
  \epsilon_m(t)
\end{bmatrix} = \hat{\mathbf{x}}(t) - \hat{\mathbf{x}}(t) \tag{8}
\]

which represents the propagated error in the state vector due to both an error in formulation of the laws of motion and an error \(\eta(t_0)\) in initial conditions from which the reference trajectory was calculated.

**Equations of Disturbed Motion**

Differentiating Eq. 8 and substituting from Eqs. 1 and 4,
then

\[
\frac{d}{dt} \dot{\epsilon}(t) = \dot{f}(\ddot{X}; \dot{a}, t) - \dot{f}(\ddot{X}; \dot{a}, t) \tag{9}
\]

which represents the true or precise equation of disturbed motion with respect to the reference solution \( \ddot{X}(t) \) as depicted in Fig. 2.

Using the definition of \( \eta \), Eq. 9 may be rewritten as

\[
\frac{d}{dt} \dot{\epsilon}(t) = \dot{f}(\ddot{X}; \dot{a}, t) - \dot{f}(\ddot{X}; \dot{a}, t) + \eta(\ddot{X}, t) \tag{10}
\]

with no loss of generality.

First Variational Equations

Expanding the right-hand terms in Eq. 10 by Taylor's series about the point \( \ddot{X}(t) \) on the reference solution gives

\[
\eta(\ddot{X}, t) = \eta(\ddot{X}, t) + \left( \frac{\partial f_1}{\partial x_j} \right) (\ddot{X} - \ddot{X}) + \sigma [(\ddot{X} - \ddot{X})^2] \tag{11}
\]

\[
\dot{f}(\ddot{X}; \dot{a}, t) - \dot{f}(\ddot{X}; \dot{a}, t) = \left( \frac{\partial f_1}{\partial x_j} \right) (\ddot{X} - \ddot{X}) + \sigma [(\ddot{X} - \ddot{X})^2] \tag{12}
\]

\[
\frac{d}{dt} \dot{\epsilon}(t) = \left( \frac{\partial f_1}{\partial x_j} \right) (\ddot{X} - \ddot{X}) + \eta(\ddot{X}, t) + \sigma [(\ddot{X} - \ddot{X})^2] \tag{13}
\]

Denoting the difference \( \ddot{X} - \ddot{X} \) by \( \Delta \ddot{X} \) and ignoring the second-order and higher terms, the first (free) variational equations or first-order perturbation equations are obtained.

\[
\frac{d}{dt} \Delta \ddot{X} = \left( \frac{\partial f_1}{\partial x_j} \right) \Delta \ddot{X} + \eta(\ddot{X}, t) \tag{14a}
\]

\[
\Delta \ddot{X}(t_o) = \dot{\epsilon}(t_o) \tag{14b}
\]
where

\[
\begin{pmatrix}
\frac{\partial \hat{f}_1}{\partial x_1} & \frac{\partial \hat{f}_1}{\partial x_2} & \cdots & \frac{\partial \hat{f}_1}{\partial x_n} \\
\frac{\partial \hat{f}_2}{\partial x_1} & \frac{\partial \hat{f}_2}{\partial x_2} & \cdots & \frac{\partial \hat{f}_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial \hat{f}_n}{\partial x_1} & \frac{\partial \hat{f}_n}{\partial x_2} & \cdots & \frac{\partial \hat{f}_n}{\partial x_n}
\end{pmatrix}
\]

and \( \Delta \dot{X}(t) = \dot{\epsilon}(t) \) since

\[
\frac{d}{dt} \Delta \dot{X}(t) \equiv \frac{d}{dt} \dot{\epsilon}(t)
\]

The solution of Eqs. 14 approximates the true disturbed motion of the space vehicle with respect to the reference solution \( \dot{X}(t) \).

Denoting the coefficient matrix \( \frac{\partial \hat{f}_i}{\partial x_j} \) by \( B(t) \) it is noted that

\[
\frac{d}{dt} \Delta \dot{X} = B(t) \cdot \Delta \dot{X}
\]

represents the homogeneous form of the approximate equations of disturbed motion. The approach is now to find the general solution to the homogeneous equations and also a particular solution to the nonhomogeneous Eqs. 14. The sum of these solutions then represents the general solution of the linear approximate differential equations of disturbed motion.

Fundamental Solution Matrix

Noting that the first variational equations represent a nonhomogeneous first-order system of linear ordinary-differential equations with time varying coefficients \( B(t) \) and forcing function \( \dot{\eta}(t) \), it is known (from elementary theorems on multiplication and superposition of solutions of linear equations) that if any \( n \) linearly independent solution vectors can be obtained to the homogeneous equations and arrayed as column vectors of a time-dependent matrix, then that matrix may be called a fundamental solution matrix.
A fundamental solution matrix (or matrix of linearly independent solutions to the homogeneous variational equations) which at time \( t = t_0 \) takes on the numerical value of \( I \), the identity matrix, has the property that given any arbitrary set of initial conditions \( \Delta X_0 \) the solution vector \( \Delta X(t) \) to the homogeneous perturbation equations can be written as

\[
\Delta X(t) = \pi(t) \cdot \Delta X(t_0)
\]  \[16\]

where \( \pi(t_0) = I \) and the fundamental solution matrix normalized to \( I \) at \( t_0 \) is

\[
\pi(t) = \\
\begin{bmatrix}
\pi_{11}, & \pi_{12}, & \cdots, & \pi_{1n} \\
\pi_{21}, & \pi_{22}, & \cdots, & \pi_{2n} \\
\vdots & \vdots & & \vdots \\
\pi_{n1}, & \pi_{n2}, & \cdots, & \pi_{nn}
\end{bmatrix}
\]

In short, Eq. 16 represents the general solution to the homogeneous first variational equations. It should be noted that Eq. 16 contains \( n \) arbitrary constants which are the \( n \) elements of \( \Delta X(t_0) \), the initial disturbance vector. To be specific, the \( k \)th column vector of \( \pi(t) \) satisfies the following linear differential equation:

\[
\frac{d}{dt} \begin{bmatrix}
\pi_{1k} \\
\pi_{2k} \\
\vdots \\
\pi_{nk}
\end{bmatrix} = B(t) \begin{bmatrix}
\pi_{1k} \\
\pi_{2k} \\
\vdots \\
\pi_{nk}
\end{bmatrix}, \quad \begin{bmatrix}
\pi_{1k}(t_0) \\
\pi_{2k}(t_0) \\
\vdots \\
\pi_{nk}(t_0)
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix} 
\]  \[17\]

where \( k = 1, 2, 3, \ldots, n \).

Since \( \pi(t) \) is composed of \( n \) solution vectors of Eq. 17, it may be more easily considered as being the solution matrix generated by (numeric or analytic) integration of the following matrix differential equation from initial conditions given at time \( t_0 \):

\[
\frac{d}{dt} \pi(t) = B(t) \cdot \pi(t) 
\]  \[18a\]

\[
\pi(t_0) = I 
\]  \[18b\]
The purpose of this section is to demonstrate a method (the adjoint method) to obtain the inverse matrix of the fundamental solution matrix $\pi(t)$ without resorting to numerical matrix inversion techniques. The usefulness of this method is obvious in later portions of this paper.

The matrix differential equation adjoint (or adjunct) to Eq. 18 is

$$\dot{A}(t) = -A(t) \cdot B(t)$$

The solution matrix $A(t)$ to the differential equation, Eq. 19a, may be determined either by numerical or analytic integration and is defined as the adjoint matrix. It is asserted that the unique property of the adjoint matrix $A(t)$ is that it is the inverse fundamental solution matrix. That is

$$A(t) \cdot \pi(t) = I$$

This can be proved by first differentiating Eq. 20, then

$$\frac{d}{dt} (A \cdot \pi) = \left[ \frac{d}{dt} A \right] \cdot \pi + A \cdot \left[ \frac{d}{dt} \pi \right]$$

From Eqs. 18 and 19

$$\frac{d}{dt} (A \cdot \pi) = -A \cdot B \cdot \pi + A \cdot B \cdot \pi$$

$$= 0,$$ the null matrix

Integrating

$$A(t) \cdot \pi(t) = A(t_0) \cdot \pi(t_0) = I \cdot I = I$$
Therefore Eq. 20 holds.

It is obvious then from Eq. 20 that \( \Lambda(t) \) is the inverse fundamental solution matrix of the homogeneous variational equations. From the theory of matrices, it is also known that

\[
\Lambda(t) \cdot \pi(t) = \pi(t) \cdot \Lambda(t) = I \tag{21}
\]

General Solution of the First Order Perturbation Equations

To recapitulate, the general solution of the homogeneous equations, Eqs. 15, and a particular solution to the nonhomogeneous equations, Eqs. 14, are sought. The general solution to the homogeneous equations has been demonstrated by Eq. 16.

Now, consider the time derivative of the product \( \Lambda(t) \cdot \Delta X(t) \).

\[
\frac{d}{dt} \left[ \Lambda(t) \cdot \Delta X(t) \right] = \left[ \frac{d}{dt} \Lambda(t) \right] \cdot \Delta X(t) + \Lambda(t) \cdot \frac{d}{dt} \Delta X(t) \tag{22}
\]

Substituting from Eqs. 14 and 19

\[
\frac{d}{dt} \left[ \Lambda(t) \cdot \Delta X(t) \right] = -\Lambda(t) \cdot B(t) \cdot \Delta X(t) + \Lambda(t) \cdot \hat{\eta}(t) = \Lambda(t) \cdot \hat{\eta}(t) \tag{23}
\]

Integrating both sides

\[
\Lambda(t) \cdot \Delta X(t) = \Lambda(t_o) \cdot \Delta X(t_o) + \int_{t_o}^{t} \Lambda(r) \cdot \hat{\eta}(r) \, dr \tag{24}
\]

Multiplying both sides of Eq. 24 by \( \pi(t) = \Lambda^{-1}(t) \) and recalling that \( \Lambda(t)=I \)

\[
\Delta \dot{X}(t) = \pi(t) \cdot \Delta X(t_o) + \int_{t_o}^{t} \pi(t) \cdot \Lambda(r) \cdot \hat{\eta}(r) \, dr \tag{25}
\]

Where \( \pi(t) \cdot \Lambda(r) \) in the integral is recognized as the Green's function, \( K(t,r) \) of the nonhomogeneous equations, Eqs. 14.
Eq. 25 then represents the general solution to the approximate equations of disturbed motion. The integral term in Eq. 25 is recognized as a particular solution of the nonhomogeneous equations, as can be seen as follows

Let

\[
\mathbf{T}(t) = \mathbf{I}(t) \mathbf{r}^2(t) / K(t) - \eta(\tau) \, dt
\]

Then

\[
\frac{d}{dt} \mathbf{T}(t) = \mathbf{I}(t) \mathbf{r}^2(t) / K(t) - \eta(\tau) \, dt
\]

Substituting from Eq. 18

\[
\mathbf{T}(t) = B(t) \cdot \mathbf{T}(t) + \eta(t)
\]

where \( \mathbf{T}(t_o) = 0 \), the null vector.

Eq. 27 is recognized as the nonhomogeneous variational equations, Eqs. 14, where \( \Delta \mathbf{X}(t_o) = \mathbf{T}(t) = 0 \) merely generates a particular solution and hence, Eq. 25 is shown to be the general solution to the nonhomogeneous approximate differential equations of disturbed motion. The general solution Eq. 25 may be more easily interpreted when written in the form

\[
\Delta \mathbf{X}(t) = \mathbf{Ax}_1(t) \Delta x_1(t) \quad \Delta x_2(t) \quad \Delta x_n(t) = \mathbf{Ax}_1(t) + \mathbf{T}(t)
\]

The first term in Eq. 28 represents the propagated disturbances due to initial disturbances \( \Delta \mathbf{X}(t_o) \) in the generalized coordinates \( \mathbf{X}(t_o) \) at time \( t_o \). The second term represents a bias error introduced by ignored disturbing forces acting upon the particle in configuration space over the time interval \( [t_o, t] \).

Change of Independent Variable

The previous derivation determined the general solution to
the approximate equations of disturbed motion in terms of disturbances in initial conditions and a bias term introduced by disturbing forces acting over the time of flight from initial time $t_o$. It is sometimes necessary to obtain the general solution of the nonhomogeneous equations which relates a terminal error $\Delta \vec{X}(T)$ in terms of initial conditions specified at any arbitrary time $\Delta \vec{X}(t)$.

Specification of terminal disturbed motion $\Delta \vec{X}(T)$ is most easily accomplished by transforming the first variational equations under a change of independent variable from time $t$ to time to go $t_g$, i.e., let

$$t_g = T - t = \text{time to go}$$

where $T = t_o + t_f = \text{terminal time}$

$$t_f = \text{time of flight of the particle along the reference trajectory from initial time } t_o \text{ to target time or terminal time.}$$

From the above definition, it follows that the approximate equations of disturbed motion with respect to reverse time are

$$\frac{d}{dt_g} \Delta \vec{X}(t_g) = \frac{d}{dt} \Delta \vec{X} \cdot \frac{dt}{dt_g} = \left[-B(t_g) \cdot \Delta \vec{X}(t_g) + \vec{\eta}(t_g)\right]$$

The matrix differential equation which generates the fundamental solution matrix with respect to reverse time is

$$\frac{d}{dt_g} \pi(t_g) = -B(t_g) \cdot \pi(t_g)$$

$$\pi(o) = I$$

The adjoint matrix differential equation that generates the inverse solution matrix to the transformed system of linear homogeneous differential equations is

$$\frac{d}{dt_g} \Lambda(t_g) = +\Lambda(t_g) \cdot B(t_g)$$

$$\Lambda(o) = I$$
The general solution to the nonhomogeneous equations is (as before)
\[ \Delta \tilde{X}(t_g) = \pi(t_g) \cdot \Delta \tilde{X}(o) + \pi(t_g) \int_o^{t_g} A(t'_g) \cdot \eta(t'_g) dt'_g \] [33]

Multiplying both sides by \( \pi^{-1}(t_g) = A(t_g) \) and rearranging terms,
\[ \Delta \tilde{X}(t_g=0) = A(t_g) \cdot \Delta \tilde{X}(t_g) - \int_o^{t_g} A(t'_g) \cdot \eta(t'_g) dt'_g \] [34]

where \( t_g = T - t \)

It is noted that \( A(t_g) \) represents the solution matrix of Eq. 32 and not Eq. 19. If the independent variable is transformed from \( t_g \) back to real time \( t \) again, the desired prediction equation is obtained as
\[ \Delta \tilde{X}(T) = A^*(t) \cdot \Delta \tilde{X}(t) + \int_{T-t}^{T} A^*(\tau) \cdot \eta(\tau) d\tau \] [35]

where, in order to distinguish between the two sets of fundamental solution matrices and their inverses, the following notation has been adopted
\[ \pi^*(t) = \pi(t_g) = \pi(T-t) ; \text{ solution of Eq. 31} \] [36a]
\[ A^*(t) = A(t_g) = A(T-t) ; \text{ solution of Eq. 32} \] [36b]

Examining the integral term above, recalling that in the previous derivation it was shown to be a simple bias term \( \Gamma(t_g) \) is again defined as
\[ \Gamma(t_g) = -\int_o^{t_g} A(t'_g) \cdot \eta(t'_g) dt'_g \]
Then
\[ \frac{d}{dt} \Gamma(t_g) = + A(t_g) \cdot \eta(t_g) \] [37a]
\[ \Gamma(t_g=0) = 0 \] [37b]
GUIDANCE AND CONTROL

\[ \Delta \mathbf{x}(T) = \begin{bmatrix} \Delta x_1(T) \\ \Delta x_2(T) \\ \vdots \\ \Delta x_n(T) \end{bmatrix} = \Lambda^*(t) \cdot \Delta \mathbf{x}(t) + \hat{\mathbf{r}}^*(t) \]  

[38]

where \( \Lambda^*(t) = \Lambda(t_{0\infty}) \) and \( \hat{\mathbf{r}}^*(t) = \hat{\mathbf{r}}(t_{0\infty}) \) are obtained from Eqs. 32 and 37, respectively.

Variation of Independent Variable

One of the chief sources of error which must be considered in any analysis is the effect of timing errors. The derivations of basic variational relationships above implicitly assumed that the time of flight along the reference path was constant. The manner in which initial, ascent path, or mid-course timing errors propagate along a trajectory to result in terminal errors in position, velocity, mass, attitude, etc., may be critical for some missions. For example, if a lunar impact vehicle is injected into an astroballistic trajectory, a timing error of one minute at injection is approximately equivalent to a target displacement (a point on the moon's surface) of 38 statute miles.

The basic relationships derived previously need not be modified if the original nonlinear equations of motion are defined by transforming them into an autonomous system through introduction of a new variable called system time, \( \tau \). This new variable simply replaces the old independent variable absolute time \( t \) wherever it appears in \( f(t) \), and thus forces the right-hand side of the equations of motion to represent an autonomous system, i.e., time independent.

In order to be able to perform this substitution, the relationship between absolute time \( t \) and system time \( \tau \) must also be defined by means of a new differential relationship as follows:

Let

\[ \frac{d}{dt} r(t) = 1 \]  

[39a]

\[ r(t_0) = r_0 \]  

[39b]

\[ r(t) = r_0 + t \]  

[39c]

If the new variable \( r \) is considered to represent an \( x_{n+1} \).
coordinate in configuration space, the equations of motion may now be written as

\[
\frac{d}{dt} \vec{X}(t) = \frac{d}{dt} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \\ x_{n+1}(t) \end{bmatrix} = f(\vec{X}) \tag{40}
\]

where \( f(\vec{X}) \) is of \((n+1)\) dimensions and replaces \( f(\vec{X},t) \) in Eq. 4 and

\[
\frac{d}{dt} x_{n+1}(t) = f_{n+1}(x_1, x_2, \ldots, x_n, x_{n+1}) = 1 \tag{41}
\]

For convenience of notation, the vector \( \vec{X}(t) \) is henceforth considered as an \( \eta \)-dimensional vector rather than \( n+1 \) by tacitly assuming that the new differential equation, Eq. 41, is included in the original system of differential equations of motion.

One immediate effect of this assumption is that the last row of the \( B(t) \) matrix, Eq. 14, becomes \((0, 0, \ldots, 0)\) and also, because of the form of Eq. 41, timing errors \( \Delta r(t) \) are propagated as constants along a trajectory in the absence of forcing functions. This does not mean that the time-error induced effects in position, velocity, mass, attitude, etc., are also propagated as constants.

The forcing function acting on system time \( \eta(t) \) may in one application represent the time-dilation of non-Newtonian mechanics, or in another application it may represent the frequency drift of mechanical origin such as that from a vehicle-borne clock.

**IGNORED FORCES**

The treatment of ignored forces in the analysis of the motion of a body or vehicle under the force laws of celestial mechanics (astrodynamics) usually results in one of two basic types of investigation.

In one form of investigation, it is assumed that the force laws are adequately known to a specified degree of accuracy, and it is desired to determine the effects of uncertainties in the force laws upon the precomputed trajectory and, in particular, the induced terminal error or miss components.
GUIDANCE AND CONTROL

of the state vector \( \hat{X}(t) \). The second form of investigation is concerned with the problem of determining the unknown disturbing forces themselves which cause observed disturbed motion of a space vehicle.

Both types of investigation will be described; the first type, the treatment of assumed ignored forces, will be considered in some detail, whereas the second type, the determination of the disturbing forces, will be considered only briefly.

Treatment of Assumed Ignored Forces

The problem at hand is the prediction of effects propagated in time along a reference trajectory which are induced by uncertainties or inaccuracies in the assumed force laws. Implicitly in this statement, it is assumed that the adopted force laws are coincident with the unknown laws of nature.

Practically, this discussion is concerned with the effects of ignoring certain terms in the equations of motion (such as planetary perturbations), the effects of numerical integration algorithm (or machine induced) truncation and roundoff errors (which may be thought of as fictitious disturbing forces), and the effects of uncertainties in the physical constants in the adopted equations of motion. It is noted that variations in the force laws may be considered as conceptual variations, such as an inverse-cubed law of gravitational attraction, or they may pertain to a variation in the physical constants only, leaving the mathematical form of the equations intact; or they may be a combination of the two possibilities mentioned above.

Eq. 7 defines the difference between the unknown laws of nature and the formulated or adopted laws as

\[
\eta(t) = \hat{f}(\hat{X}; \hat{u}, t) - f(\hat{X}; \hat{u}, t)
\]

where \( \eta(t) = \eta(\hat{X}, t) \) since \( \hat{X} = \hat{X}(t) \) = the reference trajectory. For consistency, \( \hat{\eta}(t) \) is now defined as

\[
\hat{\eta}(t) = \hat{f}(\hat{X}; \hat{u}, t) - \hat{f}(\hat{X}; \hat{u}, t) + \Delta\eta(t)
\]  

[42]

where \( \Delta\eta(t) \) represents the difference between human knowledge of the force laws and the actual laws and assume that this difference is of second order in effect and, hence, ignorable. The quantity \( \hat{f}(\hat{X}, t) \) then represents the adopted laws, and \( f(\hat{X}, t) \) represents the practical laws used for precomputation of reference paths in lunar mission problems.
In studying the effects of uncertainties in the physical constants $\vec{\alpha}$, as defined by Eq. 6, it is convenient to expand $f(\vec{X};\vec{\alpha},t)$ about the point $(\vec{X};\vec{\alpha},t)$ in Taylor's series so that

$$\hat{\eta}(t) = \hat{\eta}(\vec{X};\vec{\alpha},t) \approx \left( \frac{\partial f_1}{\partial \alpha_k} \right) \cdot \Delta \vec{\alpha}$$

where the disturbance in the physical constants vector is

$$\Delta \vec{\alpha} = \begin{bmatrix} \hat{\alpha}_1 - \alpha_1 \\ \hat{\alpha}_2 - \alpha_2 \\ \vdots \\ \hat{\alpha}_m - \alpha_m \end{bmatrix}$$

and the generalized gradient of $\vec{f}$ with respect to the $\vec{\alpha}$ coordinates is

$$\left( \frac{\partial f_i}{\partial \alpha_k} \right) = \begin{bmatrix} \frac{\partial f_1}{\partial \alpha_1}, \frac{\partial f_1}{\partial \alpha_2}, \ldots, \frac{\partial f_1}{\partial \alpha_m} \\ \frac{\partial f_2}{\partial \alpha_1}, \frac{\partial f_2}{\partial \alpha_2}, \ldots, \frac{\partial f_2}{\partial \alpha_m} \\ \vdots \\ \frac{\partial f_n}{\partial \alpha_1}, \frac{\partial f_n}{\partial \alpha_2}, \ldots, \frac{\partial f_n}{\partial \alpha_m} \end{bmatrix}$$

Numerical Integration Algorithm Errors

The problem of predicting or estimating numerical integration algorithm errors can be treated under the category of ignored forces. Conceptually it is easy to visualize truncation and roundoff errors, which propagate in time through the closed loop system, as representing a disturbed motion about a reference (analytic) trajectory and induced by a fictitious disturbing force $\vec{\eta}(t)$. Eq. 14 is used to estimate the error as a function of time once $\vec{\eta}(t)$ is specified. Specifically, $\vec{\eta}(t)$ is the vector whose $n$ coordinates are the $n$ stepwise truncation and roundoff error estimates. Care must be used in specifying the stepwise truncation error for a coordinate which represents a double integration.
GUIDANCE AND CONTROL

Refs. 11-16 in the English literature demonstrate the details of this type of numerical integration error analysis, whereas the Russians V. P. Myachin and A. S. Sochilina in Refs. 17 and 18 have demonstrated a more detailed analysis, with numerical examples, using Cowell's integration algorithm applied to the equations of motion of Jupiter, Uranus, and Saturn.

Determination of Disturbing Forces

The determination of unknown disturbing forces which cause a space vehicle to depart from its reference trajectory into disturbed motion implies that observations of the disturbed path \( \vec{X}(t) \) at arbitrary instants of time \( t_1, t_2, \ldots, t_m \) are available. These observations may be complete at any time instant or they may be incomplete, i.e., some or all observations on the \( n \)-coordinates of \( \vec{X}(t) \) may be available. The numerical difference between the observed state vector \( \vec{X}(t) \) and the reference state vector at the same time instant \( \vec{X}(t) \) represents the observed disturbed motion, i.e.

\[
\Delta \vec{X}_{\text{obs}}(t) = \vec{X}_{\text{obs}}(t) - \vec{X}_{\text{ref}}(t)
\]

If observations are commenced at a particular instant of time \( t_o \) from which a complete reference trajectory is determined, then it may be assumed that \( \Delta \vec{X}_{\text{obs}}(t_o) \) is identically zero and recalling Eq. 25, then

\[
\Delta \vec{X}_{\text{obs}}(t) = \int_{t_o}^{t} K(t,r) \cdot \vec{\eta}(r) \, dr
\]  \[44\]

where the Green's function of Eq. 14 is

\[
K(t,r) = \pi(t) \cdot \Lambda(r)
\]

The matrices \( \pi(t) \) and \( \Lambda(t) \) are generated by solution of the matrix differential Eqs. 18 and 19. The disturbing vector \( \vec{\eta}(t) \) above is defined by Eq. 42 and is the object of the investigation.

Eq. 44 is a homogeneous linear matrix integral equation of Volterra type. The equation may be solved numerically without undue difficulty for \( \vec{\eta}(t) \) using finite difference methods.

This application of the calculus of perturbations has been reported on in detail by the Russian A. A. Deberdeev in both thesis work (1955) and more recently in Ref. 20. Methods of solution of the integral equation, Eq. 44, are summarized and compared in Ref. 21.
NEWTONIAN EQUATIONS OF MOTION IN THE EARTH-MOON SYSTEM

N-body Problem

The differential equations of motion for the N-body problem will first be defined in both absolute and relative form and then modified to include atmospheric and nonpoint mass gravitational field forces. The derivation essentially follows the excellent treatment of the subject found in Chap. 7 of Ref. 10.

The absolute form is with respect to an arbitrary inertial or space stable cartesian coordinate frame of reference either in uniform motion of pure translation or at rest in inertial space. The relative form of the equations of motion is relative to an inertial or space stable cartesian coordinate frame of reference whose origin coincides with the center of mass of one of the N-bodies. The relative form of the equations of motion will be the most useful since the positions of the sun, planets, moons, asteroids, and other massive bodies in the solar system are known tabular quantities in one or more coordinate frames whose origin is usually assumed to be located at a dynamical center, i.e., the center of mass of the moon or the planet Earth.

Absolute Form of Equations of Motion

A Newtonian frame of reference is assumed which is composed of a right-handed and orthogonal set of axis with origin located initially at any convenient point in three-dimensional space as indicated in Fig. 3.

It is assumed that within the N-body system, i.e., the volume of space occupied by the N bodies, there exist forces which act on each of the bodies. These forces are of two types, internal and external forces. The internal forces are defined to be the Newtonian gravitational forces caused by the mutual gravitational attraction of each of the bodies for each of the remaining bodies. Newton's Third Law is also assumed so that the attraction of one body for any other body is equal and opposite to the attraction of the second body for the first. Newton's Law of Gravitation then states that the force acting on the ith body due to the gravitational attraction of the jth body is

\[
(\vec{F}_{ij})^{\text{int}} = \frac{k^2 m_i m_j}{r_{ij}^3} \vec{r}_{ij}
\]  

[45]
GUIDANCE AND CONTROL

where \( \mathbf{r}_{ij} \) is the position vector directed from the \( i \)th body to the \( j \)th body

\[
k^2 = \text{the universal or Gaussian gravitational constant}
\]

\[
m_i = \text{mass of the } i \text{th body}
\]

\[
m_j = \text{mass of the } j \text{th body}
\]

\[
r_{ij} = r_{ji} = \text{magnitude of the vector } \mathbf{r}_{ij}
\]

It should be noted that the units of mass, force, and displacement must be chosen to be consistent in the above equation.

The total internal gravitational force \( F_{i}^{\text{int}} \) on the \( i \)th mass point is the sum of the individual gravitational forces acting on the \( i \)th mass caused by the \((N-1)\) remaining masses; i.e.

\[
\sum_{k=1, k \neq i}^{N} \frac{k^2 m_i m_k}{r_{ik}^3} \mathbf{r}_{ik}
\]

[46]

where \( k \) is a dummy variable of summation and should not be confused with \( k^2 \).

The total external force \( F_{i}^{\text{ext}} \) acting on the \( i \)th body may be the sum of such forces other than the gravitational forces, such as atmospheric drag or lift forces, electromagnetic radiation forces, powered rocket thrust forces. The total force acting on the \( i \)th body \( \mathbf{F}_i \) is then

\[
\mathbf{F}_i = F_{i}^{\text{int}} + F_{i}^{\text{ext}}
\]

or the differential equations of motion for the \( N \)-body problem in absolute form is

\[
m_i \frac{d^2 \mathbf{r}_i}{dt^2} = k^2 m_i \sum_{k=1, k \neq i}^{N} \frac{m_k}{r_{ik}^3} \mathbf{r}_{ik} + F_{i}^{\text{ext}}
\]

[47]

Relative Form of Equations of Motion

The relative form of the equations of motions of a space vehicle under external forces and in the gravitational field
GUIDANCE AND CONTROL

of N-bodies is more commonly used in trajectory and guidance sensitivity studies, since the equations are usually defined relative to a coordinate or reference frame whose origin coincides with the center of mass of a planet or the sun or moon. These coordinate frames are especially convenient, since ephemerides or tables of position coordinates of the bodies in the solar system, as well as stars, are referenced to either a heliocentric or geocentric coordinate frame. A standard set of coordinate frames will be defined and discussed later.

From Fig. 1 it is evident that

$$\dot{\mathbf{r}}_{i,j} = \dot{\mathbf{r}}_j - \dot{\mathbf{r}}_i \quad [48]$$

Differentiating Eq. 48 twice

$$\frac{d}{dt} \dot{\mathbf{r}}_{i,j} = \frac{d}{dt} \dot{\mathbf{r}}_j - \frac{d}{dt} \dot{\mathbf{r}}_i \quad [49]$$

$$\frac{d^2}{dt^2} \dot{\mathbf{r}}_{i,j} = \frac{d^2}{dt^2} \dot{\mathbf{r}}_j - \frac{d^2}{dt^2} \dot{\mathbf{r}}_i \quad [50]$$

Using Eq. 47 then

$$\frac{d^2}{dt^2} \mathbf{r}_{i,j} = k^2 \left[ \sum_{k=1}^{N} \frac{m_k}{r_{jk}^3} \mathbf{r}_{jk} - \sum_{k=1}^{N} \frac{m_k}{r_{ik}^3} \mathbf{r}_{ik} \right]$$

$$+ \left[ \frac{(\mathbf{F}_j)^{\text{ext}}}{m_j} - \frac{(\mathbf{F}_i)^{\text{ext}}}{m_i} \right] \quad [51]$$
Rewriting Eq. 51 gives

\[
\frac{d^2}{dt^2} \mathbf{r}_{ij} = k^2 \left[ \left( \frac{m_i}{r_{ji}} \mathbf{r}_{ji} - \frac{m_j}{r_{ij}} \mathbf{r}_{ij} \right) \right]
\]

\[
+ \sum_{k=1}^{N} \sum_{k \neq i,j} \frac{m_k}{dy} \left( \frac{\mathbf{r}_{jk} - \mathbf{r}_{ik}}{r_{jk}^3} \right) \left[ \frac{(\mathbf{F}_j)_{\text{ext}}}{m_j} - \frac{(\mathbf{F}_i)_{\text{ext}}}{m_i} \right] \tag{52}
\]

But

\[
\mathbf{r}_{ji} = -\mathbf{r}_{ij}; r_{ji} = r_{ij} = \left| \mathbf{r}_{ji} \right| = \left| \mathbf{r}_{ij} \right| \tag{58}
\]

Therefore, Eq. 52 may be written as

\[
\frac{d^2}{dt^2} \mathbf{r}_{ij} = \frac{-k^2(m_i + m_j)}{r_{ij}^3} \mathbf{r}_{ij} + k^2 \sum_{k=1}^{N} \sum_{k \neq i,j} \frac{m_k}{dy} \left( \frac{\mathbf{r}_{jk} - \mathbf{r}_{ik}}{r_{jk}^3} \right) \left[ \frac{(\mathbf{F}_j)_{\text{ext}}}{m_j} - \frac{(\mathbf{F}_i)_{\text{ext}}}{m_i} \right] \tag{54}
\]

which is the differential equation of relative motion for the N-body problem of the jth body (or space vehicle) relative to a Newtonian frame at the ith body.

It is convenient, at times, to separate the accelerations contained in the right-hand member of Eq. 54 into two parts: one part corresponding to the acceleration producing Keplerian or two-body motion, and the second part considered as Keplerian perturbing accelerations acting on the two-body reference motion; i.e.

\[
\frac{d^2}{dt^2} \mathbf{r}_{ij} = \dot{\mathbf{r}}_{ij} + \ddot{\mathbf{r}}_{ij}
\]
or

\[
\frac{d^2}{dt^2} \mathbf{r}_{ij} = -\frac{k^2 \mu_{ij}}{r_{ij}^3} \mathbf{r}_{ij} + k^2 \sum_{k=1}^{N} m_k \left( \frac{\mathbf{r}_{jk}}{r_{jk}^3} - \frac{\mathbf{r}_{ik}}{r_{ik}^3} \right)
\]

\[
\quad + \frac{\mathbf{F}_{j}^{\text{ext}}}{m_j} - \frac{\mathbf{F}_{i}^{\text{ext}}}{m_i}
\]

[56]

and the two-body acceleration \( \dot{\mathbf{r}} \) and the Keplerian perturbing accelerations of the N-body problem \( \ddot{\mathbf{r}} \) are

\[
\dot{\mathbf{r}} = \frac{-k^2 \mu_{ij}}{r_{ij}^3} \mathbf{r}_{ij}
\]

[57]

\[
\ddot{\mathbf{r}} = k^2 \sum_{k=1}^{N} m_k \left( \frac{\mathbf{r}_{jk}}{r_{jk}^3} - \frac{\mathbf{r}_{ik}}{r_{ik}^3} \right) + \frac{\mathbf{F}_{j}^{\text{ext}}}{m_i} - \frac{\mathbf{F}_{i}^{\text{ext}}}{m_i}
\]

[58]

where \( k^2 = \text{Gaussian or universal gravitational constant} \)

\( \mu_{ij} = (m_i + m_j) = \text{reduced mass of the two-body problem} \)

\( m_k = \text{mass of any one of the (N-2) remaining bodies} \)

The first term under the summation sign in Eq. 52 is commonly referred to in celestial mechanics as the direct term or direct acceleration of the kth body on the space vehicle, whereas the second term is called the indirect term or indirect acceleration of the kth body on the ith body. In numerical computation of the solution to Eq. 56, the indirect term oftentimes is a tabular quantity, since it may be pre-computed regardless of the motion of the space vehicle, and stored in the memory unit of a digital computer or in tables for hand calculations.

**Modified Relative Form Due to Non-N-body Gravitational Forces**

The treatment of the N-body problem described so far is to consider each of the N-bodies as a gravitating point-mass in the possible presence of external forces acting on the system.
GUIDANCE AND CONTROL

It is now assumed that each of the massive bodies, i.e., planets or sun, has the shape and gravitational field of an oblate ellipsoid and the necessary added terms are developed in the relative form of the generalized equations of motion. The only exceptions to the above assumptions will be the shape and gravitational field of the moon and a space vehicle.

The vehicle will be considered as a mass point of negligible mass so that \( \mu_{ij} = m_i + m_j \gg m_1 \), and the moon will be treated as a rigid body with triaxial ellipsoid symmetry with a resulting triaxial gravitational field vector intensity function. The assumption is made that the gravitational accelerations caused by one massive body upon any other massive body are simply as defined by Eq. 45; i.e., the moon, planets, and sun attract one another as if they were point masses, by reason of the distances involved. The negligible mass space vehicle, however, is assumed to be acted on by nonpoint mass gravitational forces arising from the oblate ellipsoidal gravitational field of the massive body (planet or sun) at the dynamical center or origin of the coordinate frame in use. The single exception will be the lunar non-point mass forces when the geocentric coordinate frame is in use or when the center of mass of the moon is at the dynamical center.

The standard form for the gravitational potential of an oblate ellipsoid (planet) is

\[
\Phi_i(\hat{r}) = \frac{k^2 m_i}{r} \left[ \frac{J_i a_i}{3 r^2} \left( 1 - \frac{3 z^2}{r^2} \right) + \frac{H_i a_i^3}{5 r^5} \left( \frac{3}{r} - \frac{5 z^3}{r^3} \right) \right.
\]

\[
+ \frac{K_i a_i^4}{30 r^4} \left( 3 - 30 \frac{z^2}{r^2} + 35 \frac{z^4}{r^4} \right) \left. \right] + \ldots
\]

where

\( \hat{r} = \hat{r}_{ij} = x_i \hat{x} + y_i \hat{y} + z_i \hat{z} \), position vector of the jth body or space vehicle of negligible mass relative to the ith body

170
Guidance and Control

The gravitational field intensity vector \( \mathbf{g}_i(\mathbf{r}) \) of the \( i \)th oblate planet is defined in cartesian inertial coordinates as

\[
\mathbf{g}_i(\mathbf{r}) = \nabla \Phi_i(x,y,z) = \frac{\partial \Phi_i}{\partial x} \mathbf{\hat{x}} + \frac{\partial \Phi_i}{\partial y} \mathbf{\hat{y}} + \frac{\partial \Phi_i}{\partial z} \mathbf{\hat{z}}
\]

[60]

The standard oblate planet potential function as given by Eq. 60 may then be used to obtain

\[
\frac{\partial \Phi_i}{\partial x} = (\mathbf{g} \cdot \mathbf{\hat{x}}) = \frac{-k_1 a_1^3 x}{r^3} \left[ 1 + \frac{J_{11} a_1^2}{r^2} \left( 1 - \frac{3 z^2}{r^2} \right) \right] + \frac{H_{11} a_1^3}{r^4} \left( 3 - 7 \frac{z^2}{r^2} \right) \left( \frac{z^2}{r^2} - \frac{21 z^4}{r^4} \right) + \cdots
\]

[61a]

\[
\frac{\partial \Phi_i}{\partial y} = (\mathbf{g} \cdot \mathbf{\hat{y}}) = \frac{y}{x} \cdot \frac{\partial \Phi_i}{\partial x}
\]

[61b]

\[
\frac{\partial \Phi_i}{\partial z} = (\mathbf{g} \cdot \mathbf{\hat{z}}) = \frac{-k_1 a_1^3 z}{r^3} \left[ 1 + \frac{J_{11} a_1^2}{r^2} \left( 3 - 5 \frac{z^2}{r^2} \right) \right] - \frac{H_{11} a_1^3}{r^2 z} \left( \frac{3}{5} - 6 \frac{z^2}{r^2} + 7 \frac{z^4}{r^4} \right) \left( \frac{5}{2} - \frac{35 z^2}{3 r^2} + \frac{21 z^4}{2 r^4} \right) + \cdots
\]

[61c]

where care must be taken in the numerical evaluation of the above expressions for the case \( z = 0 \) so as to avoid numerical instability.

The standard relative form of the generalized equations of motion of a point mass relative to a Newtonian cartesian reference frame whose origin coincides with the center of
mass of the sun or any planet or planetoid in the solar
system may then be written as
\[
\frac{d^2}{dt^2} \vec{r}_{ij} = \vec{g}_i(\vec{r}_{ij}) + \sum_{\substack{k=1 \\ k \neq i,j}}^N k^2 m_k \left( \frac{\vec{r}_{jk}}{r_{jk}^3} - \frac{\vec{r}_{ik}}{r_{ik}^3} \right)
\]
\[
+ \left[ \frac{(\vec{F}_j)^{\text{ext}}}{m_j} - \frac{(\vec{F}_i)^{\text{ext}}}{m_i} \right]
\]
where \( \vec{g}_i \) is given by Eq. 61.

It should be noted that the above equations Eqs. 62 do not
contain terms corresponding to the perturbing gravitational
accelerations imposed upon the space vehicle (the 4th body)
by Earth's triaxial moon when the ith body is taken to be
Earth or when the ith body (and hence, the dynamical center)
is taken to be the moon. These cases are considered as
exceptions to the standard relative form, Eq. 62, and the
last case is treated next.

The standard form for the gravitational potential function
for the triaxial ellipsoidal moon is
\[
\Phi(\vec{r}_\Delta) = \frac{k^2 m}{r_\Delta^2} \left[ 1 + \frac{J_\Delta^1}{3r_\Delta^2} \cdot \left( 1 - 3 \frac{z_\Delta^2}{r_\Delta^2} \right) \right. \\
+ \frac{J_\Delta^2}{3r_\Delta^2} \cdot \left. \left( 1 - 3 \frac{y_\Delta^2}{r_\Delta^2} \right) \right]
\]
where \( \vec{r}_\Delta = \) position vector of a point relative to the
selenographic coordinate frame with cartesian
components \( x_\Delta, y_\Delta, \) and \( z_\Delta. \)
\( m_\Delta = \) moon's mass
\( J_\Delta^1, J_\Delta^2 = \) spherical harmonic coefficients
GUIDANCE AND CONTROL

The corresponding gravitation intensity vector, \( \vec{g}_\Delta(r_\Delta) \), relative to the cartesian (noninertial) selenographic coordinate frame is

\[
\vec{g}_\Delta(r_\Delta) = \nabla \Phi_\Delta = \frac{\partial \Phi_\Delta}{\partial x_\Delta} \hat{x}_\Delta + \frac{\partial \Phi_\Delta}{\partial y_\Delta} \hat{y}_\Delta + \frac{\partial \Phi_\Delta}{\partial z_\Delta} \hat{z}_\Delta
\]

[64]

Where the unit vector triad of the selenographic frame is denoted by \( \hat{x}_\Delta \), \( \hat{y}_\Delta \), and \( \hat{z}_\Delta \) and

\[
\vec{r}_\Delta = x_\Delta \hat{x}_\Delta + y_\Delta \hat{y}_\Delta + z_\Delta \hat{z}_\Delta
\]

[65]

Carrying out the indicated partial differentiation

\[
\frac{\partial \Phi_\Delta}{\partial x_\Delta} = (\vec{g}_\Delta \cdot \hat{x}_\Delta) = \frac{-k^2 m x_\Delta}{r_\Delta^3} \left[ 1 + \frac{J_{1}}{r_\Delta^2} \left( 1 - 5 \frac{z_\Delta^2}{r_\Delta^2} \right) ight]
\]

[66a]

\[
\frac{\partial \Phi_\Delta}{\partial y_\Delta} = (\vec{g}_\Delta \cdot \hat{y}_\Delta) = \frac{-k^2 m y_\Delta}{r_\Delta^3} \left[ 1 + \frac{J_{1}}{r_\Delta^2} \left( 1 - 5 \frac{z_\Delta^2}{r_\Delta^2} \right) ight] + \frac{J_{2}}{r_\Delta^2} \left( 3 - 5 \frac{y_\Delta^2}{r_\Delta^2} \right)
\]

[66b]

\[
\frac{\partial \Phi_\Delta}{\partial z_\Delta} = (\vec{g}_\Delta \cdot \hat{z}_\Delta) = \frac{-k^2 m z_\Delta}{r_\Delta^3} \left[ 1 + \frac{J_{1}}{r_\Delta^2} \left( 3 - 5 \frac{z_\Delta^2}{r_\Delta^2} \right) \right]
\]

[66c]
GUIDANCE AND CONTROL

It is apparent from the above equations that the point mass field expressions may be separated from the remainder of Eq.
66 so that

\[ \ddot{\mathbf{r}}_\Delta = \frac{-k^3 m_\Delta}{r_\Delta^3} \dot{\mathbf{r}}_\Delta + \hat{g}_\Delta^* \dot{\mathbf{r}}_\Delta \quad [67] \]

Since the foregoing expression for \( \ddot{\mathbf{r}}_\Delta \) refers to the non-inertial selenographic coordinate frame that is defined by the rotating and librating principal axis of symmetry of the moon's figure, the relative equations of motion of a space vehicle are written relative to a Newtonian frame with common origin to the selenographic frame. This choice simplifies the resulting expressions and calculations but requires the introduction of the selenographic rotation matrix \( \Theta(t) \), which is discussed later. The equations of motion of a space vehicle relative to an inertial frame with origin at the moon's center of mass and with \( \hat{i}_x \) and \( \hat{i}_y \) coplanar to Earth's equatorial plane are

\[
\frac{d^2}{dt^2} \mathbf{r}_\Delta = \frac{-k^3 m_\Delta}{r_\Delta^3} \dot{\mathbf{r}}_\Delta + \Theta(t) \cdot \hat{g}_\Delta^* \dot{\mathbf{r}}_\Delta - \hat{g}_\Theta \hat{\Theta}_\Delta + \hat{g}_\Theta \hat{\Theta}_\Delta + \mathbf{F}_{\text{ext}}^\Delta \quad [68]
\]

where

\( \mathbf{r}_\Delta \) = position vector from moon center to vehicle

\( \dot{\mathbf{r}}_\Delta \) = position vector from earth center to vehicle

\( \dot{\mathbf{r}}_\Delta^k \) = position vector from vehicle to the kth body (planet or sun)

\( \dot{\mathbf{r}}_\Delta^k \) = position vector from the moon center to the kth body (planet or sun)

The equations of motion of a space vehicle relative to an inertial frame with origin at Earth's center of mass under the perturbing accelerations of both the oblate earth gravitational field and the triaxial moon gravitation field are
Atmospheric Reaction

Atmospheric reaction forces, i.e., drag, side, and lift forces are considered as external forces acting only on the vehicle and not on any of the remaining bodies in the N-body problem. Hence, the term $\vec{F}_{\text{ext}}^m$ is assumed to be identically zero in the equations of motion developed up to this point. If the space vehicle, which was considered as a point mass in the preceding discussion, is in reality a "winged" vehicle, then a relative wind-fixed coordinate frame must be defined as well as the force laws.

By convention, drag forces are assumed to be acting in the direction of the relative wind where the relative wind velocity vector is simply the apparent velocity of the atmosphere with respect to an assumed motionless vehicle. It should be noted that other conventions exist in which drag forces are defined as the vector component of the total atmospheric reaction force resolved along the longitudinal axis of a vehicle. Using the adopted convention, the remaining two orthogonal components of reaction force must lie by necessity in the plane normal to the relative wind vector. These two components, which will be defined, are the lift force and side force vector components of the total reaction force.

The atmosphere of a planet is assumed to be rotating with the planet's rotational velocity vector, $\vec{\Omega}_1$. Hence, the
velocity of the atmosphere with respect to inertial space at
the position of a vehicle is

\[ \dot{V}_A(r_{ij}) = \Omega \times r_{ij} \]  

[70]

The velocity of the atmosphere \( \dot{V}_R \), relative to a vehicle
which has an inertial velocity \((d/dt)r_{ij}\) is the velocity of
the relative wind

\[ \dot{V}_R = \left[ \Omega \times r_{ij} - \frac{d}{dt} r_{ij} \right] \]  

[71]

In inertial geocentric-equatorial coordinates, assuming \( \Omega_i \)
to be directed along \( \hat{i}_z \), Eq. 71 may be written

\[ \dot{V}_R = - \left[ \hat{i}_x(v_x + \Omega y) + \hat{i}_y(v_y - \Omega x) + \hat{i}_z v_z \right] \]  

[72]

where

\[ \hat{\Omega}_i = \Omega \hat{z} \]

\[ \frac{d}{dt} r_{ij} = V_x \hat{i}_x + V_y \hat{i}_y + V_z \hat{i}_z \]

The projections of the total atmospheric reaction force
vector onto the plane normal to the relative wind velocity
vector in the directions of \( \dot{V}_R \times \hat{r} \) and \( \dot{V}_R \times (\dot{V}_R \times \hat{r}) \) are defined
as the side-force \( \hat{S} \) and the lift-force vectors, \( \hat{L} \), respec-
tively. The remaining component of the total atmospheric
reaction vector is in the direction of the relative wind
vector and is defined as the drag-force \( \hat{D} \). The triad \( \hat{D}, \hat{S}, \)
and \( \hat{L} \) forms a mutually orthogonal set with unit vectors \( \hat{i}_D, \hat{i}_S, \) and \( \hat{i}_L \) where

\[ \hat{i}_D = \frac{\dot{V}_R}{V_R} = \hat{i}_x \left( \frac{-v_x - \Omega y}{V_R} \right) + \hat{i}_y \left( \frac{-v_y + \Omega x}{V_R} \right) + \hat{i}_z \left( \frac{-v_z}{V_R} \right) \]  

[73]
GUIDANCE AND CONTROL

\[ \hat{\mathbf{S}} = \frac{\mathbf{v}_R \times \mathbf{r}}{v_R r} = \hat{x} \left[ \frac{yv_z - zv_y + \Omega xz}{v_R \cdot r} \right] + \hat{y} \left[ \frac{zv_x - xv_z + \Omega yz}{v_R \cdot r} \right] \]

\[ + \hat{z} \left[ \frac{xv_y - yv_x - \Omega (x^2 + y^2)}{v_R \cdot r} \right] \]  

\[ \hat{\mathbf{L}} = \hat{\mathbf{D}} \times \hat{\mathbf{S}} = \frac{1}{v_R^2 r} \left\{ \hat{x} \left[ -(v_y - \Omega x) \left[ (xv_y - yv_x) - \Omega(x^2 + y^2) \right] \right] \right. 

\[ + v_z \left[ (zv_x - xv_z) + \Omega yz \right] \]

\[ + \hat{y} \left[ -(v_z - \Omega x) \left[ (v_y - xv_y) + \Omega xz \right] \right] \]

\[ + \hat{z} \left[ -(v_x + \Omega y) \left[ (zv_x - xv_z) + \Omega yz \right] \right] \]

\[ + (v_y - \Omega x) \left[ (yv_z - zv_y) + \Omega xz \right] \} \]  

[74]

with \( v_R = \sqrt{(v_x + \Omega y)^2 + (v_y - \Omega x)^2 + v_z^2} \)

\[ r = \sqrt{x^2 + y^2 + z^2} \]  

[76a]

[76b]

Drag, side, and lift force laws are then defined in following manner

\[ D = \frac{1}{2} C_D A_D \rho v_R^2 \]  

[77a]

\[ S = \frac{1}{2} C_S A_S \rho v_R^2 \]  

[77b]

\[ L = \frac{1}{2} C_L A_L \rho v_R^2 \]  

[77c]

with \( \hat{\mathbf{D}} = D \hat{\mathbf{D}}, \hat{\mathbf{S}} = S \hat{\mathbf{S}}, \hat{\mathbf{L}} = L \hat{\mathbf{L}} \)
where $C_D$, $S$, $L$ = drag, side, and lift force coefficients

$A$ = projected reference area of the vehicle

$V_R$ = speed of the relative wind

$\rho$ = density of the atmosphere at the point $r$

The drag, side, and lift coefficients are functions of flight regime, i.e., mean free path of molecular motion, as well as functions of the orientation of the axis of symmetry of the vehicle to the relative wind and are assumed as empirical data derived from wind tunnel tests.

The external accelerations acting on the vehicle due to atmospheric reaction forces $D$, $\dot{S}$, and $L$ are then

\[
\begin{align*}
\ddot{r}_D &= \frac{1}{2} \frac{C_D Ap}{m} V_R^2 \dot{\gamma} \\
\ddot{r}_S &= \frac{1}{2} \frac{C_S Ap}{m} V_R^2 \dot{\gamma} \\
\ddot{r}_L &= \frac{1}{2} \frac{C_L Ap}{m} V_R^2 \dot{\gamma}
\end{align*}
\]

and the equations of motion of a space vehicle relative to an inertial frame with origin at Earth's center of mass under the perturbing accelerations of the oblate Earth gravitational field, the triaxial lunar field, solar, and planetary point mass perturbing accelerations, and atmospheric reaction accelerations are
The equations of motion Eqs. 80 are now defined as the precise or adopted equations of motion as discussed in earlier sections. The practical equations of motion are those from which reference trajectories are obtained and will be defined as

\[
\frac{d^2}{dt^2} \vec{r}_{\Delta} = \frac{-k^2 m_\oplus}{r_{\oplus \Delta}^3} \vec{r}_{\oplus \Delta} + \vec{g}_\oplus(\vec{r}_{\oplus \Delta}) + \vec{r}_D + \vec{r}_S + \vec{r}_L
\]

\[+ k^2 m_c \left( \frac{\vec{r}_{\Delta C}}{r_{\Delta C}^3} - \frac{\vec{r}_{\oplus C}}{r_{\oplus C}^3} \right) + \Theta(t) \cdot \vec{\gamma}^* (\vec{r}_{\Delta \Theta}) \]

\[+ \sum_{k=1}^{N} k^2 m_k \left( \frac{\vec{r}_{\Delta k}}{r_{\Delta k}^3} - \frac{\vec{r}_{\oplus k}}{r_{\oplus k}^3} \right) \]

where the practical model excludes the N-body planetary attractions as expressed by the adopted equations. As discussed earlier, this restriction automatically forces the disturbing vector \( \vec{\eta} (t) \) to represent the planetary attractions as part of the total vector besides the effect of variations in the physical constants, i.e.
\[ \dot{\eta}(t) = \left[ \left( \frac{\partial f_i}{\partial \alpha_k} \right) \cdot \Delta \alpha + \sum_{k=1}^{N} k^2 m_k \left( \frac{r_{\Delta k}}{r_{\Delta k}^3} \frac{r_{\Delta k}}{r_{\Delta k}^3} \right) + \Delta \bar{u}(t) \right] \]

where \(\Delta \bar{u}(t)\) now may represent numerical integration forces, effects of higher than fourth harmonic terms in the Earth's gravitational field representation, etc., which are, for the moment, dismissed as being ignorable.

The vector form of the equations of motion must now be reduced to the canonical form, \(\frac{d}{dt}X = f(x, x', \cdots, x_n)\), i.e.

\[ \frac{d}{dt} x_{\Theta} = f_1 = \dot{x}_{\Theta} \]

\[ \frac{d}{dt} y_{\Theta} = f_2 = \dot{y}_{\Theta} \]

\[ \frac{d}{dt} z_{\Theta} = f_3 = \dot{z}_{\Theta} \]

\[ \frac{d}{dt} x_{\Theta} = f_4 = \frac{-k^2 m x_{\Theta}}{r_{\Theta}^3} \left[ 1 + \frac{J_a^2}{r_{\Theta}^2} \left( 1 - 5 \frac{z_{\Theta}^2}{r_{\Theta}^2} \right) + \frac{H_{a}^2}{r_{\Theta}^4} \left( 3 - 7 \frac{z_{\Theta}^2}{r_{\Theta}^2} \right) + \frac{K_{a}^4}{r_{\Theta}^4} \left( \frac{1}{2} - 7 \frac{z_{\Theta}^2}{r_{\Theta}^2} + \frac{21 z_{\Theta}^4}{2 r_{\Theta}^4} \right) \right] \]
GUIDANCE AND CONTROL

\[ + \ k^2 m_\Delta \left( \frac{\dot{x}_\Delta}{r_\Delta^3} - \frac{\dot{y}_\Delta}{r_\Theta^3} \right) \]

\[ + \ \left[ \theta_{11}(r) \frac{\partial \Phi}{\partial x_\Delta} + \theta_{12}(r) \frac{\partial \Phi}{\partial y_\Delta} \right] \]

\[ + \ \theta_{13}(r) \frac{\partial \Phi}{\partial z_\Delta} \]

\[ + \ \left( \frac{\rho V_R^2}{2m} \right) \left[ C_D(\dot{\gamma}_D \cdot \dot{\gamma}_x) + C_S(\dot{\gamma}_S \cdot \dot{\gamma}_x) + C_L(\dot{\gamma}_L \cdot \dot{\gamma}_x) \right] \]

\[ + \ k^2 m_\Theta \left( \frac{\dot{x}_\Theta}{r_\Theta^3} - \frac{\dot{y}_\Theta}{r_\Theta^3} \right) \]  \hspace{1cm} \text{[86]}

\[ \frac{d}{dt} \hat{y}_\Theta = f_6 = \left\{ \begin{array}{c}
-k^2 m_\Theta \hat{y}_\Theta \\
1 + \frac{J_\Theta^2}{r_\Theta^2} \left( 1 - 5 \frac{z_\Theta^2}{r_\Theta^2} \right) \\
+ \frac{H a^3}{r_\Theta^4} \left( 3 - 7 \frac{z_\Theta^2}{r_\Theta^2} \right) \\
+ \frac{K a^4}{r_\Theta^2} \left( \frac{1}{2} - 7 \frac{z_\Theta^2}{r_\Theta^2} + 21 \frac{z_\Theta^4}{r_\Theta^4} \right) \\
+ \ k^2 m_\Theta \left( \frac{y_\Delta}{r_\Delta^3} - \frac{\dot{y}_\Theta}{r_\Theta^3} \right) \end{array} \right. \]
GUIDANCE AND CONTROL

\[ + \begin{bmatrix} \theta_{21}(r) \frac{\partial \Phi}{\partial x_{\Delta}} & \theta_{22}(r) \frac{\partial \Phi}{\partial y_{\Delta}} & \theta_{23}(r) \frac{\partial \Phi}{\partial z_{\Delta}} \end{bmatrix} \]

\[ + \left( \frac{A \nu R^2}{2m} \right) \left[ C_D(\hat{\Delta}_D \cdot \hat{\Delta}_y) + C_S(\hat{\Delta}_S \cdot \hat{\Delta}_y) + C_L(\hat{\Delta}_L \cdot \hat{\Delta}_y) \right] \]

\[ + k^2 \left( \frac{\gamma_{\Delta \Theta} - \gamma_{\Theta \Theta}}{r_{\Delta \Theta}^3 - r_{\Theta \Theta}^3} \right) \]

\[ \frac{d}{dt} \begin{bmatrix} z_{\Delta} \\ \end{bmatrix} = f_e = \begin{bmatrix} -k^2 m z_{\Delta} \\ \end{bmatrix} + \begin{bmatrix} 1 + \frac{\Theta \Theta}{r_{\Theta \Theta}^2} \left( 3 - \frac{1}{r_{\Theta \Theta}^2} \right) \\ \end{bmatrix} \frac{\partial \Phi}{\partial x_{\Delta}} + \begin{bmatrix} \frac{3}{5} - 6 \frac{z_{\Delta}^2}{r_{\Theta \Theta}^2} + \eta \frac{z_{\Theta}^4}{r_{\Theta \Theta}^4} \\ \end{bmatrix} \frac{\partial \Phi}{\partial y_{\Delta}} + \begin{bmatrix} \frac{5}{2} - \frac{35}{3} \frac{z_{\Delta}^2}{r_{\Theta \Theta}^2} + \frac{21}{2} \frac{z_{\Theta}^4}{r_{\Theta \Theta}^4} \\ \end{bmatrix} \frac{\partial \Phi}{\partial z_{\Delta}} \]

\[ + k^2 \left( \frac{z_{\Delta} \Theta - z_{\Theta} \Theta}{r_{\Delta \Theta}^3 - r_{\Theta \Theta}^3} \right) \]

\[ + \begin{bmatrix} \theta_{31}(r) \frac{\partial \Phi}{\partial x_{\Delta}} & \theta_{32}(r) \frac{\partial \Phi}{\partial y_{\Delta}} & \theta_{33}(r) \frac{\partial \Phi}{\partial z_{\Delta}} \end{bmatrix} \]

\[ + \left( \frac{A \nu R^2}{2m} \right) \left[ C_D(\hat{\Delta}_D \cdot \hat{\Delta}_z) + C_S(\hat{\Delta}_S \cdot \hat{\Delta}_z) + C_L(\hat{\Delta}_L \cdot \hat{\Delta}_z) \right] \]

\[ + k^2 \left( \frac{z_{\Delta} \Theta - z_{\Theta} \Theta}{r_{\Delta \Theta}^3 - r_{\Theta \Theta}^3} \right) \]

[87]

[88]
Specific Analysis Equations

Proceeding as described in the section on derivation of basic relationships, calculate the coefficient matrix $B(t)$ of the first variational equations whose elements are $\frac{\partial f_i}{\partial x_j}$, where $x_j$ in the new notation is, successively $\chi^{\Delta}$, $\chi^{\Delta'}$, $\zeta^{\Delta}$, $\dot{x}^{\Delta}$, $\dot{y}^{\Delta}$, $\dot{z}^{\Delta}$, and $\tau$.

Certain of the partial derivatives are identically zero for obvious reasons as seen by the following:

$$B(t) = \frac{\partial f_i}{\partial x_j} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} & \frac{\partial f_4}{\partial \chi^{\Delta}} \\
\frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} & \frac{\partial f_5}{\partial \chi^{\Delta}} \\
\frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} & \frac{\partial f_6}{\partial \chi^{\Delta}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Certain of the remaining indicated partial derivatives are symmetric in the state coordinates so that the tedious task of partial differentiation of the $f_i$'s is not as onerous as it appears to be. Nevertheless, the resulting expressions are lengthy and, as a result, will not be reproduced here since their derivation is straightforward.
GUIDANCE AND CONTROL

In component form, the first variational equations pertaining to the "practical" set of equations of motion are

\[
\frac{d}{dt} \Delta x_{\Theta\Delta} = \Delta \dot{x}_{\Theta\Delta} \tag{91}
\]

\[
\frac{d}{dt} \Delta y_{\Theta\Delta} = \Delta \dot{y}_{\Theta\Delta} \tag{92}
\]

\[
\frac{d}{dt} \Delta z_{\Theta\Delta} = \Delta \dot{z}_{\Theta\Delta} \tag{93}
\]

\[
\frac{d}{dt} \Delta x = \Delta \dot{x} + \Delta \nu + \Delta \zeta + \Delta \chi \tag{94}
\]

\[
\frac{d}{dt} \Delta y = \Delta \dot{y} + \Delta \nu + \Delta \zeta + \Delta \chi \tag{95}
\]

\[
\frac{d}{dt} \Delta z = \Delta \dot{z} + \Delta \nu + \Delta \zeta + \Delta \chi \tag{96}
\]

184
\[
\frac{d}{dt} \Delta r = 0 \Rightarrow \Delta r(t) = \Delta r(t_0)
\]  \[97\]

It is assumed that a reference solution of the practical equations is available which yields values of \(x_{\Theta}(t), y_{\Theta}(t), \cdots, \dot{z}_{\Theta}(t)\), which can then be used in evaluating the time-varying coefficients in Eqs. 91-97.

The adjoint set of variational equations to the previous set of equations is

\[
\frac{d}{dt} \lambda_1 = - \left[ \lambda_4 \frac{\partial f_4}{\partial x_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial x_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial x_{\Theta}} \right]
\]  \[98\]

\[
\frac{d}{dt} \lambda_2 = - \left[ \lambda_4 \frac{\partial f_4}{\partial y_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial y_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial y_{\Theta}} \right]
\]  \[99\]

\[
\frac{d}{dt} \lambda_3 = - \left[ \lambda_4 \frac{\partial f_4}{\partial z_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial z_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial z_{\Theta}} \right]
\]  \[100\]

\[
\frac{d}{dt} \lambda_4 = - \left[ \lambda_1 + \lambda_4 \frac{\partial f_4}{\partial \dot{x}_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial \dot{x}_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial \dot{x}_{\Theta}} \right]
\]  \[101\]

\[
\frac{d}{dt} \lambda_5 = - \left[ \lambda_2 + \lambda_4 \frac{\partial f_4}{\partial \dot{y}_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial \dot{y}_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial \dot{y}_{\Theta}} \right]
\]  \[102\]

\[
\frac{d}{dt} \lambda_6 = - \left[ \lambda_3 + \lambda_4 \frac{\partial f_4}{\partial \dot{z}_{\Theta}} + \lambda_5 \frac{\partial f_5}{\partial \dot{z}_{\Theta}} + \lambda_6 \frac{\partial f_6}{\partial \dot{z}_{\Theta}} \right]
\]  \[103\]
\[
\frac{d}{dt} \lambda_7 = - \lambda_4 \frac{\partial f_4}{\partial r} + \lambda_5 \frac{\partial f_5}{\partial r} + \lambda_6 \frac{\partial f_6}{\partial r}
\]

where \( \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \)

= the row vector of the \( \Lambda(t) \) matrix.

The variation in the physical constant vector \( \delta(t) \), described in earlier sections, in this case is

\[
\Delta \delta = \begin{bmatrix}
\Delta (k^2 m_\oplus) \\
\Delta a_\oplus \\
\Delta J_\oplus \\
\Delta H_\oplus \\
\Delta K_\oplus \\
\Delta (k^2 m_\odot) \\
\Delta J_\odot^1 \\
\Delta J_\odot^2 \\
\Delta \Omega_\odot \\
\Delta C_D \\
\Delta C_S \\
\Delta C_L \\
\Delta \rho \\
\Delta m \\
\Delta x_\Theta \\
\Delta y_\Theta \\
\Delta z_\Theta \\
\Delta (k^2 m_\oplus) \\
\Delta x_\odot \\
\Delta y_\odot \\
\Delta z_\odot 
\end{bmatrix}
\]

More than the 21 coordinates of \( \Delta \delta \) could have been specified since the errors in the selenographic rotation matrix elements could have been added. Numerical values for most of the coordinates of \( \Delta \delta \) may be found in Ref. 10. The procedure in determining the vehicle sensitivity to uncertainties in these constants is then to solve (numerically or by analytic means) the linear differential equation for \( \tilde{f}(t) \), the bias
term in the general solution for $\Delta \dot{x}(t)$ where

$$\frac{d}{dt} \mathbf{\dot{x}(t)} = \mathbf{B(t)} \cdot \mathbf{\dot{x}(t)} +$$

$$\begin{bmatrix}
0, 0, 0, 0, 0, 0, \cdots, 0 \\
0, 0, 0, 0, 0, 0, \cdots, 0 \\
0, 0, 0, 0, 0, 0, \cdots, 0 \\
\frac{\partial f_4}{\partial (k^2 m^2 \oplus)} \frac{\partial f_4}{\partial z^2 \oplus} \frac{\partial f_5}{\partial H^2 \oplus} \frac{\partial f_6}{\partial K^2 \oplus} \frac{\partial f_7}{\partial \Delta \Theta \oplus} \frac{\partial f_8}{\partial \Delta \Theta \oplus} \frac{\partial f_9}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{10}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_5}{\partial (k^2 m^2 \oplus)} \frac{\partial f_5}{\partial z^2 \oplus} \frac{\partial f_6}{\partial H^2 \oplus} \frac{\partial f_7}{\partial K^2 \oplus} \frac{\partial f_8}{\partial \Delta \Theta \oplus} \frac{\partial f_9}{\partial \Delta \Theta \oplus} \frac{\partial f_{10}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{11}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_6}{\partial (k^2 m^2 \oplus)} \frac{\partial f_6}{\partial z^2 \oplus} \frac{\partial f_7}{\partial H^2 \oplus} \frac{\partial f_8}{\partial K^2 \oplus} \frac{\partial f_9}{\partial \Delta \Theta \oplus} \frac{\partial f_{10}}{\partial \Delta \Theta \oplus} \frac{\partial f_{11}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{12}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_7}{\partial (k^2 m^2 \oplus)} \frac{\partial f_7}{\partial z^2 \oplus} \frac{\partial f_8}{\partial H^2 \oplus} \frac{\partial f_9}{\partial K^2 \oplus} \frac{\partial f_{10}}{\partial \Delta \Theta \oplus} \frac{\partial f_{11}}{\partial \Delta \Theta \oplus} \frac{\partial f_{12}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{13}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_8}{\partial (k^2 m^2 \oplus)} \frac{\partial f_8}{\partial z^2 \oplus} \frac{\partial f_9}{\partial H^2 \oplus} \frac{\partial f_{10}}{\partial K^2 \oplus} \frac{\partial f_{11}}{\partial \Delta \Theta \oplus} \frac{\partial f_{12}}{\partial \Delta \Theta \oplus} \frac{\partial f_{13}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{14}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_9}{\partial (k^2 m^2 \oplus)} \frac{\partial f_9}{\partial z^2 \oplus} \frac{\partial f_{10}}{\partial H^2 \oplus} \frac{\partial f_{11}}{\partial K^2 \oplus} \frac{\partial f_{12}}{\partial \Delta \Theta \oplus} \frac{\partial f_{13}}{\partial \Delta \Theta \oplus} \frac{\partial f_{14}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{15}}{\partial \Delta \Theta \oplus} \\
\frac{\partial f_{10}}{\partial (k^2 m^2 \oplus)} \frac{\partial f_{10}}{\partial z^2 \oplus} \frac{\partial f_{11}}{\partial H^2 \oplus} \frac{\partial f_{12}}{\partial K^2 \oplus} \frac{\partial f_{13}}{\partial \Delta \Theta \oplus} \frac{\partial f_{14}}{\partial \Delta \Theta \oplus} \frac{\partial f_{15}}{\partial \Delta \Theta \oplus} \cdots \frac{\partial f_{16}}{\partial \Delta \Theta \oplus} \\
\end{bmatrix}$$

$$\begin{bmatrix}
\Delta(k^2 m^2 \oplus) \\
\Delta a^2 \oplus \\
\Delta \Theta \oplus \\
\Delta \Theta \oplus \\
\Delta \Theta \oplus \\
\Delta \Theta \oplus \\
\end{bmatrix}$$

with $\mathbf{\dot{x}(t_0)} =$

$$\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \end{bmatrix}$$

Since the calculus of perturbations discussed is concerned only with first-order linear differential equations, the principles of superposition and scaling will also hold. Thus, the evaluation of the bias term introduced by planetary perturbations separately is possible. To evaluate the effect of planetary perturbations, the following set of linear differential equations are solved:
\[ \frac{d}{dt} \tilde{\mathbf{r}}(t) = \mathbf{B}(t) \cdot \tilde{\mathbf{r}}(t) + \mathbf{f}(t) \]

\[ \mathbf{f}(t) = \sum_{k=1}^{N} k^2 m_k \left( \begin{array}{c} \Delta x_k \frac{r_{\Delta k}}{r_{\Delta k}^3} - X_k \frac{r_{\Delta k}}{r_{\Delta k}^3} \\ \Delta y_k \frac{r_{\Delta k}}{r_{\Delta k}^3} - Y_k \frac{r_{\Delta k}}{r_{\Delta k}^3} \\ \Delta z_k \frac{r_{\Delta k}}{r_{\Delta k}^3} - Z_k \frac{r_{\Delta k}}{r_{\Delta k}^3} \end{array} \right) \]

with

\[ \tilde{\mathbf{r}}(t_0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \]

**Error Sensitivities**

In the perturbation equations developed for the practical equations, a perturbation vector is assumed of the form

\[ \Delta \mathbf{X}(t) = \begin{bmatrix} \Delta x_{\Delta}(t) \\ \Delta y_{\Delta}(t) \\ \Delta z_{\Delta}(t) \\ \Delta \theta_{\Delta}(t) \\ \Delta \hat{\theta}_{\Delta}(t) \\ \Delta \hat{\theta}_{\Delta}(t) \\ \Delta r(t) \end{bmatrix} \]
GUIDANCE AND CONTROL

The prediction equation developed previously, Eq. 28, is, in this case

\[
\Delta \mathbf{X}(t) = \\
\begin{bmatrix}
\Delta x_{\Theta}(t) \\
\Delta y_{\Theta}(t) \\
\Delta z_{\Theta}(t) \\
\Delta r_{\Theta}(t)
\end{bmatrix} = \\
\begin{bmatrix}
\pi_{11}(t), \pi_{12}(t), \cdots, \pi_{17}(t) \\
\pi_{21}(t), \pi_{22}(t), \cdots, \pi_{27}(t) \\
\vdots \\
\pi_{61}(t), \pi_{62}(t), \cdots, \pi_{67}(t) \\
0, 0, \cdots, 1
\end{bmatrix} \cdot \\
\begin{bmatrix}
\Delta x_{\Theta}(t_o) \\
\Delta y_{\Theta}(t_o) \\
\Delta z_{\Theta}(t_o) \\
\Delta r_{\Theta}(t_o)
\end{bmatrix} + \tilde{\mathbf{p}}(t)
\]

where reference is made to the prediction from the initial error problem. It is obvious on reflection of Eq. 109 that the elements of the fundamental solution matrix \(\pi(t)\) are, in reality, error sensitivities. It can be shown, for example, that

\[
\pi_{11}(t) = \frac{\Delta x_{\Theta}(t)}{\Delta x_{\Theta}(t_o)} = \frac{\partial x_{\Theta}(t)}{\partial x_{\Theta}(t_o)} [110a]
\]

\[
\pi_{12}(t) = \frac{\Delta y_{\Theta}(t)}{\Delta y_{\Theta}(t_o)} = \frac{\partial y_{\Theta}(t)}{\partial y_{\Theta}(t_o)} [110b]
\]

\[
\vdots
\]

\[
\pi_{67}(t) = \frac{\Delta r_{\Theta}(t)}{\Delta r_{\Theta}(t_o)} = \frac{\partial z_{\Theta}(t)}{\partial r_{t_o}} [110c]
\]

Similarly, in the linear prediction equation for predicting \(\Delta \mathbf{X}(T)\), it can be shown that the elements of the \(\Lambda^*\) matrix are error sensitivities to terminal error, i.e.

189
GUIDANCE AND CONTROL

\[ \lambda_{11}(t) = \frac{\Delta x_{\Delta}}{\Delta x_{\Delta}(t)} \approx \frac{\partial x_{\Delta}}{\partial x_{\Delta}(t)} \]

\[ \lambda_{12}(t) = \frac{\Delta x_{\Delta}}{\Delta y_{\Delta}(t)} \approx \frac{\partial x_{\Delta}}{\partial y_{\Delta}(t)} \]

\[ \vdots \]

\[ \lambda_{67}(t) = \frac{\Delta z_{\Delta}}{\Delta r(t)} \approx \frac{\partial z_{\Delta}}{\partial r} \]

[111]

CONCLUSIONS

The objective of this paper has been to derive, illustrate, and interpret the calculus of perturbations as applied to the lunar mission. It is obvious that the application can be easily interpreted in terms of an interplanetary or satellite mission as well. The perturbation calculus has had many applications in numerical integration error propagation and error sensitivity studies in the past, as evidenced by the list of references, and it is believed by the author that it already has become a necessary analysis tool for the practicing astronautics engineer of today.

ACKNOWLEDGMENT

This paper was motivated originally by a discussion with Wallace H. Ito of the MPG Research Department during which he suggested the basic concept of perturbing the nonlinear differential operator. The concept was then expanded in detail and demonstrated by the author. The encouragement, review, and criticism offered by George D. Swanlund has been especially useful and is gratefully acknowledged.

REFERENCES


GUIDANCE AND CONTROL


6 Dzerzhinskii, F.E., Computation of Variable Trajectory Elements (Military-Technical Academy Press, 1928) (Russian).


16 Sheldon, J.W., Zondek, B., and Friedman, M., "On the time step to be used for the computation of orbits by numerical numerical integration," MTAC XI, no. 59, 181-189 (July 1957).


GUIDANCE AND CONTROL

Fig. 1 Configuration Space

Fig. 2 Disturbed motion with respect to a reference solution
GUIDANCE AND CONTROL

Fig. 3 The N-body problem