LONGITUDINAL RANGE CONTROL FOR A LIFTING VEHICLE ENTERING A PLANETARY ATMOSPHERE

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ABSTRACT

The problem of controlling the longitudinal range of a lifting vehicle entering the atmosphere at small entry angles is considered. The method proposed depends on the accurate prediction of the distance that a vehicle will travel as a function of its lift to drag ratio. The vehicle trajectory is assumed to be close to an equilibrium glide trajectory. The effect on range of deviations from the equilibrium glide trajectory is determined by using the method of adjoint functions. The effects of planetary rotation and oblateness are taken into account.

INTRODUCTION

This paper is concerned with the problem of controlling the longitudinal range of a lifting vehicle entering the atmosphere at small entry angles. A number of guidance systems have already been proposed. These systems fall into the following two categories:

1) Nominal trajectory system. A trajectory which terminates at the desired landing point is selected. The deviation of the vehicle from this trajectory is measured, and the angle of attack is adjusted as a function of the measurement in order

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to return the vehicle to the nominal trajectory.

2) Range prediction system. The range that a vehicle will travel is calculated as a function of its initial conditions and angle of attack. The angle of attack is then adjusted so that the range corresponds to the desired distance. A nominal trajectory may be used to reduce the calculations required for range prediction, but no attempt is made to follow this trajectory.

Cheatham, Young and Eggleston (1)[2] consider both of the above approaches. They choose range as a function of altitude for the nominal trajectory. At a given altitude, the range-to-go is compared with the range-to-go along the nominal at that altitude. The angle of attack is then adjusted as a function of the range error and range rate error. It is shown that oscillations in the flight path can be damped out to the desired nominal path. For their range prediction system they use calculations based on numerical solutions of the complete equations of motion. Plots are made of range as a function of angle of attack for different values of entry angle, wing loading, and flight direction. The angle of attack required can then be determined as a function of range-to-go. Reismann and Pistiner (2) also make use of a range as a function of altitude nominal trajectory. Their vehicle is basically a drag vehicle but has a deflectable tail to provide control. The nominal trajectory begins at the relatively steep entry angle of 1.6° with a velocity equal to 0.95 of circular orbital velocity. If the initial errors are limited to 0.1° in entry angle and 50 miles in range, an accuracy of 0.02 mile is obtained when the system is simulated on a digital computer. Webber (3) combines the concept of a nominal trajectory with range prediction. He selects an arbitrary nominal trajectory which traverses the desired distance. When a deviation from the nominal is measured, the effect of the deviation on range is computed and the angle of attack is adjusted so that the range at the new angle of attack corresponds to the desired distance. No attempt is made to return to the nominal trajectory. The computation of the effect on range of deviations from the nominal is facilitated by using the method of adjoint functions. Dow, Fields, and Scammell (4) describe two types of guidance systems for entry at escape velocity. The first is a range prediction system in which the landing point for a given lift to drag ratio is determined by an on-board solution of the equations of motion. The computed landing point is compared with the desired landing point and a new value of L/D

[2]Numbers in parentheses indicate References at end of paper.

is selected on the basis of the difference. This process is repeated continuously so that the L/D corresponding to the desired range is determined. The second system is a nominal trajectory system. The nominal consists essentially of a set of velocity vector directions which are stored as a function of range-to-go. An angle of attack command signal is obtained by comparing the direction of the actual velocity vector with the nominal value.

The system proposed in this paper, a range prediction system, is based on the assumption that the vehicle trajectory is always reasonably close to an equilibrium glide trajectory. The expression for range travelled along an equilibrium glide path is well known. Deviations in this range due to nonequilibrium glide initial conditions are evaluated by solving a set of linear perturbation equations. The solution of these equations is then used to evaluate the effect on range of all the approximations made in obtaining the equilibrium glide solution of the equations of motion. In this way, the effects of planetary oblateness and atmospheric rotation can easily be taken into account.

The major virtue of this system is that the range prediction scheme uses the equilibrium glide trajectory rather than an arbitrary nominal trajectory. An analytic solution is available for all the trajectory variables along an equilibrium glide path. Changes in desired range, position of entry, or orbital inclination do not require the computation of a new nominal trajectory.

EQUILIBRIUM GLIDE TRAJECTORY

The equilibrium glide trajectory, first described by Sanger (5), is an approximate solution to the two dimensional equations describing the motion of a lifting vehicle entering the atmosphere at a small entry angle. The equilibrium is established between forces in the vertical direction. Lift plus centrifugal force balance the gravitational force.

The differential equations of motion in directions perpendicular and parallel to the fight path are, respectively (see Fig. 1)

$$\frac{L}{m} + \frac{V^2 \cos\gamma}{r} = g\cos\gamma - V\dot{\gamma} \qquad [1]$$

$$\frac{dV}{dt} = -\frac{D}{m} + g\sin\gamma \qquad [2]$$

Altitude and distance travelled are related to V and γ by the expressions

$$\frac{dh}{dt} = -V \sin \gamma \qquad [3]$$

$$\frac{dX}{dt} = \frac{r_P}{r} V \cos \gamma \qquad [4]$$

and it is assumed that: 1) the planet and atmosphere are not rotating; and 2) the planet is spherical.

The equilibrium glide solution of these equations is obtained by making the following assumptions:

1) γ is small enough so that $\sin \gamma \approx \gamma$ and $\cos \gamma \approx 1$.

2) $V\dot{\gamma}$ is negligible compared to $g - (V^2/r)$.

3) The component of gravity along the flight path is negligible compared to the acceleration due to drag.

4) The rate of change of local circular orbital velocity is negligible compared to the drag acceleration.

5) Changes in radius are negligible.

6) Density is related to altitude by the expression

$$\rho = \rho_0 e^{-\beta h} \qquad [5]$$

Details of the derivation can be found in Ref. 6. The results that will be needed here are

Flight Path Angle

$$\gamma_{eg} = \frac{2}{r\beta (L/D)\bar{V}^2} \qquad [6]$$

Altitude

$$h_{eg} = \frac{1}{\beta} \log \frac{\rho_0 (C_L A/m) r\bar{V}^2}{2(1-\bar{V}^2)} \qquad [7]$$

Density

$$\rho_{eg} = \frac{2(1-\bar{V}^2)}{(C_L A/m) r\bar{V}^2} \qquad [8]$$

Range Along Surface of Planet

$$X_{eg} = X_i + \frac{1}{2} r_p \frac{L}{D} \log \frac{1-\bar{V}^2}{1-\bar{V}_i^2} \qquad [9]$$

where $\bar{V}$ is dimensionless velocity equal to velocity divided by local circular orbital velocity, the subscript eg indicates equilibrium glide value, and the subscript i indicates initial value.

The equilibrium glide expression for range given in Eq. 9 is correct only if the initial conditions in flight path angle and altitude correspond to the equilibrium glide values given in Eqs. 6 and 7. The effect on range of nonequilibrium glide initial conditions will be determined in the next section.

PERTURBATION EQUATIONS USING THE EQUILIBRIUM GLIDE TRAJECTORY AS THE NOMINAL FLIGHT PATH

The equations to be perturbed are Eqs. 1-4. All of the assumptions mentioned are made except that the term $V\dot{\gamma}$ is not neglected. $V\dot{\gamma}$ must be kept in Eq. 1 so that the perturbation in flight path angle can be introduced.

The two-dimensional equations are rewritten in the following form:

$$\frac{d\gamma}{dt} = \frac{g}{\bar{V} V_s} - \frac{1}{2} \rho \bar{V} V_s \frac{C_L A}{m} - \frac{\bar{V} V_s}{r} \qquad [10]$$

$$\frac{d\bar{V}}{dt} = -\frac{1}{2} \rho \bar{V}^2 V_s \frac{C_D A}{m} \qquad [11]$$

$$\frac{dh}{dt} = -\bar{V} V_s \gamma \qquad [12]$$

$$\frac{dX}{dt} = \bar{V} V_s \qquad [13]$$

The trajectory does not depend on time. Time will therefore be eliminated from the equations, and $\bar{V}$ will be used as the independent variable. Write $d\gamma/dt$ in the form

$$\frac{d\gamma}{dt} = \frac{d\gamma}{d\bar{V}} \frac{d\bar{V}}{dt} \qquad [14]$$

Substituting Eq. 14 into Eq. 10 and using Eq. 11 results in

$$\frac{d\gamma}{d\bar{V}} = \frac{1-\bar{V}^2}{\frac{1}{2}\rho r \bar{V}^3 (C_D A/m)} + \frac{L/D}{\bar{V}} \qquad [15]$$

Following the analogous procedure for h and X gives

$$\frac{dh}{d\bar{V}} = \frac{\gamma}{\frac{1}{2}\rho \bar{V} (C_D A/m)} \qquad [16]$$

$$\frac{dX}{d\bar{V}} = -\frac{1}{\frac{1}{2}\rho \bar{V} (C_D A/m)} \qquad [17]$$

Introduce the perturbation variables by means of the relations

$$\gamma = \gamma_{eg} + \delta\gamma \qquad [18]$$

$$h = h_{eg} + \delta h \qquad [19]$$

$$X = X_{eg} + \delta X \qquad [20]$$

The linearized relation between altitude and density is

$$\rho = \rho_{eg}(1 - \beta\delta h) \qquad [21]$$

Substituting Eqs. 18-21 into Eqs. 15-17 yields

$$\frac{d\,\delta\gamma}{d\bar{V}} = \frac{\beta(1-\bar{V}^2)}{\frac{1}{2}\rho_{eq}\, r\bar{V}^3(C_D A/m)}\,\delta h - \frac{d\,\gamma_{eq}}{d\bar{V}} \qquad [22]$$

$$\frac{d\,\delta h}{d\bar{V}} = \frac{1}{\frac{1}{2}\rho_{eq}\,\bar{V}(C_D A/m)}\,\delta\gamma + \frac{\beta\,\gamma_{eq}}{\frac{1}{2}\rho_{eq}\,\bar{V}(C_D A/m)}\,\delta h \qquad [23]$$

$$\frac{d\,\delta X}{d\bar{V}} = -\frac{\beta}{\frac{1}{2}\rho_{eq}\,\bar{V}(C_D A/m)}\,\delta h \qquad [24]$$

The term $d\,\gamma_{eq}/d\bar{V}$ appears in Eq. 22, because the equilibrium glide values of h and γ do not satisfy Eq. 15. The perturbation equations are finally obtained by substituting the equilibrium glide values of ρ and γ into Eqs. 22 and 24.

$$\frac{d\,\delta\gamma}{d\bar{V}} = -\frac{\beta\,(L/D)}{\bar{V}}\,\delta h - \frac{d\,\gamma_{eq}}{d\bar{V}} \qquad [25]$$

$$\frac{d\,\delta h}{d\bar{V}} = \frac{r\,(L/D)\bar{V}}{1-\bar{V}^2}\,\delta\gamma + \frac{2}{\bar{V}(1-\bar{V}^2)}\,\delta h \qquad [26]$$

$$\frac{d\,\delta X}{d\bar{V}} = -\frac{r\beta\,(L/D)\bar{V}}{1-\bar{V}^2}\,\delta h \qquad [27]$$

The resulting equations are linear with variable coefficients. They could be solved in their present form, but it is much more convenient to solve the adjoint set of equations. To see this, it is necessary to consider just what information is required from the solution of the perturbation equations. The quantity of interest is the range deviation at the terminal velocity due to nonequilibrium glide conditions at the initial velocity. In terms of the equation variables, $\delta X(\overline{V}_f)$ is desired as a function of $\delta\gamma(\overline{V}_i)$ and $\delta h(\overline{V}_i)$. If Eqs. 25-27 are solved in the conventional manner, $\delta X(\overline{V})$ is obtained for all values of $\overline{V}$ from $\overline{V}_i$ to $\overline{V}_f$. This is more information than is needed. A separate solution, however, is required for each value of $\overline{V}_i$ because of the variable coefficients in the equation. If the adjoint set of equations is used, one solution of the equation will provide the value of $\delta X(\overline{V}_f)$ due to $\delta\gamma$ or δh occurring at any initial velocity. No information is obtained at intermediate velocities, but none is required for guidance purposes.

The method of adjoint functions is discussed by Bliss (7), and an application to missile guidance is given by Tsien (8). Only the final result of the method will be given here. The expression for δX at the terminal velocity is

$$\delta X\Big|_{\overline{V}=\overline{V}_f} = (\lambda_1 \delta\gamma + \lambda_2 \delta h + \lambda_3 \delta X)\Big|_{\overline{V}=\overline{V}_i} - \int_{\overline{V}_f}^{\overline{V}_i} \left(\sum_{j=1}^{m} \lambda_j b_j\right) d\overline{V} \qquad [28]$$

The λ's are the adjoint variables that are the solution of the following adjoint differential equations:

$$\frac{d\lambda_1}{d\overline{V}} = -\frac{r(L/D)\overline{V}}{1-\overline{V}^2}\lambda_2 \qquad [29]$$

$$\frac{d\lambda_2}{d\overline{V}} = \frac{\beta(L/D)}{\overline{V}}\lambda_1 - \frac{2}{\overline{V}(1-\overline{V}^2)}\lambda_2 + \frac{r\beta(L/D)\overline{V}}{1-\overline{V}^2}\lambda_3 \qquad [30]$$

$$\frac{d\lambda_3}{d\overline{V}} = 0 \qquad [31]$$

with the initial conditions

$$\lambda_1(\bar{V}_f) = 0$$

$$\lambda_2(\bar{V}_f) = 0 \qquad [32]$$

$$\lambda_3(\bar{V}_f) = 0$$

specified at $\bar{V}_f$.

The b's are the forcing function of the original perturbation equations. $d\,\gamma_{eg}/d\bar{V}$ is an example of such a forcing function. The integral in Eq. 28 represents the contribution to range at the final velocity due to the forcing functions. In the next section, several more forcing functions will be introduced when the effect on range of the equilibrium glide assumptions is determined.

Before the adjoint equations can be solved, a value for the terminal velocity must be selected. A logical value would be the landing velocity. The range prediction system, however, is not sufficiently accurate to use all the way to touchdown. At low velocities, the trajectory differs significantly from the equilibrium glide trajectory. The range correction terms are derived from a linear perturbation equation and become inaccurate when the actual trajectory differs significantly from the nominal trajectory. The final velocity will therefore be picked as two tenths of circular orbital velocity.

The solution of Eq. 31 is $\lambda_3 = 1$. After substituting this result in Eq. 30, Eqs. 29 and 30 are numerically solved. β and r, the two planetary constants involved in the equations, are set equal to their values for Earth: $\beta = 1/23{,}500\ \text{ft}^{-1}$ and $r = 20{,}926{,}428$ ft. Figs. 2 and 3 show plots of λ_1 and λ_2, respectively, for L/D = 0.5, 1.0, and 2.0. λ_1 is the range deviation due to nonequilibrium glide flight path angle and is plotted in units of nautical miles per milliradian. λ_2 is the range deviation due to nonequilibrium glide altitude and is plotted in units of nautical miles per 1000 ft. Examination of the plots shows that λ_1 is almost independent of L/D, whereas λ_2 varies inversely as L/D. In Ref. 9, an approximate analytic solution for λ_1 and λ_2 is derived. The first term in the approximate expressions for each is

$$\lambda_{1a} = -\frac{r\bar{V}^2}{1-V^2} \qquad [33]$$

$$\lambda_{2a} = \frac{2}{(L/D)(1-\bar{V}^2)} \qquad [34]$$

Eqs. 33 and 34 verify the observed variation with L/D and give an idea of the variation with $\bar{V}$. The two range perturbation terms decrease in magnitude as velocity is reduced in proportion to the quantity $(1-\bar{V}^2)^{-1}$. From Eq. 9, it is seen that the range remaining along an equilibrium glide trajectory varies as the log $(1-\bar{V}^2)^{-1}$. The perturbation terms decrease more rapidly than does the range and therefore become less important as velocity is reduced.

The two range deviation terms have been obtained from the solution of a set of linearized equations. If the actual trajectory differs from the nominal to such an extent that the linearizing assumptions are violated, the results obtained from the linear solution will be in error. The range of validity of the linear relation between range deviation and flight path angle or altitude deviation must be determined. A oomparison between the actual and predicted range deviations is shown in Figs. 4 and 5. The straight line is the predicted value of range deviation taken from the numerical solution of the adjoint equations at $\bar{V}$ = 0.99. The circled points are obtained from numerical solutions of the complete equations of motion. The assumption required to obtain the linear perturbation equations is that density deviation varies linearly with altitude deviation. This assumption is good for values of δh small compared to the scale height[3] of the atmosphere. Flight path angle deviations of 8 mrad cause altitude deviations that exceed the scale height. The agreement between the linear solution and the complete solution is quite good in view of the extent to which the linearizing assumption is violated.

EFFECT ON RANGE OF TERMS NEGLECTED IN OBTAINING EQUILIBRIUM GLIDE SOLUTION

The effect on range of the assumptions listed at the beginning of the paper will be determined here. The method of adjoint functions provides a convenient method for taking into account those assumptions that result in terms being added

[3]The scale height of the atmosphere is the altitude change required for the density to change by a factor of e. It is the reciprocal of the atmospheric decay constant β. The scale height for Earth is approximately 23,500 ft up to an altitude of 80 miles.

to the equilibrium glide equations of motion. Assuming equilibrium glide initial conditions, Eq. 28 becomes

$$\delta X\Big|_{\bar{V}=\bar{V}_f} = -\int_{\bar{V}_f}^{\bar{V}_i} \left(\sum_{j=1}^{m} \lambda_j b_j\right) d\bar{V} \qquad [35]$$

where b_1, b_2, and b_3 are the forcing functions of the three perturbation equations, Eqs. 25-27. The value of b corresponding to each assumption must be determined.

Effect of $V\dot{\gamma}$

The term $V\dot{\gamma}$ was carried along in the derivation of the perturbation equations and appears in Eq. 25. The forcing function is

$$b_1 = -\frac{d\gamma_{eq}}{d\bar{V}} = \frac{4}{\beta r\,(L/D)\,\bar{V}^3}, \qquad [36]$$

Substituting Eq. 36 into Eq. 35, the range associated with the term $V\dot{\gamma}$ is written as

$$X_{V\dot{\gamma}}\Big|_{\bar{V}=\bar{V}_f} = -\frac{4}{\beta r(L/D)}\int_{\bar{V}_f}^{\bar{V}_i} \frac{\lambda_1}{\bar{V}^3}\, d\bar{V} \qquad [37]$$

The integral must be evaluated for each value of L/D. One such integration, however, gives $X_{V\dot{\gamma}}\big|_{\bar{V}=\bar{V}_f}$ for all values of $\bar{V}$ greater than $\bar{V}_f$. Fig. 6 is a plot of $X_{V\dot{\gamma}}\big|_{\bar{V}=\bar{V}_f}$ as a function of L/D for an initial velocity $\bar{V}_i = 0.99$. Because λ_1, to a first approximation, is independent of L/D, $X_{V\dot{\gamma}}\big|_{\bar{V}=\bar{V}_f}$ varies inversely with L/D.

Effect of Nonspherical Figure of the Planet

This section will be limited to the geometric effects of the nonspherical figure of the planet. The altitude and radius do not change at the same rate for a nonspherical planet. If flight path angle is measured with respect to the local geographic[4] horizon, it serves to specify rate of change of altitude. Eq. 3 is therefore correct as it stands. An additional term must be added to the equation of motion in a direction perpendicular to the velocity vector, because the flight path angle in that equation refers to rate of change of radius. Thus, Eq. 10 for a nonspherical planet becomes

$$\frac{d}{dt}(\gamma + f) = \frac{g}{\bar{V} V_s} - \frac{1}{2} \rho \bar{V} V_s \frac{C_L A}{m} - \frac{\bar{V} V_s}{r} \qquad [38]$$

where $(\gamma + f)$ is equal to the flight path angle measured with respect to the local geocentric horizon. f is illustrated in Fig. 7. The deviation of the vertical d is equal to

$$d = \epsilon^2 \sin L \cos L \qquad [39]$$

where ϵ is the eccentricity of the planet. ϵ^2 is 0.00672 for Earth. The angle f is related to d through the bearing angle k.

$$f = \epsilon^2 \sin L \cos L \cos k \qquad [40]$$

The bearing angle can be expressed in terms of the other trajectory parameters by the relation

$$\cos k = \frac{\sin \dot{\iota} \cos \theta}{\cos L} \qquad [41]$$

Substituting Eq. 41 into Eq. 40, and substituting for sin L, the expression

$$\sin L = \sin \dot{\iota} \sin \theta \qquad [42]$$

yields

$$f = \frac{\epsilon^2}{2} \sin^2 \dot{\iota} \sin 2\theta \qquad [43]$$

[4]The geographic horizon lies in a plane that is tangent to the reference ellipsoid representing the figure of the planet. The geocentric horizon lies in a plane that is normal to a line drawn to the center of the planet.

The derivative of f with respect to $\bar{V}$ is

$$\frac{df}{d\bar{V}} = - \frac{(L/D)\,\epsilon^2 \sin^2 i\, \bar{V} \cos 2\theta}{1 - \bar{V}^2} \qquad [44]$$

This is the forcing function that appears in the perturbation equation, Eq. 25, due to the nonspherical figure of the planet. The range effect associated with this forcing function is

$$X_{FP}\Big|_{\bar{V}=\bar{V}_f} = -\frac{L}{D}\,\epsilon^2 \sin^2 i \int_{\bar{V}_f}^{\bar{V}_i} \frac{\lambda_1 \bar{V} \cos 2\theta}{1-\bar{V}^2}\, d\bar{V} \qquad [45]$$

The total distance travelled at any velocity is assumed to be close to the equilibrium glide distance, so that θ is equal to

$$\theta = \theta_i + \frac{1}{2}\frac{L}{D} \log \frac{1-\bar{V}^2}{1-\bar{V}_i^2} \qquad [46]$$

The integral in Eq. 45 is a function of L/D and θ_i. The inclination of the orbit is outside of the integral, so that X_{FP} is an analytic function of i once the integral has been evaluated. Fig. 8 shows X_{FP} as a function of θ_i for several values of L/D and i = 90. The curves are seen to resemble sine waves of argument $2\theta_i$. X_{FP} can be expressed in the form

$$X_{FP} = B_{FP} \sin\left(2\theta_i + \phi_{FP}\right) \qquad [47]$$

B_{FP} and ϕ_{FP} are functions of L/D obtained from Fig. 8.

The physical basis for this range term is easily explained. Consider a vehicle in a polar orbit starting at the equator with equilibrium glide initial conditions. As the vehicle moves north, the radius of the planet decreases. The altitude, therefore, does not decrease as rapidly as it would for a spherical planet. The density and the associated aerodynamic force are reduced, with the result that the vehicle travels farther than for a spherical Earth.

Effect of $g\gamma$, dV_s/dt, and Nonspherical Component of Gravity

These three quantities are grouped together because they all appear as additional terms in the equation of motion in a direction parallel to the velocity vector. The complete equation, replacing Eq. 11, is

$$\frac{d\bar{V}}{dt} = \frac{1}{V_s}\left[-\frac{D}{m} + g_r(\gamma + f) - g_\theta - \bar{V}\frac{dV_s}{dt}\right] \qquad [48]$$

g_r and g_θ are the radial and tangential components of gravity, respectively. They are

$$g_r = \frac{KM}{r^2} + 6\mu\frac{KMa^2}{r^4}\left(1 - 3\sin^2 i\,\sin^2\theta\right) \qquad [49]$$

$$g_\theta = \frac{6\mu\,KMa^2}{r^4}\sin^2 i\,\sin 2\theta \qquad [50]$$

where μ is a dimensionless constant specifying the magnitude of the nonspherical gravitational field, and a is the equatorial radius of the planet. The direction of the gravitational components is shown in Fig. 9. For Earth, $6\mu = 1.638 \times 10^{-3}$. dV_s/dt is evaluated by differentiating the equation

$$V_s = \sqrt{g_r r} \qquad [51]$$

Setting a equal to r, the resulting expression is

$$\frac{dV_s}{dt} = \frac{V_s^2\bar{V}}{2r}\left[(\gamma + f) - 18\mu\sin^2 i\,\sin 2\theta\right] \qquad [52]$$

Substituting Eqs. 43, 49, 50, and 52 into Eq. 48 gives

$$\frac{d\bar{V}}{dt} = -\frac{1}{V_s}\frac{D}{m} + \frac{2V_s}{r^2\beta(L/D)\bar{V}^2}\left(1 - \frac{\bar{V}^2}{2}\right) + \frac{V_s}{r}\sin^2 i\,\sin 2\theta\left[\frac{\epsilon^2}{2}\left(1 - \frac{\bar{V}^2}{2}\right) + 3\mu\left(3\bar{V}^2 - 2\right)\right] \qquad [53]$$

Forcing functions arise when Eq. 53, rather than Eq. 11, is used in deriving the perturbation equations. The second term on the right-hand side of Eq. 53 is due to $g\gamma$ and dV_s/dt. The range associated with these terms will be called $X^s_{g\gamma}$ and is equal to

$$X_{g\gamma}\Big|_{\bar{V}=\bar{V}_f} = \frac{2\,L/D}{\beta}\int_{\bar{V}_f}^{\bar{V}_i} \frac{1-\frac{\bar{V}^2}{2}}{\bar{V}(1-\bar{V}^2)^2}\,d\bar{V} - \frac{4}{\beta^2 h}\int_{\bar{V}_f}^{\bar{V}_i}\frac{\left(1-\frac{\bar{V}^2}{2}\right)\lambda_2}{\bar{V}^3(1-\bar{V}^2)^2}\,d\bar{V} \quad [54]$$

The first integral can be evaluated analytically.

$$\int_{\bar{V}_f}^{\bar{V}_i}\frac{1-\frac{\bar{V}^2}{2}}{\bar{V}(1-\bar{V}^2)^2}\,d\bar{V} = \frac{1}{2}\log\frac{\bar{V}^2}{1-\bar{V}^2}\Bigg|_{\bar{V}_f}^{\bar{V}_i} + \frac{1}{4}\,\frac{\bar{V}^2}{1-\bar{V}^2}\Bigg|_{\bar{V}_f}^{\bar{V}_i} \quad [55]$$

The second term of Eq. 54 must be evaluated numerically. It is relatively small, having a value of 10n miles for L/D = 1, $\bar{V}_i$ = 0.99, and $\bar{V}_f$ = 0.20. The third term on the right-hand side of Eq. 53 comes about due to the nonspherical component of gravity. The range X_{NSG} associated with this term is

$$X_{NSG}\Big|_{\bar{V}=\bar{V}_f} = -\frac{\frac{L}{D}\sin^2 i}{\beta}\int_{\bar{V}_f}^{\bar{V}_i}\frac{\left[6\mu(3\bar{V}^2-2)+\epsilon^2\left(1-\frac{\bar{V}^2}{2}\right)\right]\sin 2\theta\,\lambda_2}{\bar{V}(1-\bar{V}^2)^2}\,d\bar{V}$$
$$+\frac{h\left(\frac{L}{D}\right)^2\sin^2 i}{2}\int_{\bar{V}_f}^{\bar{V}_i}\frac{\left[6\mu(3\bar{V}^2-2)+\epsilon^2\left(1-\frac{\bar{V}^2}{2}\right)\right]\bar{V}\sin 2\theta}{(1-\bar{V}^2)^2}\,d\bar{V} \quad [56]$$

X_{NSG} is plotted as a function of θ_i for several values of L/D and i = 90° in Fig. 10. The form of the plots is identical to that for X_{FP}. X_{NSG} can be added to X_{FP} so that all the information contained in Figs. 8 and 10 can be expressed in terms of the amplitude and phase of a sine wave of argument $2\theta_i$.

Numerical Check of Accuracy of Expressions Derived

Numerical solutions of the complete equations of motion were used to check the results. For an equatorial orbit, the nonspherical effects are zero. The difference in range between that predicted by the equilibrium glide range expression and that obtained from the solution of the complete equations must be due to $X_{V\dot{\gamma}}$ and $X_{g\gamma}$. For L/D = 1, the distance travelled between $\bar{V} = 0.99$ and $\bar{V} = 0.20$ is 6820n miles. The equilibrium glide expression gives a range of 6662n miles. The difference is 158n miles. The sum of $X_{V\dot{\gamma}}$ and $X_{g\gamma}$ is 161n miles.

The nonspherical effects are checked by solving the equations of motion for nonequatorial orbits and comparing the range with that for an equatorial orbit. The results are summarized in Table 1.

Table 1 Comparison of actual and predicted effect on range of the nonspherical nature of the planet

Orbital inclination, deg	Initial angle, deg	Range difference, n miles	
		actual	predicted
90	0	348	350
90	90	-302	-350
45	0	170	175

For i = 90°, the predicted nonspherical range effects for initial angles of 0° and 90° are equal in magnitude but opposite in sign. The solution of the complete equation indicates that there is a 46n mile difference in magnitude for the two cases. The explanation for this difference can be seen by referring to Fig. 4. A vehicle starting at the equator in a polar orbit on a nonspherical planet observes the altitude to be decreasing less rapidly than for a spherical planet. This corresponds to a negative flight path angle deviation. The opposite is true for a vehicle starting at the pole. In Fig. 4, it is observed that the difference between actual and predicted range is always positive. The magnitude of positive range deviations is therefore greater than the magnitude of negative range deviations. This is just the effect observed in Table 1.

Effect of Rotation of the Planet

The distance[5] that a vehicle must travel in order to reach a desired landing point depends on the time of flight, because the landing point is continuously moving. The time of flight along an equilibrium glide trajectory is given by the expression

$$t_f = t_i + \frac{r\left(\frac{L}{D}\right)}{2 V_s}\left[\log\frac{1+\bar{V}_i}{1-\bar{V}_i} - \log\frac{1+\bar{V}_f}{1-\bar{V}_f}\right] \qquad [57]$$

By using Eq. 57, the distance between the present position and the position of the landing point at the landing time can be determined.

Atmospheric rotation changes the magnitude of the aerodynamic forces. This effect can be treated in two ways:

1) The additional terms appearing in the equations of motion due to atmospheric rotation can be considered as forcing functions of the perturbation equations. This is the approach used up to now.

2) The additional terms can be included in the nominal trajectory. In this case the adjoint equations include the effect of atmospheric rotation and the equilibrium glide values of altitude and flight path angle change. The equilibrium glide value for range does not change, because it depends only on the ratio of the lift and drag forces.

The second approach is preferable. The adjoint equations are not significantly changed by the addition of the rotating atmosphere. The change in equilibrium glide flight path angle is negligible. Only the change in equilibrium glide altitude is important. The equilibrium glide altitude for a rotating atmosphere is

$$h_{eq\,ra} = h_{eq} + \frac{2}{\beta}\log\left[1 + \frac{V_A}{V_s\bar{V}}\right] \qquad [58]$$

where V_A is the velocity of the atmosphere along the flight path with respect to a nonrotating coordinate system. For an equatorial trajectory on Earth, $V_A/V_s = 0.06$. The effect on range of the rotating atmosphere can thus be determined simply by changing the equilibrium glide altitude.

[5]The distance referred to here is measured with respect to a nonrotating coordinate system.

Effect of Nonstandard Atmosphere

A nonstandard atmosphere may be represented by the equation

$$\rho = \rho_0 e^{-\beta(h) h} \quad [59]$$

where β is now a function of altitude. The effect on the equations of allowing β to be a function of h is that β is replaced wherever it appears[6] by the term $(\beta(h) + h\, d\beta/dh)$. If $\beta(h)$ is known, its effect is therefore easily computed. The difficulty, of course, is that fluctuations in density are not likely to be known in advance. The problem will be attacked in two stages. First, it can be shown that a nonstandard density has a negligible effect on all expressions except that for equilibrium glide altitude. Assume β to be a constant that differs from the predicted value by 3%. This deviation in β would produce a 50% error in density at 300,000 ft. This is a relatively large error, at least for Earth. What is the effect of this error in β on the expressions previously obtained? Equilibrium glide range is independent of β. A 3% error in β causes a 3% error in flight path angle. For L/D = 1 and $\bar{V} = 0.99$, 3% of the flight path angle is 65 microradians. The range error associated with this angle error is $65 \times 10^{-3}\ \lambda_1$, or about 9n miles. More important, however, is the fact that the uncertainty in flight path angle is likely to be larger than 65 microradians. λ_1 and λ_2 are, to a first approximation, independent of β (see Eqs. 33 and 34). X_{FP} is independent of β, and the changes in $X_{V\dot{\gamma}}$, $X_{g\gamma}$, and X_{NSG} are small. The only significant error that occurs is in the expression for h_{eg}. A 3% error in β corresponds to an altitude error of 9080 ft. The range associated with this altitude error for L/D = 1 and $\bar{V} = 0.99$ is 90n miles.

This error can be greatly reduced by eliminating the necessity for computing h_{eg}. The equilibrium glide trajectory does not really depend on altitude. It is the lift force that is important. The expression for h_{eg} is obtained from the equilibrium glide relation which can be written in the form

$$a_{L_{eg}} = \frac{V_s^2}{r}\left(1 - \bar{V}^2\right) \quad [60]$$

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[6] Except for β in the expression for equilibrium glide altitude. It remains simply $\beta(h)$.

where a_L is the acceleration due to lift. For a given $\bar{V}$ and vehicle configuration, there is only one altitude for which Eq. 60 is satisfied. That is the true equilibrium glide altitude, and its value depends on the particular altitude-density relation that is present. If a_L does not equal $a_{L_{eg}}$, then the altitude does not correspond to the equilibrium glide altitude. a_L can be written as

$$a_L = a_{L_{eg}} + \delta a_L$$
$$= \frac{1}{2} \rho_0 \frac{C_L A}{m} \bar{V}^2 V_s^2 e^{-\beta h_{eg}} e^{-\beta_\ell \delta h} \qquad [61]$$

where β_ℓ is the local value of $\beta(h)$. The two are equal only if $(d\beta/dh) = 0$.

Solving Eq. 61 for δh gives

$$\delta h = -\frac{1}{\beta_\ell} \log\left(1 + \frac{\delta a_L}{a_{L_{eg}}}\right) \qquad [62]$$

An accelerometer with its sensitive axis perpendicular to the velocity vector measures a_L. This measured quantity can then be compared with $a_{L_{eg}}$ to determine δa_L. β_ℓ is still unknown, but the assumed 3% error now causes a 3% error in the altitude deviation, rather than in the total altitude. Use of an accelerometer to measure δh also eliminates the need to compute the atmospheric velocity mentioned in the last section. The accelerometer measures the lift force which depends on the velocity of the vehicle with respect to the atmosphere.

The conclusion that can be reached, then, is that reasonable uncertainties in knowledge of density do not have an appreciable effect on the accuracy of range prediction.

INSTRUMENTATION REQUIREMENTS

An expression for the distance travelled by a lifting vehicle entering a planetary atmosphere has been developed. The problem of using this information in a range control system will now be considered.

The range expression consists of the equilibrium glide term, which is a function of L/D and $\bar{V}$, and a string of smaller terms which depend on L/D and a number of trajectory parameters. The parameters that must be available are position, altitude, velocity, flight path angle, lift acceleration, and orbital inclination.

If position and velocity information are available continuously, the range to the destination can be continuously compared with the predicted range, and L/D can be set on the basis of this comparison. In this case, it is likely that the smaller terms in the range expressions can be neglected. The closed loop nature of the system will tend to correct the effect of small errors in range prediction.

On the other hand, if L/D is to be set on the basis of information that is available at only one or two points along the trajectory, the complete range expression must be used. This situation would arise if position and velocity are obtained from ground tracking data. Such data is not presently available over much of the surface of the earth.

The range associated with a given value of L/D can easily be determined if the required information is available. It is somewhat more difficult to determine the L/D corresponding to a given range. L/D cannot be expressed explicitly as a function of range. L/D can be obtained, however, by an iteration process. It is assumed first that the total range is given by the equilibrium glide term. L/D can then be expressed as a function of range (see Eq. 9). Now the value of L/D obtained in this manner is used to evaluate all of the other range terms. L/D is adjusted so that the change in the equilibrium glide term cancels the effect of the smaller range terms. One iteration should be sufficient to provide the L/D corresponding to the desired range.

Once L/D is determined, the vehicle attitude must be adjusted to generate the desired L/D. L/D can be measured by using two accelerometers whose orientation with respect to the velocity vector is known. The angle of attack can then be adjusted until the desired ratio is obtained.

NOMENCLATURE

a = equatorial radius of planet

a_L = acceleration associated with lift

A = reference area of vehicle

b_j = forcing function of perturbation equations

B = amplitude of sine wave associated with nonspherical range terms

C_D = drag coefficient

C_L = lift coefficient

d = angle between geocentric vertical and geographic vertical

D = drag force

f = angle between geocentric vertical and projection of geographic vertical on trajectory plane

g = gravitational acceleration

h = altitude

i = orbital inclination

k = angle between north and the horizontal component of velocity

K = universal gravitational constant

L = geocentric latitude

m = mass of vehicle

M = mass of planet

r = radius from center of planet to vehicle

r_p = mean radius of planet

t = time

V = velocity with respect to nonrotating coordinate system

$\bar{V}$ = velocity divided by local circular orbital velocity

X = distance measured along surface of planet

β = exponential decay constant of the atmosphere

γ = flight path angle measured with respect to geographic horizon

ϵ = eccentricity of the planet

θ = angular range measured in trajectory plane from line of nodes

λ_j = adjoint variables

μ = constant specifying the magnitude of the nonspherical gravitational field

ρ = air density

ρ_o = intercept of straight line which best fits a curve of log ρ vs. altitude

ϕ = phase angle of sine wave associated with nonspherical range terms

Subscripts

a = approximate

A = atmosphere

eg = equilibrium glide

egra = equilibrium glide for rotating atmosphere

f = final

FP = nonspherical figure of the planet

$g\gamma$ = associated with $g\gamma$

i = initial

ℓ = local

NSG = nonspherical gravity

r = radial

s = local circular orbit

$V\dot{\gamma}$ = associated with $V\dot{\gamma}$

θ = tangential

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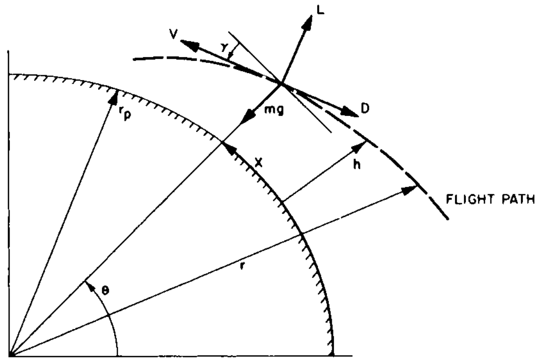

Fig. 1 Two-dimensional trajectory

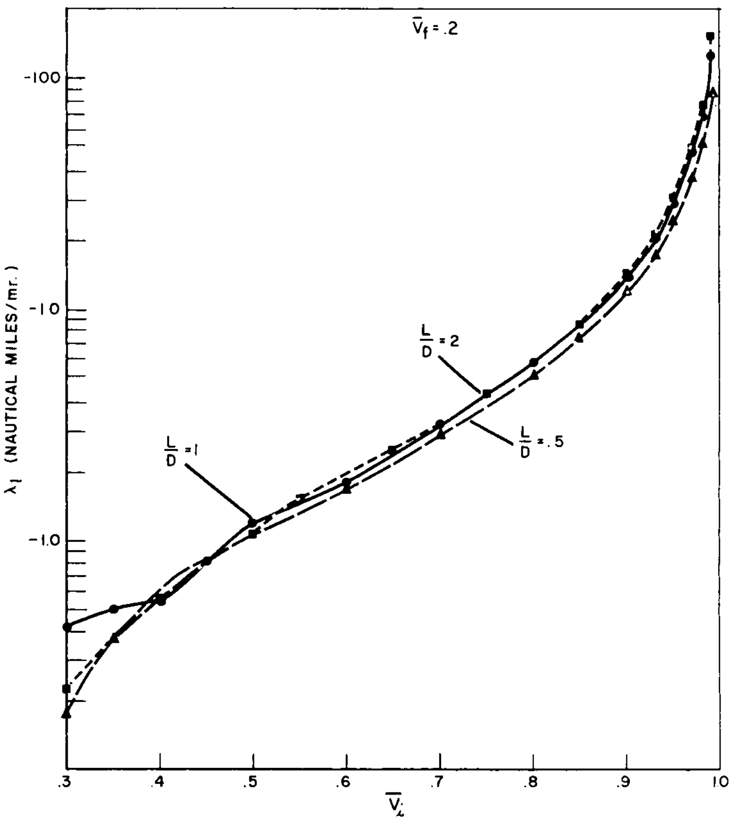

Fig. 2 Range deviation due to nonequilibrium glide initial condition in flight path angle

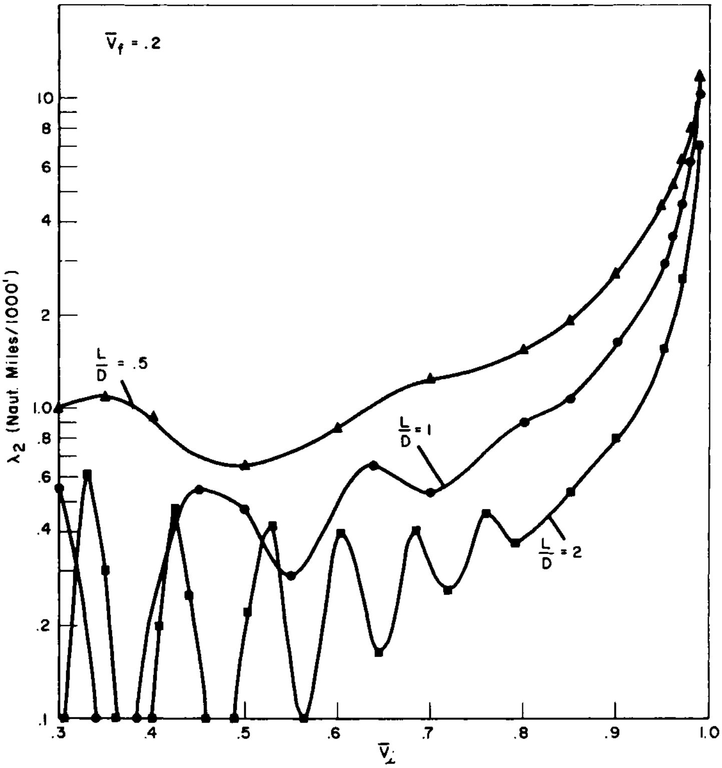

Fig. 3 Range deviation due to nonequilibrium glide initial condition in altitude

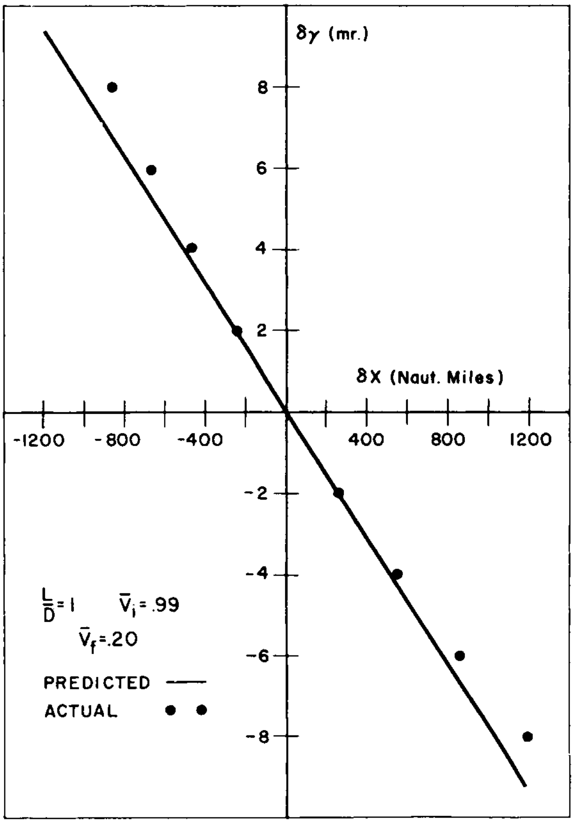

Fig. 4 Comparison between predicted and actual range deviation due to nonequilibrium glide initial condition in flight path angle

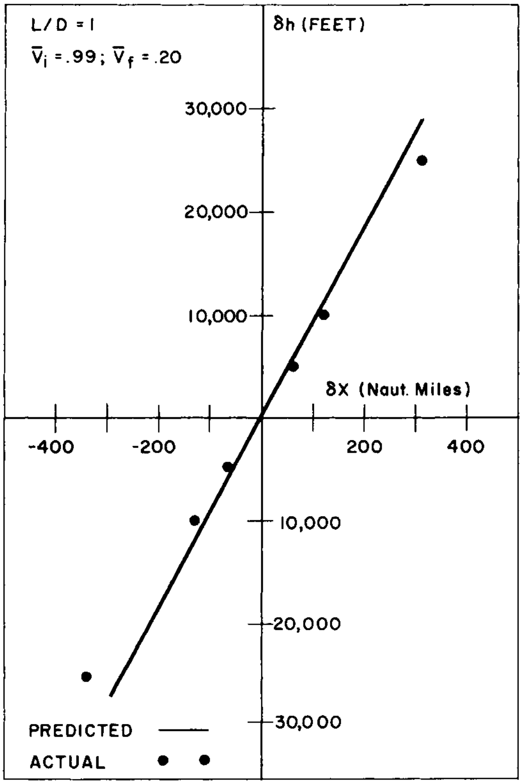

Fig. 5 Comparison between predicted and actual range deviation due to nonequilibrium glide initial condition in altitude

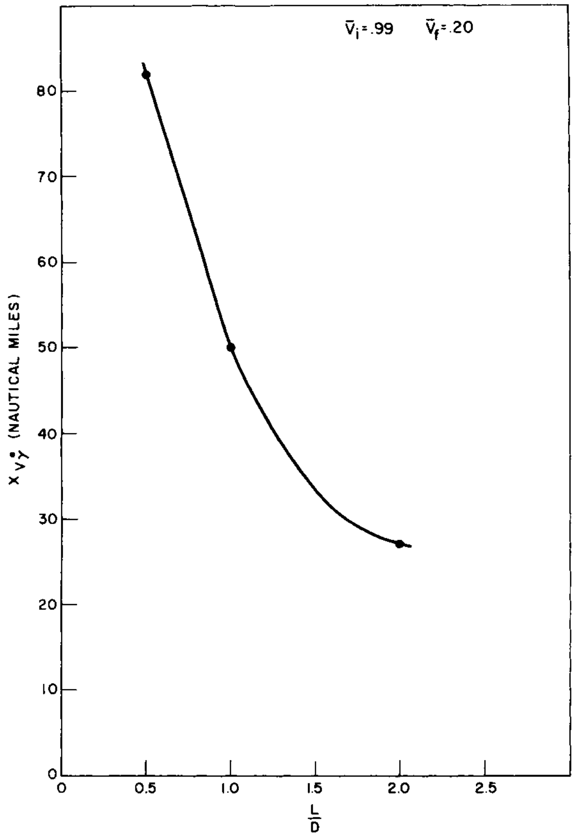

Fig. 6 Range associated with $V\dot{\gamma}$ as a function of L/D

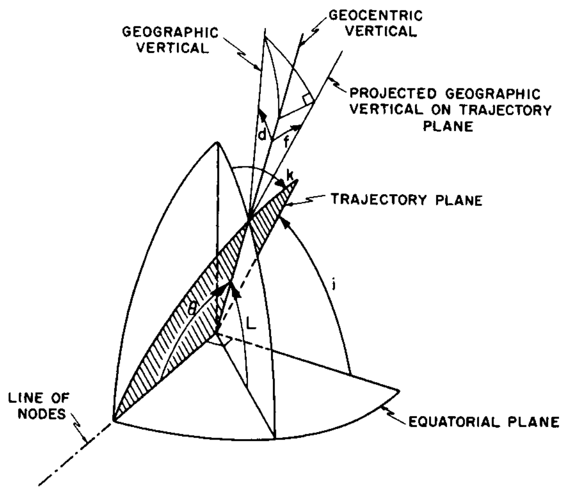

Fig. 7 Quantities required for analysis of two-dimensional trajectory on a nonspherical planet

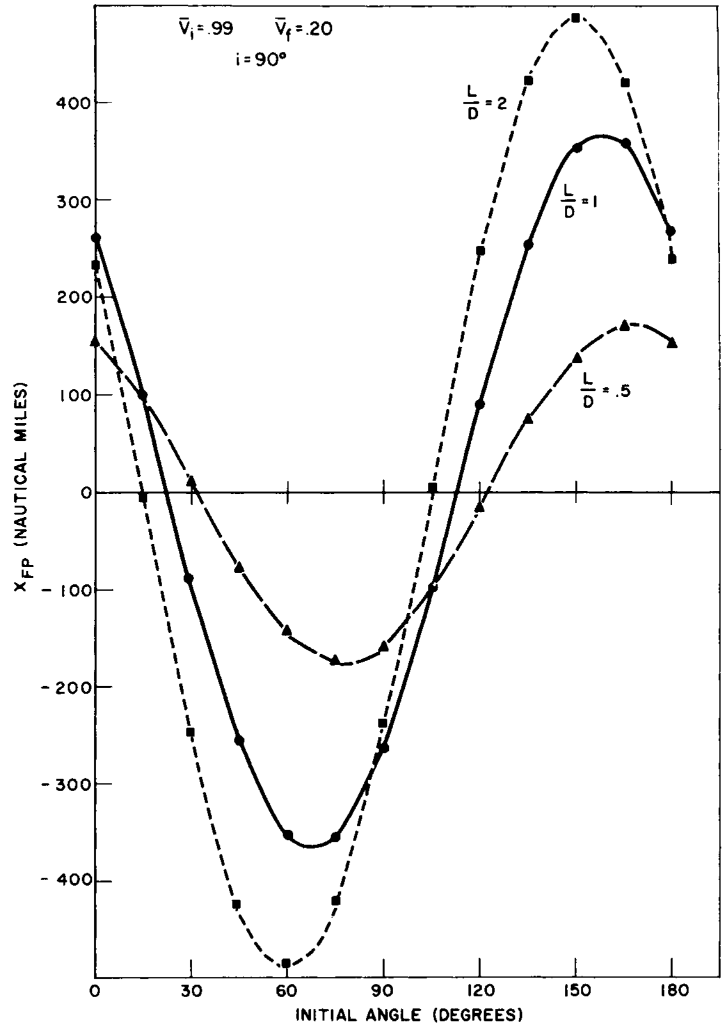

Fig. 8 Range associated with the nonspherical figure of the planet

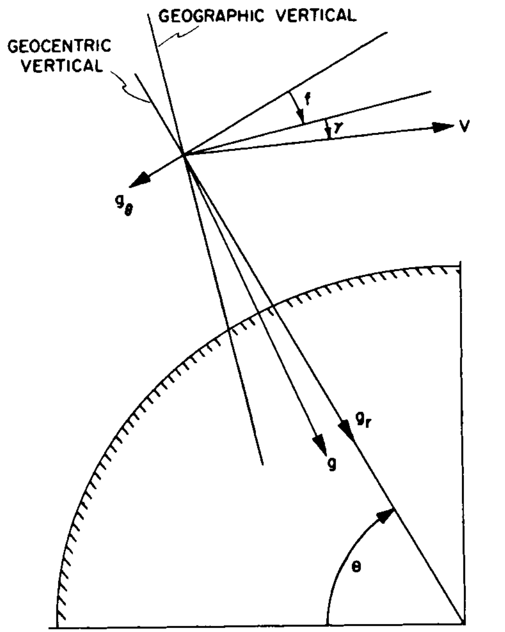

Fig. 9 Direction of gravitational components

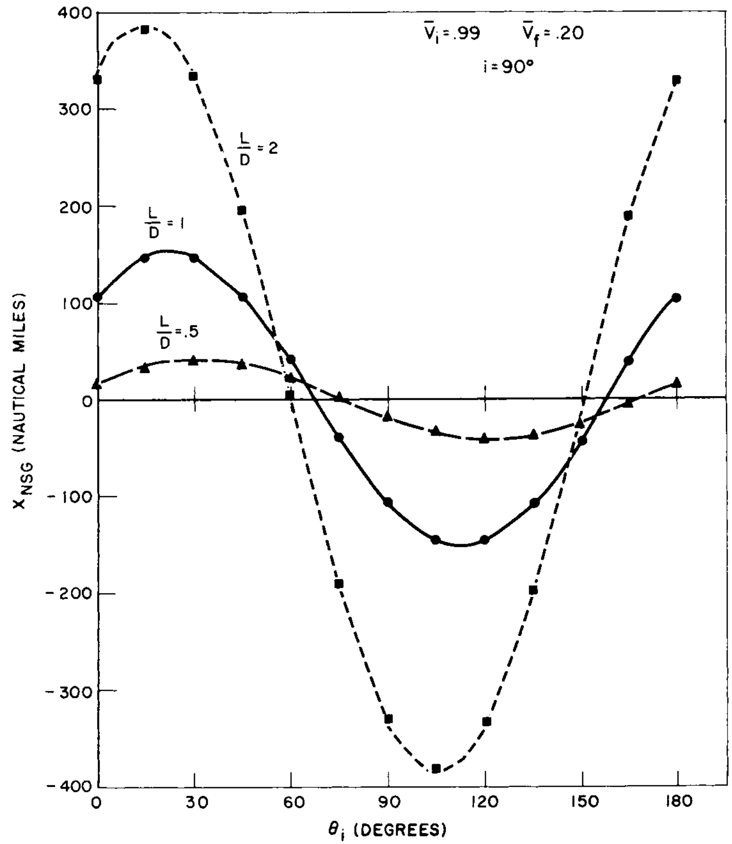

Fig. 10 Range Associated with the nonspherical component of gravity