

**SECOND-ORDER COMPRESSIBLE BOUNDARY LAYER THEORY  
WITH APPLICATION TO BLUNT BODIES IN HYPERSONIC FLOW**

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ABSTRACT

Viscous hypersonic flow near the nose of a blunt body is considered on the basis of the Navier-Stokes equations. Conventional boundary layer theory is embedded in a systematic expansion scheme. The general theory of the second approximation is developed. Seven second-order effects are identified: longitudinal curvature, transverse curvature, slip, temperature jump, entropy gradient, stagnation enthalpy gradient, and displacement. Their evaluation for a blunt body is outlined, and numerical results given for the stagnation region of a cooled sphere at infinite Mach number. In that example the increase in heat transfer due to the entropy gradient is reduced one-third by the other second-order effects.

INTRODUCTION

Ferri & Libby (Ref. 1) first pointed out that the boundary layer on a blunt body in supersonic flow is influenced by the external vorticity generated by the bow shock wave. Theories of this "vorticity interaction" have since been developed by (among others) Hayes and Probstein (Ref. 2) and Ferri, Zakkay and Ting (Ref. 3). Unfortunately, these two theories differ by a factor of more than 5 in their predictions of the increase in heat transfer due to external vorticity.

Furthermore, two objections have been raised against both theories. First, Rott and Lenard (Ref. 4) point out that the effect of external vorticity is only one of a number of second-order effects in the boundary layer, all of which should

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logically be considered concurrently. Second, study of incompressible flow (Van Dyke in Ref. 5) indicates that matching of the boundary layer with the outer rotational flow has not previously been carried out correctly.

The present study aims to clarify this situation by calculating the complete second approximation for the boundary layer near the nose of a blunt body in hypersonic flow, and in a typical case to compare the contributions to heat transfer and skin friction of all the second-order effects.

### CONTINUUM FLOW PAST A BLUNT BODY IN THE VISCOUS HYPERSONIC LIMIT

The author considers a symmetric plane or axisymmetric blunt body in a steady uniform hypersonic stream as indicated in Fig. 1. The body has a nose radius  $a$ , and is assumed to be analytic at least past the limiting characteristic that reaches the subsonic region. The gas is taken to be perfect with constant specific heats and Prandtl number. The viscosity is assumed to depend only on temperature, in this section as its  $\omega$  power.

Dimensional analysis shows that for a given body, gas and dimensionless surface temperature condition, the flow depends upon only the free stream Mach number and nose Reynolds number

$$M_\infty = \frac{U_\infty}{C_\infty} \quad R_\infty = \frac{U_\infty a}{\nu_\infty} \quad (1.1)$$

The applicability of the Navier-Stokes equations are considered when both these parameters are large or, more formally, as they both tend to infinity at rates whose relationship is to be deduced. The inviscid stagnation temperature is needed, which is given by the energy equation as

$$\frac{T_0}{T_\infty} = 1 + \frac{\gamma-1}{2} M_\infty^2 \propto M_\infty^2 \text{ as } M_\infty \rightarrow \infty \quad (1.2)$$

(The meaning of the various subscripts is indicated in Fig. 1.) This is true provided  $\gamma$  does not simultaneously approach unity; and one may definitely exclude that unrealistic Newtonian limit, with its attendant non-uniformities.

### Thicknesses of Shock Wave and Boundary Layer

It is known from various theories that at infinite Mach number the thickness  $\Delta$  of the full shock layer is some fraction of the nose radius

$$\frac{\Delta}{a} \propto 1 \text{ as } M_\infty \rightarrow \infty \quad (1.3)$$

The thickness  $d$  of the detached shock wave is of the order (Adams and Probstein in Ref. 6) of  $\nu/c$  evaluated at sonic conditions (subscript  $*$ ). Hence

$$\frac{d}{a} \propto \frac{\nu_*}{ac_*} = \frac{\mu_*}{\mu_\infty} \frac{c_\infty}{c_*} \frac{\rho_\infty}{\rho_*} \frac{\rho_\infty}{\rho_0} \frac{\nu_\infty}{U_\infty a} \frac{U_\infty}{c_\infty} \propto \left(\frac{T_0}{T_\infty}\right)^{\omega-1/2} \frac{M_\infty}{R_\infty} \propto \frac{M_\infty^{2\omega}}{R_\infty} \quad (1.4)$$

This estimate is confirmed by the detailed analysis of Grad (Ref. 7).

The thickness  $\delta$  of the boundary layer can be estimated with the help of Stewartson's (Ref. 8) transformation, which gives an equivalent incompressible problem with kinematic viscosity  $\nu_0$  (the value at the inviscid stagnation point). Hence

$$\frac{\delta}{a} \propto \left(\frac{\nu_0}{U_\infty a}\right)^{1/2} = \left(\frac{\mu_0}{\mu_\infty}\right)^{1/2} \left(\frac{\rho_\infty}{\rho_0}\right)^{1/2} \sqrt{\frac{\nu_\infty}{U_\infty a}} \propto \frac{M_\infty^\omega}{\sqrt{R_\infty}} \quad (1.5)$$

#### Viscous Hypersonic Similitude for Blunt Bodies

These results show that in the hypersonic limit ( $M_\infty \rightarrow \infty$ ) the flow field depends not upon  $M_\infty$  and  $R_\infty$  separately, but only upon the combination

$$\epsilon = \frac{[(\gamma-1)M_\infty^2]^{\frac{\omega}{2}}}{\sqrt{R_\infty}} \quad (1.6)$$

where the factor  $(\gamma-1)$  has been inserted for later convenience. This is the viscous hypersonic similarity parameter for blunt bodies. This similitude is implicit in Hayes and Probstein's discussion (Ref. 9) of the extension to real gases of Oswatitsch's Mach number independence principle.

The boundary layer thickness is  $O(\epsilon)$  and the shock wave thickness is  $O(\epsilon^2)$ . This means that there can never exist the "viscous layer regime" proposed by Hayes and Probstein (Ref. 2) in which the shock layer is nearly all viscous but the shock wave still negligibly thin. Instead, the situation is that envisioned by Guiraud (Ref. 10), in which boundary layer theory can be embedded in a systematic expansion scheme. Successive terms of an expansion in powers of  $\epsilon$  correspond to inviscid flow, the "boundary layer regime," and the "vorticity

interaction regime." Beyond that point one must inquire into the limits of continuum theory.

### Limits of Continuum Theory

The relative error involved in using the Navier-Stokes equations is  $o(\epsilon)$ . It is  $O(\epsilon^2)$  for the Burnett equations, but they are increasingly regarded as suspect. In any case, a second approximation in the boundary layer can be calculated using the Navier-Stokes equations together with first-order slip and temperature jump at the surface.

Formidable complications arise in approximations beyond the second. It would be necessary to use the Burnett equations together with higher-order slip conditions, or a more acceptable alternative. As will be seen later, second viscosity and creep velocity due to the surface temperature gradient appear in the third approximation. Moreover, it would be necessary to consider the effects of shock structure.

Nevertheless, the possibility that the Navier-Stokes equations (with appropriate slip conditions) remain useful even when  $\epsilon = O(1)$  has been suggested by Gilbarg and Paolucci (Ref. 11) and Liepmann and Roshko (Ref. 12, sec. 14.13), and merits investigation. Cheng (Ref. 13) has made that assumption in his interesting study of the present problem based on the approximation of a thin shock layer.

### SECOND-ORDER COMPRESSIBLE BOUNDARY LAYER THEORY

Second-order boundary layer theory is developed in a general form, free of the restrictions associated with the problem just discussed (though with that application in mind). Thus the oncoming flow need not be isoenergetic, nor the viscosity vary as a power of temperature, nor the Mach number be infinite.

The analysis is carried out using the technique of inner and outer expansions that has been developed by Lagerstrom, Kaplun and Cole for treating singular perturbation problems in fluid mechanics. It parallels that given for incompressible flow by Van Dyke in Ref. 5, where reference may be made for details omitted here. The chief innovation, aside from modifications appropriate to compressibility, is the use of individual velocity components rather than a stream function. That is desirable for various reasons, and would be essential if one were to consider three-dimensional boundary layers.

Dimensionless Variables

An orthogonal coordinate system  $(s,n)$  is used (see Fig. 2) where  $n$  is the distance normal to the surface and  $s$  is the distance along the surface (measured from the stagnation point in the blunt body problem). The components of the velocity vector  $\vec{q}$  are  $(u,v)$ . The meridian curve has curvature  $\kappa(s)$ , reckoned positive for a convex body, and in axisymmetric flow  $\theta(s)$  is its angle with and  $r(s)$  its distance from the axis.

$U_\infty$  is taken to be some characteristic free stream speed,  $M_\infty$  is the corresponding Mach number, and  $a$  is a characteristic length (the nose radius in the blunt body problem). It is convenient to introduce dimensionless variables that remain bounded in the stagnation region if  $M_\infty$  becomes infinite. In that case the density is of order  $\rho_\infty$ , the pressure is  $\rho_\infty U_\infty^2$  and the temperature is  $(\gamma - 1) M_\infty^2 T_\infty = U_\infty^2 / c_p$ . All lengths are therefore referred to  $a$ , velocities to  $U_\infty$ , pressure to  $\rho_\infty U_\infty^2$ , density to  $\rho_\infty$ , temperature to  $U_\infty^2 / c_p$ , entropy to  $c_p$ , enthalpy to  $U_\infty^2$  and viscosities to the value of  $\mu$  at  $T = U_\infty^2 / c_p$ . Henceforth it is assumed (without changing notation) that this has been done.

This is entirely equivalent to choosing the four fundamental units of, say, mass, length, time and temperature such that

$$a = U_\infty = \rho_\infty = c_p = 1 \tag{2.1}$$

and replacing the second coefficient of viscosity  $\lambda$  by its ratio to the first.

Full Navier-Stokes Problem

With  $j = 0$  for plane and  $j = 1$  for axisymmetric flow, the length element  $d\ell$  is given by

$$d\ell^2 = (1 + \kappa n)^2 ds^2 + dn^2 + (r + n \cos \theta)^2 j d\phi^2 \tag{2.2}$$

where  $\phi$  is the cylindrical coordinate in plane flow and the azimuthal angle in axisymmetric flow. Then from the usual relations for vector operators in orthogonal coordinates, the continuity equation  $\text{div}(\rho \vec{q}) = 0$  is found to be

$$[(r + n \cos \theta)^j \rho u]_s + [(1 + \kappa n)(r + n \cos \theta)^j \rho v]_n = 0 \tag{2.3}$$

where subscripts indicate differentiation.

The Navier-Stokes momentum equations are found after some calculation, for example, from Tsien (Ref. 14), to be

$$\begin{aligned}
 \epsilon^{-2} \left[ \rho \left( u \frac{u_s}{1+\kappa n} + v u_n + \frac{\kappa}{1+\kappa n} uv \right) + \frac{p_s}{1+\kappa n} \right] &= \frac{\partial}{\partial n} \mu \left( u_n + \frac{v_s - \kappa u}{1+\kappa n} \right) \\
 + \frac{2}{1+\kappa n} \frac{\partial}{\partial s} \mu \frac{u_s + \kappa v}{1+\kappa n} + \mu \left( \frac{2\kappa}{1+\kappa n} + \frac{j \cos \theta}{r+n \cos \theta} \right) \left( u_n + \frac{v_s - \kappa u}{1+\kappa n} \right) \\
 + \frac{2j\mu}{(1+\kappa n)(r+n \cos \theta)} \left[ \frac{u_s + \kappa v}{1+\kappa n} - \frac{u}{(1+\kappa n)(r+n \cos \theta)} (r+n \cos \theta)_s \right. \\
 \left. + \frac{v \cos \theta}{r+n \cos \theta} \right] (r+n \cos \theta)_s + \frac{1}{1+\kappa n} \frac{\partial}{\partial s} \lambda \left[ \frac{u_s + \kappa v}{1+\kappa n} + v_n \right. \\
 \left. + \frac{j}{r+n \cos \theta} \left\{ \frac{u}{1+\kappa n} (r+n \cos \theta)_s + v \cos \theta \right\} \right]
 \end{aligned} \tag{2.4}$$

$$\begin{aligned}
 \epsilon^{-2} \left[ \rho \left( u \frac{v_s}{1+\kappa n} + v v_n - \frac{\kappa}{1+\kappa n} u^2 \right) + p_n \right] &= 2(\mu v_n)_n \\
 + \frac{1}{1+\kappa n} \frac{\partial}{\partial s} \mu \left( u_n + \frac{v_s - \kappa u}{1+\kappa n} \right) + 2\mu \left( \frac{\kappa}{1+\kappa n} + \frac{j \cos \theta}{r+n \cos \theta} \right) v_n \\
 - 2\mu \frac{\kappa}{1+\kappa n} \frac{u_s + \kappa v}{1+\kappa n} - \frac{2j\mu \cos \theta}{(r+n \cos \theta)^2} \left[ \frac{u}{1+\kappa n} (r+n \cos \theta)_s + v \cos \theta \right] \\
 + \frac{j\mu}{(1+\kappa n)(r+n \cos \theta)} \left( u_n + \frac{v_s - \kappa u}{1+\kappa n} \right) (r+n \cos \theta)_s + \frac{\partial}{\partial n} \lambda \left[ \frac{u_s + \kappa v}{1+\kappa n} \right. \\
 \left. + v_n + \frac{j}{r+n \cos \theta} \left\{ \frac{u}{1+\kappa n} (r+n \cos \theta)_s + v \cos \theta \right\} \right]
 \end{aligned} \tag{2.5}$$

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The energy equation is

$$\begin{aligned} \epsilon^{-2} \left[ \rho \left( u \frac{T_s}{1+\kappa n} + v T_n \right) - \left( u \frac{P_s}{1+\kappa n} + v P_n \right) \right] &= \frac{\sigma^{-1}}{1+\kappa n} \left( \frac{\mu T_s}{1+\kappa n} \right)_s \\ &+ \sigma^{-1} (\mu T_n)_n + \frac{j \sigma^{-1} \mu T_s}{(1+\kappa n)^2 (r+n \cos \theta)} (r+n \cos \theta)_s \\ &+ \sigma^{-1} \left( \frac{\kappa}{1+\kappa n} + \frac{j \cos \theta}{r+n \cos \theta} \right) \mu T_n + \Phi \end{aligned} \quad (2.6a)$$

where  $\sigma = \mu c_p / k$  is the Prandtl number, and the dissipation function is

$$\begin{aligned} \Phi = \mu \left[ 2 \left( \frac{u_s + \kappa v}{1+\kappa n} \right)^2 + 2v_n^2 + \frac{2j}{(r+n \cos \theta)^2} \left\{ \frac{u}{1+\kappa n} (r+n \cos \theta)_s \right. \right. \\ \left. \left. + v \cos \theta \right\}^2 + \left( u_n + \frac{v_s - \kappa u}{1+\kappa n} \right)^2 \right] + \lambda \left[ \frac{u_s + \kappa v}{1+\kappa n} + v_n \right. \\ \left. + \frac{j}{r+n \cos \theta} \left\{ \frac{u}{1+\kappa n} (r+n \cos \theta)_s + v \cos \theta \right\} \right]^2 \end{aligned} \quad (2.6b)$$

The equation of state is

$$p = \frac{\gamma-1}{\gamma} \rho T \quad (2.7)$$

which, together with the viscosity law

$$\mu = \mu(T) \quad (2.8)$$

completes the system of equations.

The tangency condition at the surface of a solid body is

$$v(s,0) = 0 \quad (2.9)$$

Street (Ref. 15) points out that all first-order slip flow conditions proposed have (when converted to dimensionless variables) the form

$$\left. \begin{aligned} u &= \epsilon^2 \frac{\mu}{p} \left( a_1 \sqrt{\frac{\gamma-1}{\gamma}} T u_n + b_1 \frac{\gamma-1}{\gamma} T_s \right) \\ T &= T_b + \epsilon^2 c_1 \frac{\mu}{p} \sqrt{\frac{\gamma-1}{\gamma}} T T_n \end{aligned} \right\} \text{at } n = 0 \quad \begin{matrix} (2.10a) \\ (2.10b) \end{matrix}$$

where  $a_1$ ,  $b_1$  and  $c_1$  are dimensionless constants of order unity.

The upstream boundary conditions depend on the problem considered. For uniform parallel flow they are

$$\vec{q} \equiv (u, v) \rightarrow \vec{i}, \quad p \rightarrow \frac{1}{\gamma M_\infty^2}, \quad \rho \rightarrow 1 \quad \text{as } n \rightarrow \infty \quad (2.10c)$$

where  $\vec{i}$  is a unit vector in the streamwise direction.

The Perturbation Parameter

The parameter  $\epsilon$  in the momentum and energy equations is given by

$$\epsilon = \left[ \frac{\mu(U_\infty^2/c_p)}{\rho_\infty U_\infty a} \right]^{1/2} = \frac{1}{\sqrt{R_\infty}} \left[ \frac{\mu([\gamma-1]M_\infty^2 T_\infty)}{\mu(T_\infty)} \right]^{1/2} \quad (2.11)$$

where here  $\mu$  is the actual (dimensional) viscosity. The two similarity parameters  $M_\infty$  and  $R_\infty$  appear otherwise in the problem as  $1/\gamma M_\infty^2$  in the upstream condition 2.10c on pressure. Hence in the hypersonic limit, as  $M_\infty \rightarrow \infty$ ,  $\epsilon$  becomes the only similarity parameter. This is the general form of the similitude discussed previously for the special case of viscosity proportional to a power  $\omega$  of temperature, in which case Eq. 2.11 reduces to Eq. 1.6.

At the other extreme, as  $M_\infty$  vanishes, the dimensionless variables adopted here become inappropriate. However, one can formally recover the incompressible flow problem, for which temperature variations are negligible, by regarding the coefficients of viscosity as constants. Then  $\epsilon$  is simply  $R_\infty^{-1/2}$ , and the present analysis reproduces that of Van Dyke in Ref. 5.



Outer Expansion

It appears that outside the boundary layer (and shock wave) the flow quantities can, for an analytic body, be expanded asymptotically in integral powers of  $\epsilon$ . Thus the outer expansion has the form

$$\begin{aligned}
 u(s,n;\epsilon) &- U_1(s,n) + \epsilon U_2(s,n) + \dots \\
 v(s,n;\epsilon) &- V_1(s,n) + \epsilon V_2(s,n) + \dots \\
 p(s,n;\epsilon) &- P_1(s,n) + \epsilon P_2(s,n) + \dots \\
 \rho(s,n;\epsilon) &- R_1(s,n) + \epsilon R_2(s,n) + \dots \\
 T(s,n;\epsilon) &- T_1(s,n) + \epsilon T_2(s,n) + \dots
 \end{aligned}
 \tag{2.12}$$

It is implied that all these functions and their derivatives are of order unity in the outer region. The leading terms represent the basic inviscid flow, secondary terms the outer flow due to displacement thickness of the boundary layer, etc.

Substituting into the equations of motion Eqs. 2.3 to 2.8 gives for the first approximation

$$[(r+n \cos \theta)^j R_1 U_1]_s + [(1+\kappa n)(r+n \cos \theta)^j R_1 V_1]_n = 0 \tag{2.13a}$$

$$R_1 (U_1 \frac{U_{1s}}{1+\kappa n} + V_1 U_{1n} + \frac{\kappa}{1+\kappa n} U_1 V_1) = 0 \tag{2.13b}$$

$$R_1 (U_1 \frac{V_{1s}}{1+\kappa n} + V_1 V_{1n} - \frac{\kappa}{1+\kappa n} U_1^2) + P_{1n} = 0 \tag{2.13c}$$

$$R_1 (U_1 \frac{T_{1s}}{1+\kappa n} + V_1 T_{1n}) - (U_1 \frac{P_{1s}}{1+\kappa n} + V_1 P_{1n}) = 0 \tag{2.13d}$$

$$P_1 = \frac{\gamma-1}{\gamma} R_1 T_1 \tag{2.13e}$$

which are the inviscid flow equations. Eliminating  $T_1$  between the last two shows that they can be replaced by

$$\left( \frac{U_1}{1+\kappa n} \frac{\partial}{\partial s} + V_1 \frac{\partial}{\partial n} \right) \frac{P_1}{R_1 \gamma} = 0 \tag{2.14}$$

This means that  $P_1/R_1^{\gamma}$  is constant along streamlines, and hence so is the entropy  $S_1$ . For this reason it is convenient to satisfy the continuity equation 2.13a by introducing a stream function in the usual way, setting

$$(r+n \cos \theta)^j R_1 U_1 = \Psi_{1n} \quad \text{III} \quad (1+\kappa n)(r+n \cos \theta)^j R_1 V_1 = -\Psi_{1s} \quad (2.15)$$

Then the entropy can be written as

$$S_1 = S_1(\Psi_1) \quad (2.16)$$

Likewise, the inviscid equations can be manipulated to show that the stagnation enthalpy  $H_1$  is constant along streamlines

$$H_1 \equiv T_1 + \frac{1}{2} U_1^2 = H_1(\Psi_1) \quad (2.17)$$

For the second approximation one obtains a corresponding, more complicated, system of equations, of which that for tangential momentum is

$$R_1 \left( \frac{U_1 U_{2s} + U_2 U_{1s}}{1+\kappa n} + V_1 U_{2n} + V_2 U_{1n} + \kappa \frac{U_1 V_2 + U_2 V_1}{1+\kappa n} \right) + R_2 \left( \frac{U_1 U_{1s}}{1+\kappa n} + V_1 U_{1n} + \frac{\kappa}{1+\kappa n} U_1 V_1 \right) + \frac{P_{2s}}{1+\kappa n} = 0 \quad (2.18)$$

These are also inviscid equations, because viscous terms in the Navier-Stokes equations appear multiplied by  $\epsilon^2$  and so affect only the third approximation. They describe small perturbations of the basic inviscid flow given by the first approximation.

The outer expansion is not valid at the surface, but the leading term does satisfy the tangency condition of Eq. 2.9, which gives

$$V_1(s,0) = 0 \quad (2.19)$$

### Basic Flow Near Surface

Conventional boundary layer theory requires the values of the inviscid velocity and temperature  $U_1$  and  $T_1$  at the surface of the body. The second approximation involves also  $P_1$  at the surface. It is therefore assumed that the basic inviscid flow has been calculated (for example, by numerical

methods of solving the blunt body problem), which provides at least, say, the surface pressure distribution and the density at one point. Then all the other inviscid values at the surface can be calculated from Eqs. 2.14, 2.13e and 2.17.

Second-order boundary layer theory requires also the normal derivatives, at the surface, of the basic inviscid flow quantities. These can be expressed in terms of the surface values themselves together with the gradients across streamlines, at the surface, of two quantities that are constant along streamlines: the entropy  $S_1$  and stagnation enthalpy  $H_1$ .

Two of these desired relations are found at once by evaluating Eqs. 2.13a and 2.13c at the surface

$$\left. \begin{aligned} V_{1n} &= -(r^j R_1)^{-1} \frac{\partial}{\partial s} (r^j R_1 U_1) \\ P_{1n} &= \kappa R_1 U_1^2 \end{aligned} \right\} \text{ at } n = 0 \quad (2.20)$$

The relation for  $U_1$  involves the vorticity of the basic flow, which has only the component in the  $\phi$  direction

$$\Omega_1 \equiv \phi\text{-component } (\text{curl } \vec{Q}_1) = \frac{V_{1s} - \kappa U_1}{1 + \kappa n} - U_{1n} \quad (2.21)$$

Evaluating Crocco's vortex theorem

$$\vec{Q}_1 \times \vec{\Omega}_1 = \text{grad } H_1 - T_1 \text{ grad } S_1 \quad (2.22)$$

at the surface, using Eqs. 2.15, 2.16 and 2.17, gives

$$\Omega_1 = r^j R_1 (T_1 S_1' - H_1') \quad \text{at } n = 0 \quad (2.23)$$

(where  $S_1'$  means  $ds_1/d\Psi_1$ , etc.), so that the desired relation is

$$U_{1n} = -\kappa U_1 - r^j R_1 (T_1 S_1' - H_1') \quad \text{at } n = 0 \quad (2.24)$$

Finally, the relations for  $T_1$  and  $R_1$  are found, by differentiating Eqs. 2.17 and 2.13e with respect to  $n$ , as

$$\left. \begin{aligned} T_{1n} &= \kappa U_1^2 + r^j R_1 U_1 T_1 S_1' \\ R_{1n} &= R_1 U_1 \left( \frac{\kappa}{\gamma-1} \frac{U_1}{T_1} - r^j R_1 S_1' \right) \end{aligned} \right\} \text{ at } n = 0 \quad (2.25)$$

Inner Expansion

The outer expansion fails at the surface because, through loss of the highest derivatives in the Navier-Stokes equation, it cannot satisfy the condition of Eq. 2.10a on the tangential velocity. As a result, it is invalid within a distance of order  $\epsilon a$  of the surface. Prandtl made a separate investigation of this boundary layer by magnifying the normal coordinate, introducing the boundary layer variable

$$N = n/\epsilon \tag{2.26}$$

and correspondingly reducing the normal velocity.

This is generalized to higher approximations by introducing a second asymptotic expansion that replaces the outer expansion near the surface. For an analytic body this inner expansion has the form

$$\begin{aligned} u(s,n;\epsilon) &= u_1(s,N) + \epsilon u_2(s,N) + \dots \\ v(s,n;\epsilon) &= \epsilon v_1(s,N) + \epsilon^2 v_2(s,N) + \dots \\ p(s,n;\epsilon) &= p_1(s,N) + \epsilon p_2(s,N) + \dots \\ \rho(s,n;\epsilon) &= \rho_1(s,N) + \epsilon \rho_2(s,N) + \dots \\ T(s,n;\epsilon) &= t_1(s,N) + \epsilon t_2(s,N) + \dots \end{aligned} \tag{2.27}$$

The leading terms represent conventional boundary layer theory, and secondary terms represent its extension with which the investigator is concerned here.

The viscosity law can be correspondingly expanded in a Taylor series to give

$$\mu(T) = \mu(t_1) + \epsilon \mu'(t_1) t_2 + \dots \tag{2.28}$$

Substituting the inner expansion into the full equations of motion gives for the first approximation

$$(r^j \rho_1 u_1)_s + (r^j \rho_1 v_1)_N = 0 \tag{2.29a}$$

$$\rho_1 (u_1 u_{1s} + v_1 u_{1N}) + p_{1s} - (\mu u_{1N})_N = 0 \tag{2.29b}$$

$$p_{1N} = 0 \tag{2.29c}$$

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$$\rho_1(u_1 t_{1s} + v_1 t_{1N}) - (u_1 p_{1s} + v_1 p_{1N}) - \sigma^{-1} (\mu t_{1N})_N - \mu u_{1N}^2 = 0 \quad (2.29d)$$

$$p_1 = \frac{\gamma-1}{\gamma} \rho_1 t_1 \quad (2.29e)$$

which are the conventional compressible boundary layer equations. Similarly, the second-order equations are found to be

$$\begin{aligned} & [r^j (\rho_1 u_2 + \rho_2 u_1 + j \frac{\cos \theta}{r} N \rho_1 u_1)]_s + [r^j \rho_1 v_2 + \rho_2 v_1 \\ & + (\kappa + j \frac{\cos \theta}{r} N \rho_1 v_1)]_N = 0 \end{aligned} \quad (2.30a)$$

$$\rho_1(u_1 u_{2s} + u_2 u_{1s} + v_1 u_{2N} + v_2 u_{1N}) + \rho_2(u_1 u_{1s} + v_1 u_{1N}) + p_{2s} - (\mu u_{2n} +$$

$$\mu' u_{1n} t_2)_N = \rho_1 u_1 \kappa (N u_{1s} - v_1) + \kappa N p_{1s} - \kappa (\mu u_1)_N + \mu (2\kappa + j \quad (2.30b)$$

$$\frac{\cos \theta}{r}) u_{1N}$$

$$p_{2N} = \kappa \rho_1 u_1^2 \quad (2.30c)$$

$$\rho_1(u_1 t_{2s} + u_2 t_{1s} + v_1 t_{2N} + v_2 t_{1N}) + \rho_2(u_1 t_{1s} + v_1 t_{1N}) - (u_1 p_{2s} + u_2 p_{1s})$$

$$- (v_1 p_{2N} + v_2 p_{1N}) - \sigma^{-1} (\mu t_{2N} + \mu' t_{1N} t_2)_N - 2\mu u_{1N} u_{2N} - \mu' u_{1N}^2 t_2$$

$$= \kappa N u_1 (\rho_1 t_{1s} - p_{1s}) + \sigma^{-1} (\kappa + j \frac{\cos \theta}{r}) \mu t_{1N} - 2\kappa \mu u_1 u_{1N} \quad (2.30d)$$

$$p_2 = \frac{\gamma-1}{\gamma} (\rho_1 t_2 + \rho_2 t_1) \quad (2.30e)$$

where  $\mu = \mu(t_1)$  and  $\mu' = \mu'(t_1)$ .

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It is remarkable that the second coefficient of viscosity does not appear, its effect being of third order. The second-order equations can be modified using the first-order ones, and this will be done later.

Substituting the inner expansion into the conditions of Eqs. 2.9 and 2.10a on velocity at the surface gives the boundary conditions

$$u_1(s,0) = v_1(s,0) = 0 \tag{2.31}$$

$$u_2(s,0) = a_1 \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 u_{1N} \right]_{N=0} \tag{2.32a}$$

$$v_2(s,0) = 0 \tag{2.32b}$$

The second term in the slip condition 2.10a, which gives a contribution to slip velocity ("creep velocity") due to a temperature gradient along the surface, does not appear here because it is a third-order effect.

If the temperature of the body is prescribed as  $T_b(s)$ , the temperature jump condition 2.10b provides the boundary conditions

$$t_1(s,0) = T_b(s) \tag{2.33}$$

$$t_2(s,0) = c_1 \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 t_{1N} \right]_{N=0} \tag{2.34}$$

However, one may impose other boundary conditions on the temperature at the surface. For example, if there is zero heat transfer, the boundary conditions are

$$t_{1N}(s,0) = t_{2N}(s,0) = 0 \tag{2.35}$$

In such cases the temperature jump condition serves instead to determine the body temperature as

$$T_b(s) - t_1(s,0) - \epsilon c_1 \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 t_{1N} \right]_{N=0} + \dots \tag{2.36}$$

Effects of slip and temperature jump disappear to second order for a very highly cooled body. More precisely, they are demoted to third order in the present expansion scheme if the

ratio of body to stagnation temperatures is  $O(\epsilon^{2\omega-1})$ , so that the ratio of the second to the first term in Eq. 2.36 is  $O(\epsilon^2)$ . However, in the usual situation that ratio, though perhaps small, is of order unity (e.g., one-tenth). Then slip effects remain formally of second order. Indeed, their contribution will be numerically significant even for small values of the temperature ratio, because Eqs. 2.32a and 2.36 show that only its  $(\omega - 1/2)$  power is relevant. These remarks agree with the conclusions of a forthcoming note by Lenard and Rott.

Second-order effects of slip and temperature jump vanish for an insulated wall because  $t_{1N} = 0$ .

Matching Conditions

The outer expansion violates some of the boundary conditions at the surface, where it is invalid. Conversely, the inner expansion is invalid far from the surface, and so in general violates the upstream conditions. Hence, neither expansion has sufficient boundary conditions. The missing ones are supplied by matching the two expansions in the overlap region (the outer fringes of the boundary layer) where both are valid.

In this problem it is possible to apply the restricted matching principle (Lagerstrom, Ref. 16). This requires that

$$\begin{aligned} & \text{m-term inner expansion of (p-term outer expansion)} = \\ & \text{p-term outer expansion of (m-term inner expansion)} \end{aligned} \tag{2.37}$$

and is applied with  $m=p$  for the  $m^{\text{th}}$  approximation in the boundary layer, and with  $m = p - 1$  for the  $p^{\text{th}}$  term in the outer flow.

Applying the matching principle with  $m = p = 1$  gives the matching conditions for conventional boundary layer theory

$$\left. \begin{aligned} u_1(s,N) - U_1(s,0) \\ P_1(s,N) - P_1(s,0) \\ \rho_1(s,N) - R_1(s,0) \\ t_1(s,N) - T_1(s,0) \end{aligned} \right\} \text{ as } N \rightarrow \infty \tag{2.38}$$

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Note that no condition is imposed upon  $v_1$ . Its behavior for large  $N$  is found in solving the boundary layer problem. Because the equation of state holds uniformly, only two of the three thermodynamic variables need actually be matched.

Next, matching with  $m = 1$  and  $p = 2$  gives a single matching condition for the outer flow due to displacement thickness

$$V_2(s,0) = \lim_{N \rightarrow \infty} (v_1 - N v_{1N}) \tag{2.39}$$

This expresses the fact that the effect of the boundary layer upon the outer flow is that of a distribution of sources over the body whose strength is proportional to the slope of the displacement thickness.

Finally, matching with  $m = p = 2$ , using analyticity in  $n$  of the basic inviscid flow at the surface, gives the matching conditions for the second boundary layer approximation

$$\left. \begin{aligned} u_2(s,N) - N U_{1n}(s,0) + U_2(s,0) & \tag{2.40a} \\ v_1(s,N) - N V_{1n}(s,0) + V_2(s,0) & \tag{2.40b} \\ p_2(s,N) - N P_{1n}(s,0) + \bar{P}_2(s,0) & \tag{2.40c} \\ \rho_2(s,N) - N R_{1n}(s,0) + R_2(s,0) & \tag{2.40d} \\ t_2(s,N) - N T_{1n}(s,0) + T_2(s,0) & \tag{2.40e} \end{aligned} \right\} \text{ as } N \rightarrow \infty$$

These could have perhaps been written down on the basis of physical reasoning as expressing second-order joining of the boundary layer with the outer flow. The normal derivatives required here have already been evaluated in Eqs. 2.20, 2.24 and 2.25.

First-Order Boundary Layer Problem

Equations 2.29 are reduced to standard form by eliminating the pressure. Adding  $u_1$  times the momentum equation 2.29b removes  $p_1$  from the energy equation 2.29d. Integrating the normal-momentum equation 2.29c with respect to  $N$  and evaluating the function of integration from the matching condition 2.38 and 2.13e gives

$$P_1(s,N) = P_1(s,0) = \frac{\gamma-1}{\gamma} R_1(s,0) T_1(s,0) \tag{2.41}$$



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Differentiating and using Eq. 2.13b evaluated at the surface gives

$$P_{1s} = P_{1s}(s,0) = -(R_1 U_1 U_{1s})_{n=0} \quad (2.42)$$

Thus the final form of the first-order boundary layer equations is

$$(r^j \rho_1 u_1)_s + (r^j \rho_1 v_1)_N = 0 \quad (2.43a)$$

$$\rho_1 (u_1 u_{1s} + v_1 u_{1N}) - (\mu u_{1N})_N = (R_1 U_1 U_{1s})_{n=0} \quad (2.43b)$$

$$\rho_1 \left( u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N} \right) (t_1 + \frac{1}{2} u_1^2) - \frac{\partial}{\partial N} [\mu (\sigma^{-1} t_1 + \frac{1}{2} u_1^2)]_N = 0 \quad (2.43c)$$

$$\rho_1 t_1 = (R_1 T_1)_{n=0} \quad (2.43d)$$

where  $\mu = \mu(t_1)$ , and the corresponding boundary and matching conditions are

$$u_1(s,0) = v_1(s,0) = 0 \quad (2.44a)$$

$$t_1(s,0) = T_b(s), \quad t_{1N}(s,0) = 0, \text{ or the like} \quad (2.44b)$$

$$\left. \begin{array}{l} u_1(s,N) - U_1(s,0) \\ t_1(s,N) - T_1(s,0) \end{array} \right\} \text{ as } N \rightarrow \infty \quad (2.44c)$$

This is, in fact, just the standard compressible boundary layer problem with  $c_p$  set equal to unity.

Crocco pointed out that when the Prandtl number  $\sigma$  is unity a particular integral of the energy equation 2.43c, which according to Eq. 2.17 also satisfies the matching conditions 2.44c, is given by

$$t_1 + \frac{1}{2} u_1^2 = \text{const.} = H_1(0) \quad (2.45)$$

This satisfies the boundary conditions 2.44a,b only for an insulated surface.

Second-Order Boundary Layer Problem

Both  $p_1$  and  $p_2$  are now eliminated from Eqs. 2.30. Integrating Eq. 2.30c using Eqs. 2.40c and 2.20 gives

$$p_2(s, N) = N \kappa R_1(s, 0) U_1^2(s, 0) + \kappa \int_N^\infty [R_1(s, 0) U_1^2(s, 0) - \rho_1 u_1^2] dN + P_2(s, 0) \quad (2.46)$$

Hence using Eq. 2.18 evaluated at  $n = 0$  and Eq. 2.24 gives

$$P_{2s} = \frac{\partial}{\partial s} \kappa [N R_1(s, 0) U_1^2(s, 0) + \int_N^\infty \{R_1(s, 0) U_1^2(s, 0) - \rho_1 u_1^2\} dN] \\ + [r^j R_1^2 (T_1 S_1' - H_1') V_2 - R_1 (U_1 U_2)_s - R_2 U_1 U_1 s]_{n=0} \quad (2.47)$$

Thus the final form of the second-order boundary layer equations is found to be

$$C\{u_2, v_2, \rho_2\} = - [r^j (j \frac{\cos \theta}{r}) N \rho_1 u_1]_s - [r^j (\kappa + j \frac{\cos \theta}{r}) N \rho_1 v_1]_N \quad (2.48a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = \kappa [N (\mu u_{1N})_N + \mu u_{1N} - \mu' u_1 t_{1N} - \rho_1 u_1 v_1] \\ + \frac{j \cos \theta}{r} \mu u_{1N} [r^j R_1^2 (T_1 S_1' - H_1') V_2 - R_1 (U_1 U_2)_s \\ - R_2 U_1 U_1 s]_{n=0} \quad (2.48b)$$

$$E\{u_2, \rho_2, t_2\} = \kappa N \rho_1 u_1 (t_1 + \frac{1}{2} u_1^2)_s + j \frac{\cos \theta}{r} \mu (\sigma^{-1} t_1 + \frac{1}{2} u_1^2)_N \\ + \sigma^{-1} \kappa \mu t_{1N} - \kappa u_1 (\mu u_{1N})_N \quad (2.48c)$$

$$F\{\rho_2, t_2\} = \frac{\gamma}{\gamma-1} \kappa [N R_1(s, 0) U_1^2(s, 0) + \int_N^\infty \{R_1(s, 0) U_1^2(s, 0) - \rho_1 u_1^2\} dN] \\ + (R_1 T_2 + R_2 T_1)_{n=0} \quad (2.48d)$$

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Here C, D, E and F (mnemonic: continuity, dynamic, energy and fluid) are linear differential operators, with coefficients depending on the first-order boundary layer solution, given by

$$C\{u_2, v_2, \rho_2\} = [r^j(\rho_1 u_2 + \rho_2 u_1)]_s + [r^j(\rho_1 v_2 + \rho_2 v_1)]_N \quad (2.49a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = \rho_1(u_1 u_{2s} + u_2 u_{1s} + v_1 u_{2N} + v_2 u_{1N}) + \rho_2(u_1 u_{1s} + v_1 u_{1N}) - (\mu u_{2N} + \mu' u_{1N} t_2)_N \quad (2.49b)$$

$$E\{u_2, \rho_2, t_2\} = \rho_1(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N})(t_2 + u_1 u_2) + [\rho_1(u_2 \frac{\partial}{\partial s} + v_2 \frac{\partial}{\partial N}) + \rho_2(u_1 \frac{\partial}{\partial s} + v_1 \frac{\partial}{\partial N})](t_1 + \frac{1}{2} u_1^2) - \frac{\partial}{\partial N} [\mu(\sigma^{-1} t_2 + u_1 u_2)_N + \mu'(\sigma^{-1} t_1 + \frac{1}{2} u_1^2)_N t_2] \quad (2.49c)$$

$$F\{\rho_2, t_2\} = \rho_1 t_2 + \rho_2 t_1 \quad (2.49d)$$

The corresponding second-order boundary conditions are

$$u_2(s, 0) = a_1 \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 u_{1N} \right]_{N=0} \quad (2.50a)$$

$$v_2(s, 0) = 0 \quad (2.50b)$$

$$t_2(s, 0) = c_1 \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 t_{1N} \right]_{N=0} \text{ for given surface temp} \quad (2.50c)$$

= 0 for insulated surface, etc.

and the matching conditions are

$$u_2(s, N) - N[U_1(\kappa U_1 + r^j R_1 T_1 S_1')]_{n=0} + T_2(s, 0) \quad (2.51a)$$

$$t_2(s, N) - N[U_1(\kappa U_1 + r^j R_1 T_1 S_1')]_{n=0} + T_2(s, 0) \quad (2.51b)$$

Decomposition of Second-Order Problem

The linearity of the second-order problem provides opportunity for subdivision into a number of simpler problems. To this end the equations have been written with their nonhomogeneous terms on the right.

These fall naturally into three categories. Nonhomogeneous terms proportional to  $\kappa$  and  $(j \cos \theta / r)$  represent the effects of curvature of the surface. Those in the boundary conditions at the surface are the result of noncontinuum phenomena. The remainder arise from interaction of the first-order boundary layer, through its displacement thickness, with the outer inviscid flow; this induces second-order changes in pressure along the boundary layer.

It is convenient to further subdivide each of these three categories, although these secondary divisions are somewhat less natural, and in at least one case purely arbitrary. Terms proportional to  $\kappa$  arise from longitudinal curvature, whereas those in  $(j \cos \theta / r)$  arise from transverse curvature. The noncontinuum effects separate naturally into slip and temperature jump; however, for mathematical simplicity part of the temperature jump is associated with what will nevertheless be called slip, with the remainder called temperature jump.

Of the terms due to interaction with the outer flow, those involving  $(T_1 S_1' - H_1') V_2$  represent the effects of external vorticity. They are distinguished by being given by the first-order solution (which provides  $V_2$  at the surface according to Eq. 2.39), whereas the others require the much more difficult calculation of the outer flow due to displacement thickness (to obtain  $U_2, P_2, R_2$  and  $T_2$ ). It is therefore worthwhile to isolate this term, the more so because its influence is often much greater than that of the others. It provides the answer to the question raised by Ferri and Libby (Ref. 1), which may be formalized as: what is the effect of external vorticity if the surface speed and density are kept unchanged? This vorticity term separates naturally into the effects of entropy gradient and of stagnation enthalpy gradient. The remaining interaction will be called the displacement effect.

This decomposition into seven constituents is formalized by

$$u_2 = u_2^{(t)} + j u_2^{(v)} + a_1 u_2^{(s)} + (c_1 - a_1) u_2^{(T)} + S_1(0) u_2^{(e)} + H_1(0) u_2^{(H)} + u_2^{(d)} \tag{2.52}$$

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writing with corresponding expressions for  $v_2$ ,  $\rho_2$ , and  $t_2$ . The superscripts (which will often be omitted) identify the results of longitudinal curvature, transverse curvature, slip, temperature jump, entropy gradient, stagnation enthalpy gradient, and displacement.

Longitudinal Curvature

Keeping only nonhomogeneous terms that involve  $\kappa$  gives the problem for  $u_2^{(t)}$ ,  $v_2^{(t)}$ , etc.

$$C\{u_2, v_2, \rho_2\} = -\kappa r^j (N \rho_1 v_1)_N \tag{2.53a}$$

$$D\{u_2, v_2, \rho_2, t_2\} = \kappa [N(\mu u_{1N})_N + \mu u_{1N} - \mu' u_1 t_{1N} - N \rho_1 v_1 u_{1N} - \rho_1 u_1 v_1] - \frac{\partial}{\partial s} \kappa [N R_1(s, 0) U_1^2(s, 0) + \int_N^\infty \{R_1(s, 0) U_1^2(s, 0) - \rho_1 u_1^2\} dN] \tag{2.53b}$$

$$E\{u_2, \rho_2, t_2\} = \kappa [N \rho_1 u_1 (t_1 + \frac{1}{2} u_1^2)_s + \sigma^{-1} \mu t_{1N} u_1 (\mu u_1)_N] \tag{2.53c}$$

$$F\{\rho_2, t_2\} = \frac{\gamma}{\gamma-1} \kappa [N R_1(s, 0) U_1^2(s, 0) + \int_N^\infty \{R_1(s, 0) U_1^2(s, 0) - \rho_1 u_1^2\} dN] \tag{2.53d}$$

$$u_2(s, 0) = v_2(s, 0) = t_2(s, 0) = 0 \tag{2.53e}$$

$$\left. \begin{aligned} u_2(s, N) - \kappa N U_1(s, 0) \\ t_2(s, N) - \kappa N U_1^2(s, 0) \end{aligned} \right\} \text{as } N \rightarrow \infty \tag{2.53f}$$

These matching conditions indicate the absence of any term that remains bounded as  $N \rightarrow \infty$  (the remainder being, in fact, exponentially small).

Transverse Curvature

Keeping nonhomogeneous terms in  $(j \cos \theta / r)$  gives the problem for  $u_2^{(t)}$ , etc.

$$C\{u_2, v_2, \rho_2\} = - (r^j \frac{\cos \theta}{r} N \rho_1 u_1)_s - (r^j \frac{\cos \theta}{r} N \rho_1 v_1)_N \quad (2.54a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = \frac{\cos \theta}{r} \mu u_{1N} \quad (2.54b)$$

$$E\{u_2, \rho_2, t_2\} = \frac{\cos \theta}{r} \mu (\sigma^{-1} t_1 + \frac{1}{2} u_1^2)_N \quad (2.54c)$$

$$F\{\rho_2, t_2\} = 0 \quad (2.54d)$$

$$u_2(s, 0) = v_2(s, 0) = t_2(s, 0) = 0 \quad (2.54e)$$

$$u_2(s, N), t_2(s, N) = o(1) \quad \text{as } N \rightarrow \infty \quad (2.54f)$$

For Prandtl number unity and an insulated surface, a second-order counterpart of Crocco's integral (see Eq. 2.43c) is given by

$$t_2 + u_1 u_2 = 0 \quad (2.55)$$

### Slip

It is convenient, as indicated in Eq. 2.52, to include part of the temperature jump with slip. Thus the problem for  $u_2^{(s)}$ , etc., consists of the homogeneous equations of motion

$$C\{u_2, v_2, \rho_2\} = D\{u_2, v_2, \rho_2, t_2\} = E\{u_2, \rho_2, t_2\} = F\{\rho_2, t_2\} = 0 \quad (2.56a)$$

with the boundary and matching conditions

$$u_2(s, 0) = \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 u_{1N} \right]_{N=0} \quad (2.56b)$$

$$u_2(s, 0) = 0 \quad (2.56c)$$

$$t_2(s, 0) = \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 t_{1N} \right]_{N=0} \quad (2.56d)$$

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$$u_2(s,N), t_2(s,N) = o(1) \quad \text{as } N \rightarrow \infty \quad (2.56e)$$

The effects of slip and temperature jump are decomposed in this way because Nonweiler (Ref. 17) and Glauert (Ref. 18) show that the solution is then given immediately in terms of the first approximation by

$$(u_2, v_2, \rho_2, t_2) = \left[ \frac{\mu}{P_1} \sqrt{\frac{\gamma-1}{\gamma}} t_1 \right]_{N=0} \frac{\partial}{\partial N} (u_1, v_1, \rho_1, t_1) \quad (2.57)$$

Temperature Jump

For the remaining effects of temperature jump, the problem for  $u_2^{(T)}$ , etc., is found to be the preceding one with the right-hand side of Eq. 2.56b replaced by zero.

For Prandtl number unity and insulated surface, the second-order Crocco integral Eq. 2.55 satisfies the energy equation and matching conditions, but violates the boundary conditions at the surface.

Entropy Gradient

Keeping only terms in  $S_1'(0)$  gives the problem for  $u_2^{(e)}$ , etc.

$$C\{u_2, v_2, \rho_2\} = E\{u_2, \rho_2, t_2\} = F\{\rho_2, t_2\} = 0 \quad (2.58a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = -r^j (R_1^2 T_1 V_2)_{n=0} \quad (2.58b)$$

$$u_2(s,0) = v_2(s,0) = t_2(s,0) = 0 \quad (2.58c)$$

$$\left. \begin{aligned} u_2(s,N) - N r^j (R_1 T_1)_{n=0} \\ t_2(s,N) - N r^j (R_1 U_1 T_1)_{n=0} \end{aligned} \right\} \text{as } N \rightarrow \infty \quad (2.58d)$$

For Prandtl number unity and insulated surface the second-order Crocco integral Eq. 2.55 is valid.

Stagnation Enthalpy Gradient

Similarly, keeping terms in  $H_1^i(0)$  leads to the problem for  $u_2^{(H)}$ , etc.

$$C\{u_2, v_2, \rho_2\} = E\{u_2, \rho_2, t_2\} = F\{\rho_2, t_2\} = 0 \quad (2.59a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = r^j (R_1^2 V_2)_{n=0} \quad (2.59b)$$

$$u_2(s, 0) = v_2(s, 0) = t_2(s, 0) = 0 \quad (2.59c)$$

$$\left. \begin{array}{l} u_2(s, N) - N r^j (R_1)_{n=0} \\ t_2(s, N) = o(1) \end{array} \right\} \text{ as } N \rightarrow \infty \quad (2.59d)$$

The second-order Crocco integral Eq. 2.55 violates the matching conditions, because the outer flow is not isoenergetic.

Displacement

The remaining nonhomogeneous terms give the problem for  $u_2^{(d)}$ , etc.

$$C\{u_2, v_2, \rho_2\} = E\{u_2, \rho_2, t_2\} = 0 \quad (2.60a)$$

$$D\{u_2, v_2, \rho_2, t_2\} = [R_1(U_1 U_2)_s + \frac{1}{2} R_2(U_1^2)_s]_{s=n=0} \quad (2.60b)$$

$$F\{\rho_2, t_2\} = (R_1 T_2 + R_2 T_1)_{n=0} \quad (2.60c)$$

$$u_2(s, 0) = v_2(s, 0) = t_2(s, 0) = 0 \quad (2.60d)$$

$$\left. \begin{array}{l} u_2(s, N) - U_2(s, 0) \\ t_2(s, N) - T_2(s, 0) \end{array} \right\} \text{ as } N \rightarrow \infty \quad (2.60e)$$

For Prandtl number unity and an insulated surface the second-order Crocco integral Eq. 2.55 is valid.



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### APPLICATION TO BLUNT BODY

The author returns now to the original problem of a blunt body in a uniform supersonic stream, and the practical details of calculating the second-order boundary-layer solution are considered briefly.

#### Basic Flow at Surface

Values of flow quantities at the stagnation point can be found from one-dimensional relations without solving the basic inviscid flow problem. To proceed with the solution, however, one needs, say, the surface pressure distribution. Analytical methods of treating the supersonic blunt body problem are notoriously unreliable, whereas numerical procedures are now quick and accurate. For example, a number of numerical solutions for the sphere and paraboloid have been tabulated by Van Dyke & Gordon (Ref. 19). Since the boundary layer equations themselves can at best be reduced to a sequence of ordinary differential equations that must be integrated numerically, it seems appropriate to use a numerical solution also for the basic inviscid flow. The method suggested next for solving the boundary layer equations requires fitting the surface pressure distribution with a power series in distance from the nose.

The numerical solution provides also the ratio  $a$  of body to shock nose radii. Then using the oblique shock wave relations one finds that on the stagnation streamline the entropy gradient is

$$S_1'(0) = \frac{-4(\gamma-1)(M_\infty^2-1)^2 a^{1+j}}{[2\gamma M_\infty^2 - (\gamma-1)][2 + (\gamma-1)M_\infty^2]} 0^{1-j} \quad (3.1)$$

The factor of zero to the power  $(1-j)$  means that  $S_1'(0)$  vanishes in symmetric plane flow -- so that there is no second-order effect of external vorticity -- but is finite in axisymmetric flow (and would be infinite in higher dimensions!).

External vorticity arises only from entropy gradients in this problem; the stagnation enthalpy is constant across the shock wave, so that  $H_1^\dagger$  is zero everywhere, and the effect of stagnation enthalpy gradient disappears.

Boundary Layer Solution

The classical Blasius-Howarth series solution of the boundary layer problem (Schlichting in Ref. 20), although of little practical value for the slender shapes of low speed aerodynamics because of its limited convergence, seems ideally suited to blunt bodies. A few terms of the series should give good accuracy over the entire subsonic flow region, which for a sphere extends no farther back than 45 deg from the nose.

The first term in the Blasius series represents the flow in the vicinity of the stagnation point. Brown (Ref. 21) has shown how a self-similar solution of that problem can be found using the usual boundary layer variables (thus refuting the suggestion of Illingworth in Ref. 22 that this is impossible). It seems simpler to follow his lead than to apply one of the various transformations associated with the names of Dorodnitsyn, Howarth, Stewartson, Illingworth, etc.

Suppose that the inviscid solution has provided the values of thermodynamic quantities at the stagnation point

$$P_{10} \equiv P_1(0,0) \text{ III } R_{10} \equiv R_1(0,0) \text{ III } T_{10} \equiv T_1(0,0) \quad (3.2)$$

and also the pressure distribution over the body in the form

$$\frac{P_1(s,0)}{P_1(0,0)} = 1 - \pi_2 s^2 - \pi_4 s^4 - \dots \quad (3.3)$$

Then the surface speed is given by

$$U_1(s,0) = W_1 s + W_3 s^3 + \dots \quad (3.4a)$$

where

$$W_1 = \left[ \frac{2 + (\gamma-1)M_\infty^2}{\gamma M_\infty^2} \right]^{\frac{1}{2}}, \text{ etc.} \quad (3.4b)$$

and the temperature and density can likewise be found as series in even powers of  $s$ . Furthermore, the body temperature and (for axisymmetric flow) the body radius must be expanded as

$$T_b(s) = T_{10}(b_0 + b_2 s^2 + \dots) \quad (3.5a)$$

$$r(s) = s + r_3 s^3 + \dots \quad (3.5b)$$

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The continuity equation 2.43a is satisfied by introducing the usual stream function  $\psi_1$  according to

$$r^j \rho_1 u_1 = \psi_{1N}, \quad r^j \rho_1 v_1 = -\psi_{1s} \quad (3.6)$$

Then the first-order boundary layer problem is reduced to a sequence of ordinary differential equations by expanding the stream function and temperature in the Blasius series

$$\psi_1(s, N) = \left[ \frac{R_{10} \mu(T_{10})}{W_1} \right]^{\frac{1}{2}} s^{1+j} [W_1 f_1(\eta) + s^2 f_3(\eta) + \dots] \quad (3.7a)$$

$$t_1(s, N) = T_{10} [r_1(\eta) + s^2 r_3(\eta) + \dots] \quad (3.7b)$$

where the boundary layer variable is

$$\eta = \left[ \frac{W_1 R_{10}}{\mu(T_{10})} \right]^{\frac{1}{2}} N \quad (3.8)$$

Substituting into the first-order boundary layer problem Eqs. 2.43 and 2.44 yields a sequence of problems of which the first is

$$\left[ \frac{\mu(T_{10} r_1)}{\mu(T_{10})} (r_1 f_1') \right]' + (1+j)f_1(r_1 f_1')' - r_1 f_1'^2 = -1 \quad (3.9a)$$

$$\left[ \frac{\mu(T_{10} r_1)}{\mu(T_{10})} r_1' \right]' + \sigma(1+j)f_1 r_1' = 0 \quad (3.9b)$$

$$f_1(0) = f_1'(0) = 0, \quad r_1(0) = b_0 \quad (3.9c)$$

$$f_1'(\infty) = r_1(\infty) = 1 \quad (3.9d)$$

The subsequent problems involve the parameters  $W_1, W_3, T_{10}, r_3, \gamma, M_\infty, b_2$ , etc., but could be subdivided into a number of universal problems following Howarth (see Schlichting, Ref. 20). This is worthwhile, however, only if a large number of problems are to be solved.

Second Approximation

The second-order boundary layer problems can likewise be solved using Blasius series. Consider first the problem in

Eq. 2.58 for entropy gradient. The continuity equation is satisfied by introducing the second-order stream function  $\psi_2$  according to

$$r^j(\rho_1 u_2 + \rho_2 u_1) = \psi_{2N}, \quad r^j(\rho_1 v_2 + \rho_2 v_1) = -\psi_{2s} \quad (3.10)$$

Then setting

$$\psi_2(s,N) = \frac{R_{10} T_{10} \mu(T_{10})}{W_1} s^{1+j}[f_2(\eta) + \dots] \quad (3.11a)$$

$$t_2(s,N) = \frac{T_{10}^2}{W_1} \left[ \frac{R_{10} \mu(T_{10})}{W_1} \right]^{\frac{1}{2}} [r_2(\eta) + \dots] \quad (3.11b)$$

leads to a sequence of problems of which the first is

$$\begin{aligned} \mathcal{D}_1\{f_2, r_2\} &\equiv \left[ \frac{\mu(T_{10} \tau_1)}{\mu(T_{10})} (r_1 f_2' + r_2 f_1') + \frac{T_{10} \mu'(T_{10})}{\mu(T_{10})} r_2 (r_1 f_1') \right]' \\ &+ (1+j)f_1(r_1 f_2' + r_2 f_1') + (1+j)f_2(r_1 f_1') - 2r_1 f_1' f_2' - r_2 f_1'^2 \\ &= 2\beta_1 \end{aligned} \quad (3.12a)$$

$$\begin{aligned} \mathcal{E}_1\{f_2, r_2\} &\equiv \left[ \frac{\mu(T_{10} \tau_1)}{\mu(T_{10})} r_2' + \frac{T_{10} \mu'(T_{10} \tau_1)}{\mu(T_{10})} r_2 r_1' \right]' \\ &+ 2\sigma(f_1 r_2' + f_2 r_1') = 0 \end{aligned} \quad (3.12b)$$

$$f_2(0) = f_2'(0) = r_2(0) = 0 \quad (3.12c)$$

$$f_2''(\infty) = -1, \quad r_2(\infty) = 0 \quad (3.12d)$$

where

$$\beta_1 = \lim_{\eta \rightarrow \infty} (\eta f_1' - f_1) = \lim_{\eta \rightarrow \infty} (\eta - f_1) \quad (3.13)$$

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In treating curvature effects, it is necessary to expand the curvature  $\kappa$  of the meridian curve and the cosine of its angle with the axis in powers of  $s$ .

$$\kappa(s) = 1 - \kappa_2 s^2 + \dots \quad (3.14a)$$

$$\cos \theta(s) = s + C_1 s^3 + \dots \quad (3.14b)$$

Then in the problem in Eq. 2.53 for longitudinal curvature one sets

$$r^j(\rho_1 u_2 + \rho_2 u_1) = \psi_{2N} \quad (3.15)$$

$$r^j(\rho_1 v_2 + \rho_2 v_1 + \kappa N \rho_1 v_1) = -\psi_{2s}$$

$$\psi_2(s, N) = \mu(T_{10}) s^{1+j} [f_2(\eta) + \dots] \quad (3.16a)$$

$$t_2(s, N) = T_{10} \left[ \frac{\mu(T_{10})}{W_1 R_{10}} \right]^{\frac{1}{2}} [r_2(\eta) + \dots] \quad (3.16b)$$

In the first of the resulting problems, the integral appearing in the momentum equation (for example, Eq. 2.53b) can be evaluated using the fact, deduced from the first-order Eq. 3.9a, that

$$r_1 f_1'^2 = \left[ (2+j)r_1 f_1 f_1' + \eta + \frac{\mu(T_{10} r_1)}{\mu(T_{10})} (r_1 f_1')' \right]'$$

Thus the problem becomes

$$\mathcal{D}_1\{f_2, r_2\} = \left[ \frac{\mu(T_{10} r_1)}{\mu(T_{10})} r_1 f_1' \right]' - 2 \frac{1+j}{2+j} \frac{\mu(T_{10} r_1)}{\mu(T_{10})} (r_1 f_1')' + \frac{4+j}{2+j} \eta$$

$$- \eta r_1 f_1'^2 + (1+j)\eta f_1 (r_1 f_1')' - j \frac{1+j}{2+j} r_1 f_1 f_1' + 2 \frac{1+j}{2+j} \beta_1 \quad (3.17a)$$

$$\mathcal{E}\{f_2, r_2\} = \sigma(1+j)\eta f_1 r_1' - \frac{\mu(T_{10} r_1)}{\mu(T_{10})} r_1' \quad (3.17b)$$

with the boundary conditions 3.12c and d.

Similarly, in the problem Eq. 2.54 for transverse curvature, setting

$$\left. \begin{aligned} r^j(\rho_1 u_2 + \rho_2 u_1 + \frac{\cos \theta}{r} N \rho_1 u_1) &= \psi_{2N} \\ r^j(\rho_1 v_2 + \rho_2 v_1 + \frac{\cos \theta}{r} N \rho_1 v_1) &= -\psi_{2s} \end{aligned} \right\} \quad (3.18)$$

and taking the Blasius series in the form in Eq. 3.16 gives the problem

$$\mathcal{D}\{f_2, r_2\} = \left[ \frac{\mu(T_{10} r_1)}{\mu(T_{10})} (\eta r_1 f_1') \right]' \frac{\mu(T_{10} r_1)}{\mu(T_{10})} (r_1 f_1')' + 2f_1(\eta r_1 f_1')' + 2\eta f_1' - 2\eta r_1 f_1'^2 \quad (3.19a)$$

$$\mathcal{E}_1\{f_2, r_2\} = 2\sigma\eta f_1 r_1' - \frac{\mu(T_{10} r_1)}{\mu(T_{10})} r_1' \quad (3.19b)$$

$$f_2(0) = f_2'(0) = r_2(0) = 0 \quad (3.19c)$$

$$f_2''(\infty) = 1, \quad r_2(\infty) = 0 \quad (3.19d)$$

In the problem for temperature jump the stream function is introduced according to Eq. 3.6, and setting

$$\psi_2(s, N) = W_1 \mu(b_o T_{10}) \left[ \frac{\gamma}{\gamma-1} \frac{b_o}{T_{10}} \right]^{1/2} s^{1+j}[f_2(\eta) + \dots] \quad (3.20a)$$

$$r_2(s, N) = \mu(b_o T_{10}) \left[ \frac{\gamma}{\gamma-1} \frac{b_o T_{10} W_1}{R_{10} \mu(T_{10})} \right]^{1/2} [r_2(\eta) + \dots] \quad (3.20b)$$

gives the problem

$$\mathcal{D}_1 \{f_2, r_2\} = \mathcal{E}_1 \{f_2, r_2\} = 0 \quad (3.21a)$$

$$f_2(0) = f_2'(0) = 0, \quad r_2(0) = r_1'(0) \quad (3.21b)$$

$$f_2'(\infty) = r_2(\infty) = 0 \quad (3.21c)$$

Displacement is the most difficult second-order effect to calculate (though it is simpler in supersonic than subsonic flow because the region of influence is limited). Enough terms of the first-order Blasius series must be computed to give the displacement velocity  $V_2(s,0)$  accurately throughout the subsonic region. The outer flow due to displacement thickness must then be calculated, for example, by perturbing the numerical solution of the inviscid problem. One may suppose that this has been done, so that the change in surface speed is known as

$$U_2(s,0) = W_2 s + \dots \quad (3.22)$$

Then introducing the second-order stream function according to Eq. 3.6 and setting

$$\psi_2(s,N) = W_2 \left[ \frac{R_{10} \mu(T_{10})}{W_1} \right]^{\frac{1}{2}} s^{1+j} [f_2(\eta) + \dots] \quad (3.23a)$$

$$r_2(s,N) = \frac{W_2 T_{10}}{W_1} [r_2(\eta) + \dots] \quad (3.23b)$$

gives the problem

$$\mathcal{D}_1 \{f_2, r_2\} = -2 \quad \{f_2, r_2\} = 0 \quad (3.24a)$$

$$f_2(0) = f_2'(0) = r_2(0) = 0 \quad (3.24b)$$

$$f_2'(\infty) = 1 \quad r_2(\infty) = 0 \quad (3.24c)$$

Numerical Example

Numerical integration of the preceding equations has been carried out for an axisymmetric stagnation point ( $j = 1$ ) with Prandtl number  $\sigma = 0.7$ , viscosity proportional to temperature ( $\omega = 1$ ), and the body cooled to one-fifth of the stagnation temperature ( $b_0 = 0.2$ ). The relevant results are

$$f_1'(0) = 22.3911 \quad \tau_1'(0) = 2.47859 \quad (3.25a)$$

which agree with those of Cohen and Reshotko (Ref. 23), and

$$\begin{aligned} f_2'(0) &= -16.82 & \tau_2'(0) &= -0.8455 & \text{entropy gradient} \\ &= -40.25 & &= -0.7812 & \text{longitudinal curv.} \\ &= -11.10 & &= 0.8268 & \text{transverse curv.} \\ &= -519.7 & &= -37.76 & \text{temperature jump} \end{aligned} \quad (3.25b)$$

The displacement effect which, in contrast with the others, depends even at the stagnation point upon the specific body shape, has not been calculated. Fortunately, however, the displacement thickness is almost zero in this example ( $\beta_1 = -0.013095$ ), so that its effect can reasonably be neglected.

From these results the relative change in heat transfer at the stagnation point is found to be given by

$$\begin{aligned} q \propto 1 + & \left[ -3.41 S_1'(0) \frac{T_{10}}{W_1} \sqrt{\frac{R_{10}\mu(T_{10})}{W_1}} - .315 \sqrt{\frac{\mu(T_{10})}{W_1 R_{10}}} \right. \\ & \left. + .334 \sqrt{\frac{\mu(T_{10})}{W_1 R_{10}}} - .254 (c_1 - a_1) \sqrt{\frac{\gamma}{\gamma-1} \frac{W_1}{R_{10}}} \right] \epsilon \end{aligned} \quad (3.26)$$

where the terms in brackets are the contributions of entropy gradient, longitudinal curvature, transverse curvature and temperature jump (slip having no effect upon heat transfer). The entropy gradient increases the heat transfer, because  $S_1'(0)$  is negative from Eq. 3.1. The net effect of curvature



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is also positive, whereas that of slip and temperature jump is negative.

Definite numerical values are obtained by considering a specific problem. Frank Fuller of the Ames Research Center, NASA, has kindly provided the author with an accurate numerical solution for the inviscid flow past a sphere at infinite Mach number with  $\gamma = 7/5$ . This gives  $\pi_2 = 1.20$  and  $a = 0.7738$ , so that  $w_1 = 0.5855$ ,  $T_{10} = 1/2$ ,  $R_{10} = 6.4377$ , and  $S_1'(0) = -0.85531$ . The slip and temperature-jump coefficients are taken to have the simple values (Ref. 15)  $a_1 = (\pi/2)^{1/2}$  and  $c_1 = 15/8(\pi/2)^{1/2}$ . Then Eq. 3.26 and its counterpart for the skin friction give

Entr grad	Long curv	Trans curv	Slip	Temp jump	Total	
$q \propto 1 + (0.584$	$- 0.115$	$+ 0.121$	$+ 0$	$- 0.157$	$= 0.433)\epsilon$	(3.27a)

$r \propto 1 + (1.287$	$- 0.655$	$- 0.181$	$- 0.071$	$+ 0.087$	$= 0.468)\epsilon$	(3.27b)
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Here for purposes of comparison with other work, one may set

$$\epsilon = 0.6325 \frac{M_\infty}{R_\infty^{1/2}} = 1.414 \frac{1}{R_p^{1/2}} = 3.017 \frac{1}{R_f^{1/2}} \quad (3.28)$$

where

$$R_p = U_\infty a \rho_\infty / \mu_s \quad (3.29)$$

is the convenient hybrid Reynolds number used by Probstein (Ref. 24), and

$$R_f = H_1^{1/2} a \rho_o / \mu_o \quad (3.30)$$

is that used by Ferri, Zakkay and Ting (Ref. 3).

Effect of Density Ratio

These values apply only to a specific case with a density ratio across the normal shock wave of  $K = (\gamma + 1)/(\gamma - 1)$  equal to 6. However, it is impossible to estimate the effect of changing the density ratio by considering the asymptotic behavior

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as  $\gamma \rightarrow 1$ . Then for any axisymmetric body  $W_1 = (8/3K)^{1/2}$  and  $\alpha = 1$ . (See Hayes and Probststein, Ref. 2, p. 161.) Then Eq. 3.24 and its counterpart for skin friction show that the effect of entropy gradient varies as  $K^{5/4}$ , that of curvature  $K^{3/4}$ , and that of slip and temperature jump as  $K^{-1/4}$ . (These hold also if the results are expressed in terms of  $R_p$  rather than using Eq. 3.28, but are replaced by  $K^{7/4}$ ,  $K^{5/4}$  and  $K^{1/4}$  if  $R_f$  is introduced.)

These rules can be used for approximate scaling to different density ratios. They show that as the density ratio increases, the effect of entropy gradient becomes dominant. This is in accord with Cheng's conclusion (Ref. 13) that in the thin shock layer approximation the curvature and slip effects are negligible.

### Comparison with Other Work

Although the pressure gradient associated with external vorticity was ignored by previous investigators, it is unimportant in the present numerical example because the displacement thickness almost vanishes. (In fact, using the matching condition proposed by Hayes (Ref. 2, p. 371), which amounts simply to discarding the right side of Eq. 2.58b, changes  $r_2'(0)$  for the entropy gradient only from -0.8455 to -0.8617.) Nevertheless, the effect of entropy gradient calculated here is more than twice as large as that found by Probststein (Ref. 2, p. 372) for both the heat transfer and skin friction. (The fact that these are reduced by other second-order effects is irrelevant.) This discrepancy has not been resolved.

On the other hand, the present result is less than half that predicted by Ferri, Zakkay and Ting (Ref. 3), which agrees well with the present author's experiments. The present analysis would have to be re-examined if other experiments should confirm their data.

### ACKNOWLEDGMENT

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### NOMENCLATURE

a	= nose radius
$a_1, b_1, c_1$	= coefficients of slip and temperature jump
$b_0, b_2, \dots$	= coefficients in series for $T_b$
c	= speed of sound
$c_p$	= specific heat at constant pressure
$C_1, \dots$	= coefficients in series for $\cos \theta$
$C \{ \}$	= operator in continuity equation
$d \{ \}$	= thickness of shock layer
$D \{ \}$	= operator in momentum equation
$\mathcal{D} \{ \}$	= operator in momentum equation
$E \{ \}$	= operator in energy equation
$\mathcal{E} \{ \}$	= operator in energy equation
$f_1, f_3, \dots$	= functions in Blasius series for $\psi_1$
$f_2, \dots$	= functions in Blasius series for $\psi_2$
$F \{ \}$	= operator in equation of state
$H_1$	= stagnation enthalpy in basic inviscid flow (referred to $U_\infty^2$ )
j	= 0 for plane flow, 1 for axisymmetric flow
k	= coefficient of heat conduction
K	= density ratio across normal shock wave ( $> 1$ )
$M_\infty$	= reference (e.g., free stream) Mach number
n	= distance normal to body (referred to a)
N	= magnified distance normal to body, $n/\epsilon$
p	= pressure (referred to $\rho_\infty U_\infty^2$ )

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- $P_1, P_2, \dots$  = coefficients in inner expansion of  $p$   
 $P_1, P_2, \dots$  = coefficients in outer expansion of  $p$   
 $P_{10}$  =  $P_1$  at stagnation point,  $P_1(0,0)$   
 $q$  = heat transfer  
 $\vec{q}$  = velocity vector (referred to  $U_\infty$ )  
 $Q_1$  = velocity vector in basic inviscid flow (referred to  $U_\infty$ )  
 $r$  = radius of body (referred to  $a$ )  
 $r_3, \dots$  = coefficients in series for  $r$   
 $R_1, R_2, \dots$  = coefficients in outer expansion of  $\rho$   
 $R_f$  = Ferri's Reynolds number,  $H_1^{1/2} a \rho_o / \mu_o$   
 $R_p$  = Probstein's Reynolds number,  $U_\infty a \rho_\infty / \mu_s$   
 $R_{10}$  =  $R_1$  at stagnation point,  $R_1(0,0)$   
 $R_\infty$  = reference (e.g., free stream) Reynolds number,  $U_\infty a \rho_\infty / \mu_\infty$   
 $s$  = distance along body surface (referred to  $a$ )  
 $S_1$  = entropy in basic inviscid flow (referred to  $c_p$ )  
 $t_1, t_2, \dots$  = coefficients in inner expansion of  $T$   
 $T$  = temperature (referred to  $U_\infty^2 / c_p$ )  
 $T_b$  = temperature of body (referred to  $U_\infty^2 / c_p$ )  
 $T_1, T_2, \dots$  = coefficients in outer expansion of  $T$   
 $T_{10}$  =  $T_1$  at stagnation point,  $T_1(0,0)$   
 $u, v$  = velocity components in  $(s, n)$  directions (referred to  $U_\infty$ )  
 $u_1, u_2, \dots$  = coefficients in inner expansion of  $u$   
 $U_1, U_2, \dots$  = coefficients in outer expansion of  $u$

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$U_\infty$	= reference (e.g., free stream) speed
$v_1, v_2, \dots$	= coefficients in inner expansion of $v$
$V_1, V_2, \dots$	= coefficients in outer expansion of $v$
$W_1, W_3, \dots$	= coefficients in series for $U_1(s, 0)$
$W_2, \dots$	= coefficients in series for $U_2(s, 0)$
$a$	= ratio of body to shock nose radii
$\beta_1$	= displacement constant, $\lim_{\eta \rightarrow \infty} (\eta - f_1)$
$\gamma$	= adiabatic exponent
$\delta$	= thickness of boundary layer
$\Delta$	= thickness of shock wave
$\epsilon$	= perturbation parameter, Eq. 2.11
$\eta$	= boundary layer variable
$\theta$	= angle of surface
$\kappa$	= curvature of surface (referred to $a^{-1}$ )
$\kappa_2, \dots$	= coefficients in series for $\kappa$
$\lambda$	= second coefficient of viscosity (referred to value of first viscosity at temperature $U_\infty^2/c_p$ )
$\mu$	= first coefficient of viscosity (referred to its value at temperature $U_\infty^2/c_p$ )
$\nu$	= kinematic viscosity, $\mu / \rho$
$\pi_2, \pi_4, \dots$	= coefficients in series for surface pressure
$\rho$	= density (referred to $\rho_\infty$ )
$\rho_1, \rho_2, \dots$	= coefficients in inner expansion of $\rho$
$\rho_\infty$	= reference (e.g., free stream) density
$\sigma$	= Prandtl number, $\mu c_p / k$

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- $\tau$  = skin friction
- $\tau_1, \tau_3, \dots$  = coefficients in Blasius series for  $t_1$
- $\tau_2, \dots$  = coefficients in Blasius series for  $t_2$
- $\phi$  = azimuthal angle

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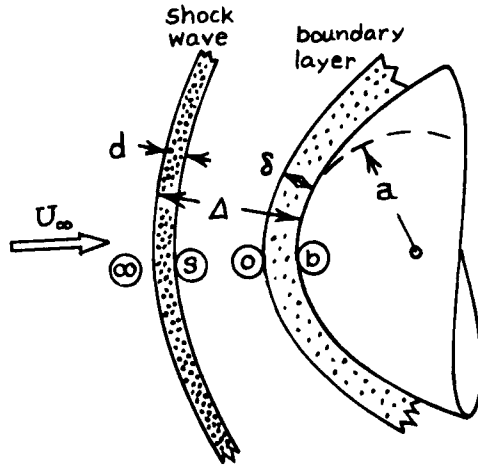


Fig. 1 Flow geometry.

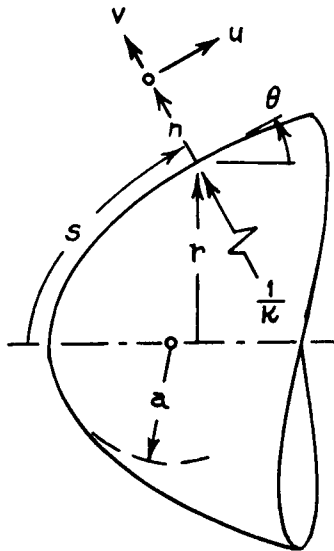


Fig. 2 Coordinate system.