SLENDER WINGS AT HIGH ANGLES OF ATTACK IN HYPERSONIC FLOWS

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ABSTRACT

Newtonian theory is worked out for a flat plate normal to a hypersonic stream. For a slender flat wing the results are applied in spanwise strips and heat transfer is calculated on the basis of boundary layer theory. Fair agreement with experiments on delta wings at angle of attack near 90 deg is found.

INTRODUCTION

The study of lifting re-entry vehicles has generated interest in wings at angles of attack up to 90 deg in hypersonic flow. It is, however, desirable to develop a method of predicting the surface pressure and heat transfer on such wings. In this paper, the limiting case of a flat wing of small aspect ratio at 90 deg angle of attack is considered. The flow over the main part of such a slender wing is essentially in the spanwise direction. By considering a narrow spanwise strip in this central region of the wing the flow can be treated as two-dimensional over the strip (see Fig. 1). Then the flow over the central region can be approximated by considering a series

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of the strips. This process requires the chordwise component of velocity be negligible. This in turn requires that the span of the strip be small compared to the distance of the strip from an edge where the flow is essentially three-dimensional in character (for a delta wing, $b(Z) << Z, L - Z$). In this connection the flow past a two-dimensional flat plate normal to a hypersonic stream is studied in this paper. A Newtonian approximate theory is carried out which provides the surface pressure and velocity distribution. Lees' boundary layer theory is then applied using the Newtonian theory to give external conditions. A correction is introduced to account for deviations from $\alpha =$ 90 deg and the heat transfer results are compared with experiments.

NEWTONIAN FLOW THEORY FOR FLAT PLATE

In this section the two-dimensional flow of a perfect gas past a flat plate normal to a high Mach number stream is calculated on the basis of Newtonian flow. The flow is calculated as a perturbation of the flow when the shock wave has infinite density ratio and lies on the body surface normal to the stream. As usual, the procedure involves a mathematical limit process in which $(\gamma \to 1, M_\infty \to \infty)$ which is carried out in a suitable coordinate system; the distance between shock and body is an appropriate function of $(\gamma - 1)$..

This work is an extension of the paper of Hida (Ref. 1) which treats axisymmetric very blunt bodies. The orders of magnitude of the various perturbations and the basic flow equations are essentially the same. Hida's analysis corresponds also to Hayes' constant stream-tube area theory (Ref. 2) and in fact leads to the same equation for shock wave shape. Hayes remarks that the two-dimensional flow does not exist. It is shown here that this difficulty can be overcome by the introduction of an inner region, adjacent to the body surface, in which the orders of magnitude are changed from that in the region near the shock. A mathematical matching of inner and outer solutions enables the complete flow field to be found.

In the Newtonian limit flow variables are expanded in terms of ϵ_o the density ratio across the very strong normal shock wave. The orders of magnitude of various quantities can be found from the requirement that the shock relation and flow equations yield a non-trivial system as $\epsilon_o \to 0$. The shock relations can be written (see Fig. 2)

$$\frac{1}{\rho_s} = \epsilon_o \left(1 + \frac{K}{1+K} \tan^2 \beta\right) \qquad (2\text{-}1a)$$

$$P_s - P_\infty = 1 - \epsilon_o - \tan^2\beta \tag{2-1b}$$

$$U_s = (\epsilon_o + \tan^2\beta)(1 - \tan^2\beta) \tag{2-1c}$$

$$V_s = (1 - \epsilon_o - \tan^2\beta)\tan\beta \tag{2-1d}$$

$$h_s = 1 + h_\infty - \tan^2\beta \tag{2-1e}$$

where

$$\lambda = \frac{\gamma - 1}{\gamma + 1}$$

$$K = \frac{1-\lambda}{\lambda M_\infty^2}$$

β = angle between normal to shock and free stream direction

$$\epsilon_o = \frac{\gamma-1}{\gamma+1} + \frac{2}{(\gamma+1)M_\infty^2}$$

Dimensionless quantities are used, and all velocities are made dimensionless by the speed at infinity q_∞, density by ρ_∞, pressure by $\rho_\infty q_\infty^2$, enthalpy by q_∞^2, and lengths by b the semi-span of the plate. If is seen from the shock relations that a distinguished case occurs when

$$\tan\beta = O(\epsilon_o)^{1/2} \text{ as } \epsilon_o \to 0$$

In this limit

$$\frac{1}{\rho_s} \to \epsilon_o + O(\epsilon_o^2)$$

$$P_s - P_\infty \to 1 + O(\epsilon_o)$$

velocity normal to plate

$$U_s \to O(\epsilon_o)$$

$$V_s \to O(\epsilon_o)^{1/2}$$

In order to arrive at non-trivial flow equations when (u, v) have different orders it is necessary to carry out the limit with $x = X/\epsilon_o^{1/2}$, y fixed, where X = distance from shock wave at $y = 0$. Consideration of entropy and the momentum equation normal to the plate shows that $P_s - P_\infty = 1 + O(\epsilon_o)$, $1/\rho_s = \epsilon_o + \ldots$ not only at the shock but in the entire flow between shock and plate. Thus the following expansion is assumed

$$U(X,Y) = \epsilon_o u(x,y) + \ldots \tag{2-2a}$$

$$V(X,Y) = (\epsilon_o)^{1/2} v(x,y) + \ldots \tag{2-2b}$$

$$P(X,Y) - P_\infty = 1 + \epsilon_o p(x,y) + \ldots \tag{2-2c}$$

$$\rho(X,Y) = \frac{1}{\epsilon_o} + \sigma(x,y) + \ldots \tag{2-2d}$$

where

$$x = \frac{X}{(\epsilon_o)^{1/2}} \qquad y = Y$$

The limiting forms of the flow equations are

$$u_x + v_y = 0 \qquad \text{continuity} \tag{2-3a}$$

$$uv_x + vv_y = 0 \qquad \text{y—momentum} \tag{2-3b}$$

$$uu_x + vu_y = -p_x \qquad \text{x—momentum} \tag{2-3c}$$

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(p - \sigma) = 0 \qquad \text{entropy} \tag{2-3d}$$

The velocities are calculated from Eqs. 2-3a and b. This field is incompressible and the pressure gradient in the transverse direction negligible. Then the pressure is found from Eq. 2-3c by integration and the correction to the density from Eq. 2-3d.

The boundary conditions at the shock are

$$v_s = \left(\frac{dx_s}{dy_s}\right) \quad \text{where} \left(\frac{dX_s}{dY_s}\right) = \tan\beta \tag{2-4a}$$

$$u_s = 1 + v_s^2 = -p_s \qquad (2\text{-}4b)$$

and at the body, a condition of tangent flow should be satisfied where $(u/v) \rightarrow 0$. It will be shown that this condition can not be satisfied and must be replaced with another.

First consider the solution of the Eq. 2-3 subject to the shock condition Eq. 2-4. The characteristics of Eqs. 2-3a and b) are the streamlines and lines $y = \text{const.}$ v is constant on streamlines. Thus (v, y) are characteristic coordinates and the equations can be transformed to canonical form in these variables. Let

$$x = x^*(v, y)$$

$$u(x, y) = u^*(v, y)$$

The Jacobian of the transformation is

$$J = \frac{\partial(x^*, y)}{\partial(v, y)} + x^*_v$$

The partial derivatives of the velocity components are

$$\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \frac{1}{J} \begin{pmatrix} u^*_v & \dfrac{\partial(x^*, u^*)}{\partial(v, y)} \\ 1 & -x^*_y \end{pmatrix}$$

Hence, the transformed equations become

$$u^*_v - x^*_y = 0 \qquad \text{continuity} \qquad (2\text{-}5a)$$

$$u^*_v(u^* - v x^*_y) + v x^*_v u^*_y = -p^*_v \qquad \text{x—momentum} \qquad (2\text{-}5b)$$

$$u^* - v x^*_y = 0 \qquad \text{y—momentum} \qquad (2\text{-}5c)$$

These are easily reduced to

$$x^*_{vy} = 0 \qquad (2\text{-}6a)$$

$$v x^*_v u^*_y = -p^*_v \qquad (2\text{-}6b)$$

$$u^* = v x^*_y \qquad (2\text{-}6c)$$

Integrating Eq. 2-6a gives

$$x^*(v, y) = f'(v) + g(y) \tag{2-7}$$

where f and g are arbitrary functions of integration. Hence

$$u^*(v, y) = vg'(y) \tag{2-8}$$

and

$$p_v^* = -v^2 f''(v)\, g''(y) \tag{2-9}$$

$$\therefore p^*(v, y) = G(y) - g''(y) \int_0^v t^2 f''(t)\, dt \tag{2-10}$$

where F(y) is also an arbitrary function of integration. It is clear from Eq. 2-8 that the boundary condition on the body cannot be satisfied, the streamlines have constant slope on $y=$ constant. But since the streamlines near the surface are ones that passed through the center region of the shock, v is small along the streamline. Hence, a new expansion for $v=O(\epsilon_o)$ must be made for the region close to the surface. This will be done later.

Now the shock relations provide one relation between f, g. The shock wave is represented parametrically by

$$y_s = y_s(v)$$

$$x_s^* = x^*(v, y_s(v)) \tag{2-11}$$

Then by the shock relations

$$v \frac{dy_s}{dv} = \frac{dx_s^*}{dv} \tag{2-12}$$

From Eq. 2-7

$$x_s^* = f'(v) + g(y_s)(v))$$

$$\therefore \{v - g'(y_s)\} \frac{dy_s}{dv} = f''(v) \tag{2-13}$$

From the shock relations Eq. 2-8 becomes

$$g'(y_s) = v + \frac{1}{v} \tag{2-14}$$

and

$$\frac{dy_s}{dv} = -\, v f''(v) \tag{2-15}$$

or

$$y_s(v) = f(v) - v f'(v) \tag{2-16}$$

On the axis, $y = 0 = v$, so that

$$\lim_{v \to 0} \{ f(v) - v f'(v) \} = 0$$

Now differentiation of Eqs. 2-14 and 2-15 gives

$$g''(y_s) = \frac{1 - v^2}{v^3 f''(v)} \tag{2-17}$$

The equation relating f and g at the shock wave are too complicated to be solved directly. However, near the axis, y and v are small so that expansions in y and v are possible, and necessary for the matching.

For $v \ll 1$, near the axis, it is assumed that $y_s(v)$ can be expanded in a power series in v. By symmetry, only the odd powers of v are contained in the expansion

$$y_s(v) = Av + Bv^3 + \ldots \qquad v \to 0$$

$$= -\, v^2 \frac{d}{dv} \left\{ \frac{f(v)}{v} \right\} \text{ by } (2-16) \tag{2-18}$$

Hence

$$f(v) = k_1 v - Av \log v - \frac{1}{2} Bv^3 + \ldots \qquad v \to 0 \tag{2-19}$$

where k_1 is an arbitrary constant of integration. Also, consideration of Eq. 2-14 shows

$$g(\xi) = A \log \xi + a_1 + \frac{A + B}{A^2} \frac{\xi^2}{2} + \ldots \qquad \xi \to 0 \tag{2-20}$$

Similarly

$$x_s^* = a_1 + k_1 + A \log A - A \left(\frac{A+B}{2} - \frac{3}{2} B \right) v^2 + \ldots, \quad v \to 0 \tag{2-21}$$

but a consideration of Eq. 2-12 shows that $B=0$, and fixing the origin yields a parametric representation of shock shape

$$y_s(v) = Av + Bv^3 + \ldots$$

$$x_s^*(v) = \left(\frac{A}{2} \right) v^2 + \ldots \tag{2-22}$$

Thus, the solution has the following expansion

$$x^*(v,y) = A \log \frac{y}{Av} + \frac{1}{2A} y^2 + \ldots - O(v^4) \qquad v, y \to 0$$

$$u^*(v,y) = v \left(\frac{A}{y} + \frac{1}{A} y + \ldots \right) \qquad y \to 0, \text{ all } v \tag{2-23}$$

$$p^*(v,y) = G(y) - \left(\frac{A}{y^2} - \frac{1}{A} - \ldots \right) \left(A \frac{v^2}{2} + \ldots \right)$$

At the shock

$$p_s^*(v,y) = G(y_s) - \frac{A}{A^2 v^2 + \ldots} - \frac{1}{A} - \ldots \; A \frac{v^2}{2} + \ldots$$

$$= G(y_s) - \left\{ \frac{1}{2} - \frac{v^2}{2} + \ldots \right\} \tag{2-24}$$

and application of the shock relations as $v \to 0$ shows $p_s^* \to -1$ so that

$$G(0) = -\frac{1}{2} \tag{2-25}$$

The "outer" solution expansion just constructed is not valid near the surface of the plate, since the orders of magnitude of the flow velocities are overestimated. It is easily seen from Eq. 2-23 that when $x^* = O(1)(u, v)$ are exponentially small (for fixed y). Alternatively, all streamlines close to the body surface pass through the shock near $y=0$. This decrease in order has the effect that the previously neglected transverse pressure gradient becomes important, even dominant, near the surface of the plate.

In order that a region can exist in which an inner expansion is valid it is necessary that the shock separation distance $D(\epsilon_o)$ be larger than the thickness of the "outer" region

$$O(1) > O(D(\epsilon_o)) > O(\epsilon_o^{1/2}) \quad \epsilon_o \to 0$$

Similarly the region adjacent to the plate must have a thickness $\mu(\epsilon_o)$ such that

$$O(D(\epsilon_o)) > O(\mu(\epsilon_o))$$

Thus an "inner" expansion is carried out keeping $(\tilde{x}, y)$ fixed where

$$\tilde{x} = \frac{X - D(\epsilon_o)}{\mu(\epsilon_o)} \tag{2-26}$$

It is expected (and can be shown) that the orders of magnitude of pressure and density are not changed from those in the outer expansion. The following expansion is assumed

$$P - P_\infty = 1 + \epsilon_o \tilde{p}(\tilde{x}, y) + \ldots \tag{2-27a}$$

$$\rho = \frac{1}{\epsilon_o} + \tilde{\sigma}(\tilde{x}, y) + \ldots \tag{2-27b}$$

$$U = a(\epsilon_o)\, \tilde{u}(\tilde{x}, y) + \ldots \tag{2-27c}$$

$$V = \nu(\epsilon_o)\, \tilde{v}(\tilde{x}, y) + \ldots \tag{2-27d}$$

The appropriate orders of D, μ, a, ν must be found to yield a consistent expansion which can satisfy the boundary conditions on the body and match with the outer expansion. The equations of motion in terms of the new variables read

$$\frac{a}{\mu} \tilde{u}_{\tilde{x}} + \nu \tilde{v}_y = 0 \qquad \text{continuity} \tag{2-28a}$$

$$\frac{a^2}{\mu} \tilde{u}\tilde{u}_{\tilde{x}} + \nu a \tilde{v} \tilde{u}_y = - \frac{\epsilon_o^2}{\mu} \tilde{p}_{\tilde{x}} \qquad \text{X-momentum} \tag{2-28b}$$

$$\frac{a}{\mu} \tilde{u} \tilde{v}_{\tilde{x}} + \nu^2 \tilde{v} \tilde{v}_y = - \epsilon_o^2\, \tilde{p}_y \qquad \text{Y-momentum} \tag{2-28c}$$

It is seen that in order to preserve the continuity equation and the effect of transverse pressure gradient

$$\nu = \epsilon_o \qquad a = \mu \epsilon_o \tag{2-29}$$

The inner equations are

$$\tilde{u}_{\tilde{x}} + \tilde{v}_y = 0 \tag{2-30a}$$

$$0 = -\tilde{p}_{\tilde{x}} \tag{2-30b}$$

$$\tilde{u}\,\tilde{v}_{\tilde{x}} + \tilde{v}\,\tilde{v}_y = -\tilde{p}_y \tag{2-30c}$$

$$\left(\tilde{u}\frac{\partial}{\partial x} + \tilde{v}\frac{\partial}{\partial y}\right)(p - \tilde{\sigma}) = 0 \tag{2-30d}$$

Equation 2-30b shows that the pressure does not vary across the inner layer so that

$$\tilde{p}(\tilde{x}, y) = \tilde{p}(y) \tag{2-31}$$

This pressure can be matched with the pressure in the outer solution by letting $x \rightarrow \infty$. For example, the body surface is $\tilde{x}=0$, $X = D(\epsilon_o)$, $x = D(\epsilon_o)/(\epsilon_o)^{1/2} \rightarrow \infty$ as $\epsilon_o \rightarrow 0$. Thus for the pressure on the body $p_b(y)$

$$p_b(y) = \tilde{p}(y) = \lim_{x \rightarrow \infty} p(x, y) \tag{2-32}$$

Further Eq. 2-30c yields a Bernoulli equation on the plate, $\tilde{u}=0$

$$p_b(y) + \frac{v_b^2(y)}{2} = \text{const} = -\frac{1}{2} = G(0) \tag{2-33}$$

Hence Eq. 2-30c can be written

$$\tilde{u}\,\tilde{v}_{\tilde{x}} + \tilde{v}\,\tilde{v}_y = v_b \frac{dv_b}{dy} \tag{2-34}$$

The solution of the inner equations satisfying the boundary conditions

$$\tilde{u}(0, y) = 0$$

$$\tilde{v}(0, y) = v_b(y) \tag{2-35}$$

is

$$\tilde{u}(\tilde{x}, y) = -\frac{1}{\lambda}\frac{dv_b}{dy}\sinh(\lambda\tilde{x})$$

$$\tilde{v}(\tilde{x}, y) = v_b(y)\cosh\lambda\tilde{x} \tag{2-36}$$

The transverse velocity as computed from the inner solution as $\tilde{x}\to-\infty$ is now matched to that computed from the outer solution as $x^*\to\infty$. Actually, both solutions should be matched to an intermediate solution but in this case there is an overlapping region of validity so that inner and outer solutions are matched directly. This can be done by writing both solutions in terms of the inner variable $\tilde{x}$. Now $x^*(y, v)\to\infty$ as $v\to 0$ for y fixed so that, using Eqs. 2-19 and 2-13

$$x^*(y, v) \to -A\log v + (k_1 - A) + \ldots + g(y)$$

or

$$v \to e^{-\frac{x-(k_1-A)-g(y)}{A}} + \ldots \quad \text{as } x\to\infty \tag{2-37}$$

Now

$$x = \frac{D(\epsilon_o) + \mu(\epsilon_o)\tilde{x}}{(\epsilon_o)^{1/2}}$$

With Eq. 2-37 written in terms of $\tilde{x}$ the matching of V, using Eq. 2-36 as $\tilde{x}\to-\infty$, reads

$$(\epsilon_o)^{1/2} e^{-\frac{1}{A}\left[\frac{D(\epsilon_o)+\mu(\epsilon_o)\tilde{x}}{(\epsilon_o)^{1/2}} - (k_1 - A) - g(y)\right]} + \ldots \tag{2-38}$$

$$= \epsilon_o v_b(y)\frac{1}{2}e^{\lambda\tilde{x}}$$

Thus $D(\epsilon_o)$, $\mu(\epsilon_o)$, λ can be chosen to make inner and outer expansions agree and $g(y)$ can be expressed in terms of $v_b(y)$

$$D = -A(\epsilon_o)^{1/2}\log(\epsilon_o)^{1/2} \tag{2-39a}$$

$$\mu = (\epsilon_o)^{1/2} \tag{2-39b}$$

$$\lambda = 1/A \tag{2-39c}$$

$$g(y) = A \log \frac{v_b(y)}{2} + \text{const} \tag{2-39d}$$

Equation 2-39d plays the role of a boundary condition for the outer solution since the velocity distribution $v_b(y)$ is related to the outer solution through the pressure matching. Letting $v \to 0$ in Eq. 2-10 identifies $F(y)$ as $p_b(y)$ so that, using Eq. 2-33

$$p^*(y, v) = -\frac{1}{2}(1 + v_b^2(y)) - g''(y)\int_0^v t^2 f''(t)\, dt \tag{2-40}$$

Thus, evaluating Eq. 2-40 on the shock $y = y_s(v)$, using Eq. 2-4b

$$\int_0^v t^2 f''(t)\, dt = -\frac{\frac{1}{2} v_b^2(y_s) - v^2 - \frac{1}{2}}{g''(y_s)}$$

But Eqs. 2-14 and 2-15 show

$$g''(y_s)\frac{dy_s}{dv} = 1 - \frac{1}{v^2}$$

$$vf''(v) = -\frac{dy_s}{dv}$$

so that

$$\int_0^v t \frac{dy_s}{dt}\, dt = \frac{\frac{1}{2} v_b^2(y_s) - v^2 - \frac{1}{2}}{1 - \frac{1}{v^2}} \frac{dy_s}{dv}$$

Differentiating with respect to v yields

$$v\frac{dy_s}{dv} = \frac{1}{2}\frac{d}{dv}\left\{\frac{v^2}{v^2 - 1}\left[v_b^2(y_s) - 2v^2 - 1\right]\frac{dy_s}{dv}\right\} \tag{2-41}$$

Equation 2-41 can be transformed to the basic equation of the problem by introducing

$$w(v) = v_b(y_s(v)) \qquad (2\text{-}42)$$

The matching condition Eq. 2-39d shows

$$g'(y_s)\frac{dy_s}{dv} = \frac{A}{w}\frac{dw}{dv}$$

Hence

$$\frac{dy_s}{dv} = \frac{v}{1+v^2}\frac{A}{w}\frac{dw}{dv} \qquad (2\text{-}43)$$

Using Eq. 2-43 in Eq. 2-41 the authors arrive at the basic equation to be integrated

$$\frac{v^2}{1+v^2}\frac{1}{w}\frac{dw}{dv} = \frac{1}{2}\frac{d}{dv}\left\{\frac{v^3}{v^4-1}(w^2-2v^2-1)\frac{1}{w}\frac{dw}{dv}\right\} \qquad (2\text{-}44)$$

w is the transverse velocity at the plate at a y-value corresponding to a given v at the shock. One may expect the solution to Eq. 2-44 to connect $(w=0, v=0)$ representing the axis of symmetry to a point representing the plate edge. Near $v \to 0$, $y_s \to Av$ so that Eq. 2-43 shows

$$\frac{1}{w}\frac{dw}{dv} \to \frac{1}{v} \quad \text{as} \quad v \to 0$$

Near $v \to 0$, Eq. 2-44 is approximated by

$$\frac{1}{v}\frac{v^3}{w}\frac{dw}{dv} = \frac{1}{2}\frac{d}{dv}\left(\frac{v^3}{w}\frac{dw}{dv}\right)$$

or

$$w = k_o v^a + \ldots \quad v \to 0 \qquad (2\text{-}45)$$

Thus, the physically reasonable solution near $v=0$ corresponds to $a=1$ with k_o thus far arbitrary. Equation 2-44 also has a singular point at $(w=3^{1/2}, v=1)$ and can be approximated in the neighborhood of this point by

$$\frac{dw^+}{dv^+} \doteq \frac{d}{dv^+}\frac{(3/2)^{1/2}w^+ - v^+}{v^+}\frac{dw^+}{dv^+} \quad \text{where} \quad w = (3)^{1/2} + w^+, \quad v = 1 + v^+ \qquad (2\text{-}46)$$

Integration gives

$$\frac{dw^+}{dv^+} = \frac{v^+(c_1 + w^+)}{(3/2)^{1/2}\, w^+ - v^+} = \frac{v^+ c_1}{(3/2)^{1/2}\, w^+ - v^+} \tag{2-47}$$

Hence for $c_1 > 0, (3^{1/2}, 1)$ is a saddle point. Further there is only one value of c_1 for which a trajectory through the saddle point approaches the origin as in Eq. 2-45 with $a = 1$. This trajectory is taken to be the unique solution to the problem; it is the only solution with $\left(\frac{dw}{dv}\right)$ finite at $v=1$. Figure 3 shows the qualitative behavior of solutions to Eq. 2-44 for which $w = k_o v$ as $v \to 0$, for various values of k_o. The heavy curve gives the final solution connecting the saddle $(3^{1/2}, 1)$ to the origin. For the numerical integration of Eq. 2-44 a digital computer was used. The solution was started from the saddle $(3^{1/2}, 1)$ with various values of $\left(\frac{dw}{dv}\right)_{v=1} = k_1$ and calculated toward $v=0$. All the resulting curves arrived at $v=0$ with either zero or infinite slope. The transition between these two types of solutions is given approximately by $k_1 = 1.92$. The details of the computed solutions near the origin are shown in Fig. 4.

Once the solution of Eq. 2-44 is found, $y_s(v)$ can be found by integration of Eq. 2-43. The scale of the solution is fixed by saying that the singular point $v=1$ on the shock should correspond to the edge of the plate. Thus $v=1$ when $y=1$ or

$$\int_0^1 \frac{dy_s}{dv}\, dv = 1 = A \int_0^1 \frac{v^2}{1+v^2} \frac{1}{w} \frac{dw}{dv}\, dv \tag{2-48}$$

The numerical integration gives $A=1.23$. Now with $w(v)$, $y_s(v)$ given $v_b(y)$ and $p_b(y)$ can be found. Figure 5 shows a plot of the surface velocity and pressure and Fig. 6 shows the shock detachment distance $D(\epsilon_o)$. The shock shape is also found by integration of Eq. 2-12 and the results are plotted in reduced and physical coordinates in Figs. 7 and 8. An integration gives a normal force coefficient for a slender wing, with zero pressure on the back

$$C_N = \frac{1}{\gamma M_\infty^2} + 1 + \epsilon_o \int_0^1 p_b(y)\, dy \doteq \frac{1}{\gamma M_\infty^2} + 1 + \epsilon_o(-0.90) \tag{2-50}$$

The results of this section are used as a basis for the heat transfer calculation of the next section.

HEAT TRANSFER TO A SLENDER FLAT WING

The results of Newtonian theory for the inviscid shock layer are used to define conditions of flow external to a boundary layer on the body. The flow, for the wing normal to the stream is considered independent in spanwise strips. The two-dimensional boundary layer flow and heat transfer is calculated from Lees' theory (Ref. 3). This affords a comparison with wind tunnel results for a perfect gas. Lees' results are simplified in terms of the Newtonian approximation.

Lees (Ref. 3) gives the two-dimensional laminar heat transfer rate distribution along a body as

$$\tilde{q} = 0.50\,(\bar{P}r)^{-2/3}\,(\tilde{\rho}_e\tilde{\mu}_e)^{1/2}\,(q_\infty)^{1/2}\,\tilde{h}_{stag}\,\tilde{F}(\tilde{s}) \tag{3-1}$$

where

$$\tilde{F}(\tilde{s}) = \frac{\tilde{p}}{\tilde{P}_{s_o}}\,\frac{\tilde{\omega}_e}{\tilde{\omega}_o}\,\frac{\tilde{V}_e}{q_\infty}\left[2\int_0^{\tilde{s}} \frac{\tilde{p}}{\tilde{P}_{s_o}}\,\frac{\tilde{\omega}_e}{\tilde{\omega}_o}\,\frac{\tilde{V}_e}{q_\infty}\,d\tilde{s}\right]^{-1/2}$$

$$\tilde{F}(0) = \left[\frac{1}{q_\infty}\left(\frac{d\tilde{V}_e}{d\tilde{s}}\right)_{s=0}\right]^{1/2}$$

$$\tilde{\omega} = \frac{\tilde{\mu}}{\tilde{R}\tilde{T}}$$

$\tilde{P}_{s_o}$ = pressure immediately behind the normal shock

$\tilde{s}$ = distance along body from stagnation point

$\tilde{(\ )}$ = have physical dimensions

Now using the Newtonian approximation of the second section

$$\frac{\tilde{\omega}_e}{\tilde{\omega}_o} = 1 + \dots$$

$$\frac{\tilde{V}_e}{q_\infty} = \epsilon_o\, v_b(y) + \dots$$

$$\frac{\tilde{P}}{\tilde{P}_{s_o}} = 1 + \frac{\epsilon_o}{2}(1 - v_b^2)$$

$$\tilde{h}_{stag_e} = \frac{1}{2} q_\infty^2 (1 + K) \qquad (3\text{-}2)$$

Hence, $\tilde{F}(\tilde{s})$ becomes

$$F(y) = (\epsilon_o)^{1/2} \left[v_b + \epsilon_o \left(\frac{1 - v_b^2}{2} \right) (v_b) \right] [I_1 + \epsilon_o I_2]^{-1/2} \qquad (3\text{-}3)$$

where

$$y = \frac{\tilde{s}}{b}$$

$$I_1 \equiv 2 \int_0^y v_b \, dy$$

$$I_2 \equiv \int_0^y v_b (1 - v_b^2) \, dy$$

The heat transfer, in terms of Stanton number, is

$$(Re_b)^{1/2} N_{ST} = 0.50 (\bar{P}r)^{-2/3} (\tilde{\mu}_e / \tilde{\mu}_\infty)^{1/2} (1 + K) \epsilon_o^{-1/2} F(y)$$

where

$$Re_b = \tilde{\rho}_\infty q_\infty b / \tilde{\mu}_\infty$$

Thus the stagnation point value is

$$(Re_b)^{1/2} N_{ST_o} = 0.50 (\bar{P}r)^{-2/3} (\tilde{\mu}_e / \tilde{\mu}_\infty)^{1/2} (1 + K) \left[\left(1 + \frac{\epsilon_o}{2} \right) v'_{b_o} \right]^{1/2} \qquad (3\text{-}4)$$

The numerical results of the second section show that the stagnation point velocity gradient is

$$v'_{b_o} \equiv \left(\frac{dv_b}{dy}\right)_{y=0} = \frac{4}{3}$$

The viscosity ratio μ_e/μ_∞ can be found from a suitable viscosity law (e.g., the Sutherland law) and the temperature variation across the shock wave, the temperature in the inviscid shock layer, is constant to order ϵ_o^2.

The heat transfer distribution in the spanwise direction, normalized by the stagnation point value is

$$\frac{N_{ST}}{N_{ST_o}} = \frac{F(y)}{F(0)} = \left[1 + \frac{\epsilon_o}{2}(1 - v_b^2)\right] v_b \left[\left(1 + \frac{\epsilon_o}{2}\right) v'_{b_o} (I_1 + \epsilon_o I_2)\right]^{-1/2} \qquad (3\text{-}5)$$

Curves illustrating the results of Eq. 3-5 are given in Fig. 9.

This heat transfer theory is now applied to the centerline of a delta wing at angle of attack α close to 90 deg.

In the calculation of heat transfer for $\alpha < 90$ deg, the spanwise variation is unchanged. The stagnation point value is found by replacing q_∞ by $q_\infty / \sin\alpha$ in the $\alpha = 90$ deg calculation. Thus

$$(Re_b)^{1/2} N_{ST_o} = \left[(Re_b)^{1/2} N_{ST_o}\right]_{\alpha = 90^\circ} (\sin\alpha)^{5/2} \qquad (3\text{-}6)$$

To compare this result with other theories and with experimental data, it is necessary to change the characteristic length in the Reynolds number from the local half-span of the wing to the centerline distance from the wing apex

$$Re_Z = \frac{\tilde{\rho}_\infty q_\infty Z}{\tilde{\mu}_\infty} = \frac{\rho_\infty q_\infty b}{\tilde{\mu}_\infty} = Re_b \tan\Lambda \qquad (3\text{-}7)$$

A specific example of the centerline heat transfer variation with angle of attack is shown in Fig. 10. The agreement with the experimental data of Bertram, et al. (Ref. 4) is good for $\alpha = 70$ deg.

The theoretical curve for low angles of attack, based on a chordwise strip theory in which the spanwise velocity is neglected, is based on Levinsky (Ref. 5).

REMARKS ON LIMITATIONS OF THE THEORY

Since the two-dimensional shock detachment distance depends only on the value of ϵ_o and is proportional to the local span of the wing, the spanwise strip theory used in this paper yields a conical shock attached to the apex of a delta wing. Assuming this shock configuration results in a chordwise velocity whose order of magnitude can be estimated.

Let β_o be the angle between the centerline of the shock and the plane normal to the free stream

$$\tan\beta_o = \frac{\tilde{D}}{\tilde{Z}} = \frac{\tilde{D}}{\tilde{L}}\frac{\tilde{L}}{\tilde{Z}} = \{1.23\,(\epsilon_o)^{1/2}\ln(\epsilon_o)^{1/2}\}\{\cot\Lambda\} \tag{4-1}$$

where

Λ = angle of sweep

Since $\tilde{V}/q_\infty = O(\epsilon_o)^{1/2}$ behind the shock, the largest $\tilde{W}/q_\infty$ permitted is $O(\epsilon_o)$. Hence, roughly

$$\cot\Lambda = O(\epsilon_o)^{1/2}$$

For $\epsilon_o = 0.10$, then $\Lambda > 70$ deg.

For angles of attack less than 90 deg, the free stream velocity has a component $\tilde{W}$ along the centerline of the wing. So long as the angle of attack is near 90 deg, this component is small. It is assumed that for these angles of attack the shock wave is still "flat" and is parallel to the wing surface. The calculations are the same as for the $\alpha = 90$ deg case except that the free stream component normal to the wing is reduced by a factor $\sin\alpha$. (See Ref. 6 and Fig. 11.)

In order to calculate the minimum angle of attack for the present theory to be applicable, considerations similar to those used in determining the minimum sweep are employed. Now $\tilde{W}_\infty/q_\infty = \cos\alpha$. Hence, in order to neglect $\tilde{W}$, $\cos\alpha = O(\epsilon_o)$. For $\epsilon_o \approx 1$, the lower limit on α is about 75 deg.

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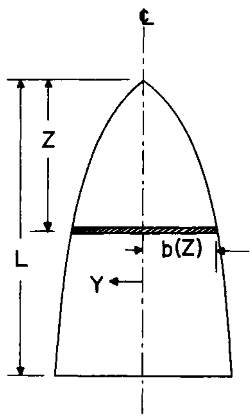

Fig. 1 Wing planform.

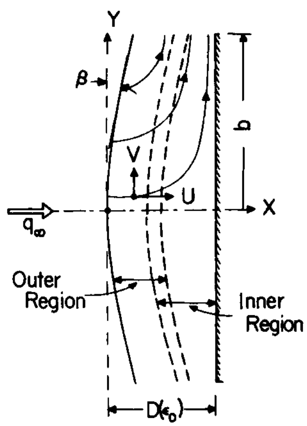

Fig. 2 Flat plate geometry.

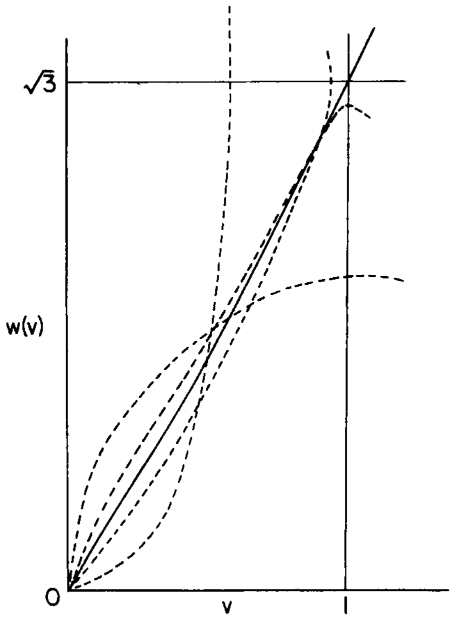

Fig. 3 Integral curves of Eq. 2-44.

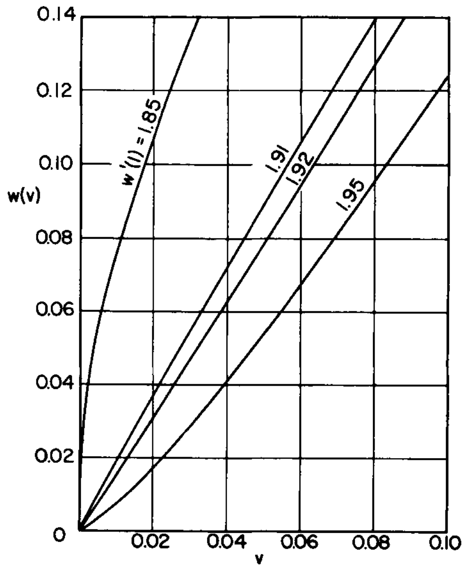

Fig. 4 Detail of integral curve of Eq. 2-44 near origin.

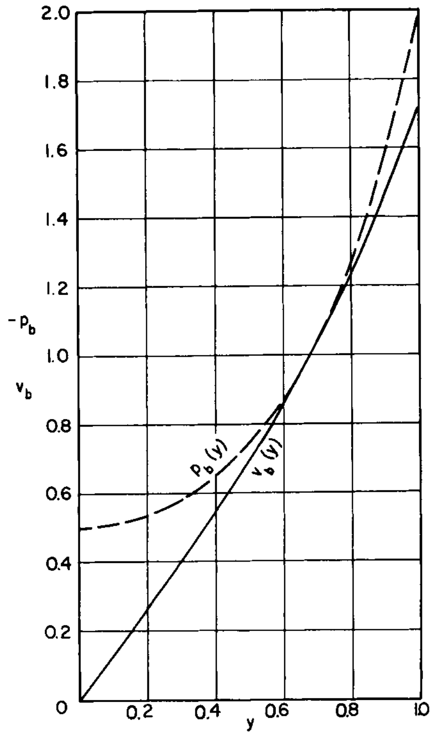

Fig. 5 Pressure and velocity distribution on flat plate.

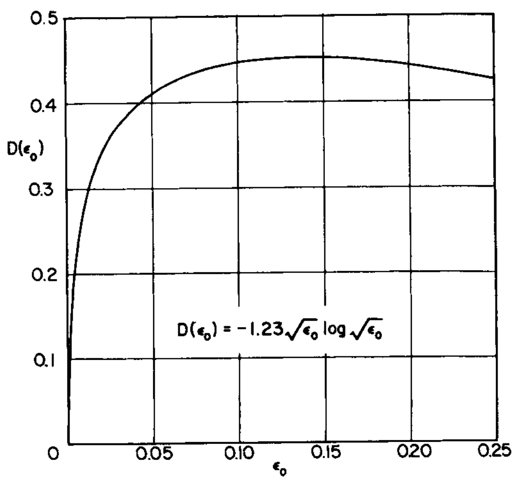

Fig. 6 Shock detachment distance.

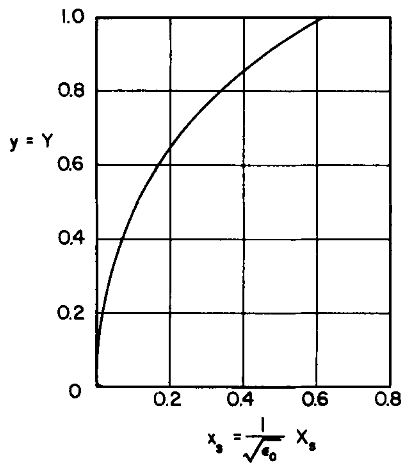

Fig. 7 Shockwave shape in reduced coordinates.

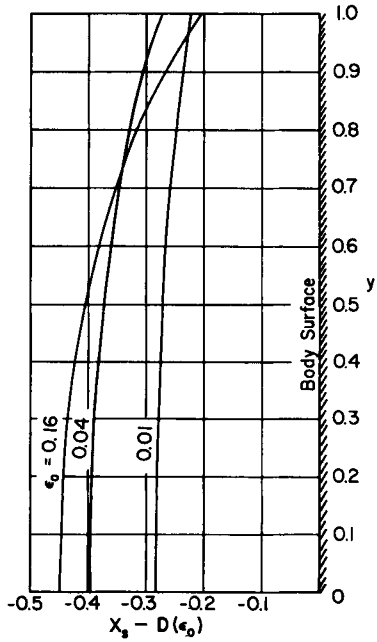

Fig. 8 Shockwave shape in physical coordinates.

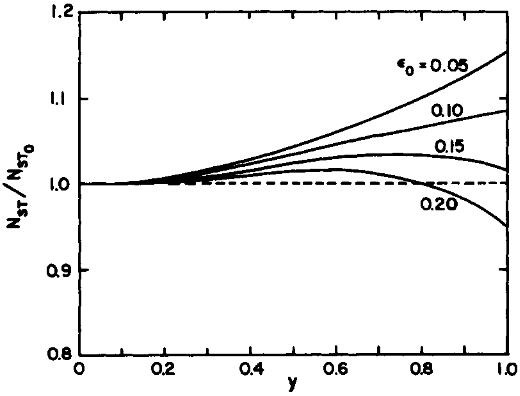

Fig. 9 Spanwise heat transfer distribution.

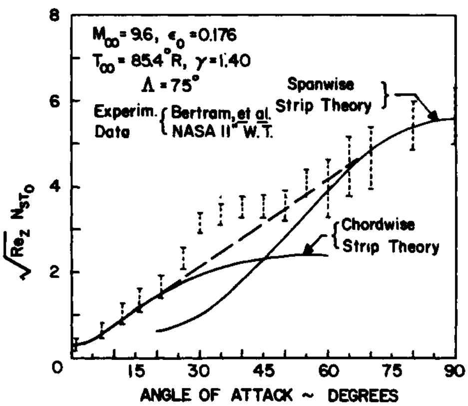

Fig. 10 Centerline heat transfer on delta wing.

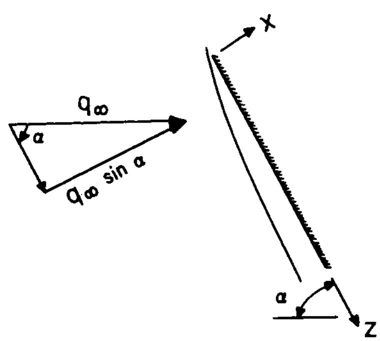

Fig. 11 Wing near 90 deg angle of attack. Side view.