# CHAPTER 1

## *The Celestial Sphere and the Nature of the Reference Systems of Astronomy*

To immediate observation, the Earth, apart from topographical irregularities, appears to be a great flat plain, upon which rests an immense hemispherical dome, the sky, that meets the Earth in the circular horizon. The observer, wherever he may be located, seems always to be at the center of this hemispherical surface, while all the celestial bodies appear to be upon it.

The impression that the celestial bodies are located upon a spherical surface is produced on the observer because his vision is unable to apprise him of the relative distances of these bodies, and consequently they appear to be all at the same distance. Only the *direction* of a celestial object can be immediately perceived; any two objects will appear to lie on a spherical surface, separated from each other by an arc on this surface equal to the angle between the two lines of sight from the observer to the objects. The body of the Earth cuts off from view all external objects in the directions below the horizon; and the part of the surface of the Earth which is within view is so small a portion of the entire surface that this visible part does not have enough curvature for its departure from a plane to be perceptible.

The seemingly spherical surface which appears to surround the Earth and to be centered at the observer is known as the *celestial sphere*. The aspect of the sky as directly observed is identically the same as it would be were the celestial bodies actually on a spherical surface centered at the Earth. Since it is much simpler and more convenient to deal with points and arcs on a sphere than with lines of sight and angles in space, it is the accepted practice to represent the immediately observed phenomena, as distinguished from their physical interpretations, *as if* the celestial sphere were physically real instead of only apparent. The systematic formal representation of the apparent positions and motions of the celestial bodies as seen projected on this sphere is the subject of *spherical astronomy*, as distinguished from *theoretical astronomy* in which the actual positions and motions in space and their physical explanation are considered. The determination of apparent positions on the sphere by observational measurements depends upon the principles of spherical astronomy, and is known as *astrometry*; the theory and use of the instruments that are employed is *practical astronomy*.

Apparent arcs on the sphere correspond to angles between lines of sight from the observer. Great circles are the intersections in which planes through the observer cut the sphere, and small circles correspond to cones with vertices at the observer.

The radius of the celestial sphere is indeterminate, but is immaterial since we are concerned only with arcs subtended on its surface by angles at its center. Since to all observers, no matter how different their locations, it always appears to each that he is at the center of the sphere, the appearances may be correctly represented by regarding the radius of the sphere as indefinitely great or *mathematically infinite*; an observer then always is at the center, no matter where he may be or how he may move about. All parallel lines and parallel planes through different positions of the observer in space will determine the same points and great circles on the celestial sphere; but as the observer alters his position among the celestial bodies, these bodies will be located in different directions from him, and will be seen in different relative positions on the sphere.

To represent the appearance of the heavens to an observer when the celestial sphere is conceived as finite, it is necessary to suppose that if the observer alters his position in space, the center of the sphere moves with him; lines and planes through the observer that move parallel to themselves continue to determine the same points and great circles on the sphere, precisely as with the infinite sphere.

Appearances as viewed by the observer are identical under the two conventions, but it is more convenient to regard the radius of the sphere as indefinitely great. In either case, however, the aspect of the celestial sphere may be diagrammatically represented on the surface of a finite sphere with any arbitrary radius by taking the center of this finite sphere as the position of the observer, and representing all parallel lines and parallel planes through different positions of the observer by a single line and single plane through the center of the sphere.

In this volume, all diagrams of constructions on the celestial sphere, unless explicitly stated otherwise, are represented as though viewed from *outside the sphere*. They are therefore analogous to the usual cartographical representations of the surface of the Earth.

#### Apparent Motions of the Celestial Bodies, and the Circles of the Sphere Which They Define

Because of the axial rotation of the Earth, the celestial sphere appears to revolve daily around the Earth, carrying all the celestial bodies around with it, in the direction opposite to the actual rotation of the Earth. The consequent daily paths of the celestial bodies across the sky appear to be arcs of concentric circles centered at two diametrically opposite points on the sphere, one above the horizon in general, and one below, which mark the ends of the axis of the apparent diurnal rotation of the sphere and which are called the *celestial poles.* The great circle of the celestial sphere midway between the poles is known as the *celestial equator.*

The poles are the points in which a line through the axis of the Earth intersects the celestial sphere; but in accordance with the conception of the sphere as mathematically infinite, these points coincide with those in which the sphere, as it appears to an observer, is intersected by *a line through his own position* which is *parallel* to the axis of the Earth. Since it appears to an observer that he is stationary and always at the center of the sphere, he interprets the relative diurnal motions of the celestial bodies as an actual rotation of a sphere which carries these bodies around in common, about an axis passing through his own position.

This illusion depends upon the great distances at which the celestial bodies are located. With few exceptions, these distances are so immensely large that the apparent motions are due almost entirely to the motion of the observer with the rotation of the Earth, and are very nearly alike for all the bodies; these bodies consequently maintain virtually the same relative configurations with one another on the sphere as they seemingly pass across the sky, as if they were fixed in position on the celestial sphere and were being carried around by a rotation of this sphere.

Actually, the diurnal motions relative to the observer are each around a center located on the axis of the Earth; the observer himself is also carried around this center, and his position among the celestial bodies consequently varies, but the slight additional motions which the accompanying variations in the directions of the lines of sight to exceptionally nearby objects superimpose upon a circular motion around a parallel axis are inappreciable without careful instrumental measurements. Further small irregularities from other causes also occur in the diurnal motions of all objects, but are likewise imperceptible to ordinary immediate observation.

The configurations therefore are not perceptibly changed by the diurnal motion; but a very few objects, notably the Moon, are readily perceived to be steadily changing their positions among the stars, due to actual motions of their own in space relative to the Earth. These objects consequently seem to be moving over the surface of the celestial sphere while being carried around in common with the stars by a rotation of the sphere. The stars are so distant that, even with highly refined instruments, long intervals of time are required to detect any effects of their own motions in space relative to the Earth; to ordinary perception they appear to remain permanently fixed on the celestial sphere, and in former times were usually called *the fixed stars,* but this term has now largely dropped out of use. The term *planet* etymologically signifies a wandering star, and was originally applied to all the objects that appear to move from place to place on the rotating celestial sphere.

The motions of the planetary bodies in space relative to the Earth, which cause their apparent motions on the rotating sphere among the stars, are partly due to the orbital motion of the Earth around the Sun; and in particular, because of this annual orbital revolution of the Earth, the Sun, while being carried around by the diurnal rotation of the celestial sphere, is simultaneously moving slowly eastward over the sphere, and completes an entire circuit of the celestial sphere each year. The annual path in which the Sun appears to be moving among the stars is called the *ecliptic*. In ancient times, this annual motion was considered to be due to an actual motion of the Sun around a stationary Earth; and as in the case of the diurnal motion, this point of view is often in practice formally retained for many purposes at the present time.

The ecliptic, the celestial equator, and the horizon are each an advantageous basis for a system of geometrical coordinates on the celestial sphere, by means of which the apparent positions of celestial objects as seen in the sky may be precisely represented in terms of their positions relative to these principal circles of the sphere.

### The Reference System of Position and Time

The horizon, dividing the visible from the invisible part of the celestial sphere, and the cardinal points of the horizon determined by the diurnal motion, constitute a natural reference system for the systematic description of the aspect of the sky. The apparent position of a celestial body at any particular time and place may be described in an exact quantitative manner by specifying (1) its angular height in the sky, vertically above the horizon, and (2) the particular point of the horizon above which it stands, relative to the cardinal directions north, east, south, and west. For this purpose, in place of the visible horizon with its local irregularities, a more precise reference line is used, known as the *astronomical horizon*, which is an exact great circle that is 90° from the point overhead toward which a plumb line is directed; this point vertically overhead is the *astronomical zenith*.

However, this alone is not adequate for a description of the phenomena that depend upon the apparent motions of the Sun, Moon, and planets over the surface of the rotating celestial sphere. For the purpose of fixing apparent positions on the sphere itself, independently of its varying aspect relative to the horizon during its diurnal rotation, a natural method is to use the celestial equator or the ecliptic as a fundamental reference circle in the same geometrical way as the horizon is used. Great circles perpendicular to the fundamental circle all pass through its poles, and constitute in each case a system of secondary reference circles. The position of any point on the sphere is represented by (1) its angular distance from the fundamental circle, measured along the secondary that passes through the point, and (2) the position of the intersection of this secondary with the fundamental circle, represented by its angular distance from an adopted cardinal point or origin on the fundamental circle (see Fig. 1). The point adopted as origin on both the equator and the ecliptic is the intersection of these two circles at which the Sun annually crosses the equator in March from south to north. This point has

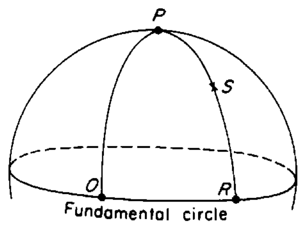


FIG. 1. Spherical coordinates: *P*, pole of the fundamental circle; *O*, origin; *OP*, secondary through *O*; *RP*, secondary through point *S* of the celestial sphere. Coordinates of *S*: *RS*, the angular distance from the fundamental circle, conventionally reckoned algebraically positive toward one pole, negative toward the other pole; and *OR*, the angular distance of *R* from *O* measured in a conventionally adopted direction from *O*.

traditionally been known as the vernal equinox or spring equinox, and the opposite intersection as the autumnal equinox, although these terms are inappropriate in the southern hemisphere of the Earth.

This procedure for expressing apparent position on the celestial sphere is essentially a specification of *direction from the observer* in terms of arcs on the sphere that measure apparent angular distances from a system of reference points and circles that correspond to selected identifiable *cardinal directions* from the observer.

Equally as essential for an exact description of the aspects of the heavens as a system of specifying position, is a method of measuring time. For this purpose, the apparent motions of the Sun, the Moon, and the stars have been the natural and accepted basis ever since the most remote historical ages. The day, marking the recurrence of light and darkness, corresponds to the diurnal circuit of the Sun; the month originally corresponded to the cycle of lunar phases during the circuit of the Moon around the celestial sphere relative to the Sun; and the year, marking the recurrence of the seasons, is determined by the annual circuit of the Sun among the stars. The passage of day and night as marked off by the progress of the celestial bodies in their diurnal circuits is indicated by a subdivision into hourly intervals, which have always been determined by observations of the positions of selected celestial bodies in their diurnal paths. Similarly, the progress of the year was determined in ancient times by observations of the position reached by the Sun among the stars in its annual circuit, as indicated by the constellations visible just above the horizon at dawn or twilight.

Modern methods of time measurement do not differ in principle from ancient practices, but have been put on a more exact basis and are facilitated by many precise mechanical aids. Since the activities of man are so largely regulated by daylight and darkness, the practical measure of time used in daily life is based upon the apparent diurnal motion of the Sun; but in the actual measurement and determination of time by astronomical observations, it is more convenient and accurate to use the apparent motions of the stars. Clocks are regulated by comparing their readings with the time determined from direct observations of the stars. Until the comparatively late period in history when clocks and watches came into general use, it was a common custom in civil life to estimate the approximate hour of the night from the positions of familiar star groups at different seasons of the year, and to determine the hour of the day either by a sundial or by estimation from the position of the Sun in the sky.

### The Practical Establishment of the Reference Systems

The formal geometric definitions of systems of coordinates based on the horizon, the equator, and the ecliptic, and of measures of time based on the apparent motions of the Sun and the stars do not depend upon any particular theoretical interpretation of the observed phenomena which determine the fundamental circles of the sphere. These reference systems are defined directly by the immediate appearances that suggest them. Wherever an observer be located, it appears to him that he is at the center of the celestial sphere; and the fundamental cardinal points and circles of the sphere are determined by the directly observed phenomena at his location. The zenith and the horizon are defined by the local plumb line and level; the poles and the equator, by the local aspect of the apparent diurnal motions of the celestial bodies; and the ecliptic and equinoxes by the apparent annual motion of the Sun. The coordinate systems were originally defined empirically on this basis in ancient times, when the prevailing point of view was geocentric and the apparent motions were considered to be the physically real motions. The later interpretation of the apparent motions as a reflection of motions of the Earth, which enables the reference systems to be dynamically defined, has not materially changed the formal geometric definitions.

Moreover, *direct observation of the apparent motions* still remains the only means of actually determining the locations of the reference circles on the celestial sphere; the primary basis of the reference systems is therefore still essentially empirical in practice. The abstractly defined geometric systems of points and circles, not being delineated in nature by visible points and lines on the celestial sphere, cannot be immediately identified, and observations cannot be directly referred to them. Only the celestial bodies themselves, and their apparent positions on the sphere *relative to one another* or *relative to visible reference marks* such as the natural horizon and the plumb line, can actually be directly perceived. Consequently, to realize the reference systems in practice, and to implement astronomical observations, means must be devised for identifying the cardinal points and circles of the sphere, and for measuring positions relative to them, entirely by observations of the apparent positions and motions of the celestial bodies relative to one another and relative to the concrete representations of the vertical and the horizontal which can be realized with the plumb line and the level.

The method employed to accomplish this is to use the stars as the immediate reference points for measurements of position. The stars, because of the near immutability of their positions relative to one another, constitute a system of visibly marked points on the celestial sphere by means of which the apparent motions that empirically define the fundamental circles may be observed. By their apparent diurnal motions, the stars show the diurnal rotation of the celestial sphere, determining the celestial poles and equator; and they provide reference points on the rotating sphere for determining the apparent motions of the Sun and other moving objects on the sphere. By selecting a system of stars distributed over the celestial sphere as standard reference points, the positions of the fundamental circles among these stars may be found by tracing the apparent motion of the Sun relative to them, and observing the diurnal motions. Systematic observations of the Sun and stars, and in practice some of the planets, extending over a long period of time, are required to determine the positions of the standard stars relative to one another, and the locations of the circles among them, from which their coordinates may be found. To determine the coordinates of further stars or of other celestial objects, the quantities that are measured in actual observation are angular distances from reference stars, which constitute the intermediary means of referring the observed position to the coordinate circles.

A network of selected stars has been the immediate reference system ever since very ancient times. The background of the stars was a natural basis of reference for the apparent motions of the Sun, Moon, and planets in the earliest ages of astronomical observation before the abstract geometric concepts of the equator and ecliptic had been explicitly formulated. The brighter individual stars near which the Moon and the planets passed came to be systematically used as reference points by the Babylonian observers; positions on the celestial sphere were represented by angular distances from these standard stars. This practice is still an indispensable means of determining positions by observation; but instead of a limitation to the explicit use of individual stars separately as independent reference points, the standard stars are further used collectively, as a connected system, to define the geometric coordinate systems. When a high order of accuracy is not required, the coordinate systems can be established, and the principal astronomical measurements in them carried on by very simple means. Successful methods for the purpose were devised early in ancient times. The refined and complex instruments of modern times, the elaborate procedures of observation, and the intricate theories and mathematical calculations upon which reductions and discussions of the observations are based, are necessary only to attain the extreme precision that is now required; the foundations of astronomy were completely established before they became available.

The next chapter is devoted to the formal geometric definitions of the coordinate systems on the empirical basis of immediate appearances. The representation of the aspects of the celestial sphere at any geographic location at any time, and of the apparent positions and motions of the celestial bodies, are the subjects of Chapters 3 to 7, inclusive. The dynamical theory of the coordinate systems, and its applications in methods of referring apparent positions to these systems, constitute Chapters 8 to 13. Systems of time measurement and their dynamical basis are treated in Chapters 14 to 16. Finally, in Chapters 17, 18, and 19, the practical techniques in use to establish the reference systems of position and time are described.

### Mathematical Methods in Spherical Astronomy

The theoretical principles of spherical astronomy and their practical applications in astrometry depend fundamentally upon the abstract geometry of a spherical surface, since immediately observed phenomena are represented in terms of relations among arcs and angles on the celestial sphere. Direct astrometric observations consist almost entirely of measuring angles between lines of sight, these angles representing the positions of celestial objects relative either to one another or to observable reference points on the sphere. The time of observation, which also is usually needed, is likewise characterized by an angle representing the position of some particular celestial body in its diurnal circle at the instant of observation; in practice, the time is found by means of a clock, but clocks must be checked at frequent intervals by direct determinations of time from observations of celestial bodies. The principal mathematical aid that is required in spherical astronomy and astrometry is therefore spherical trigonometry.

Three different kinds of angular measures must be carefully distinguished. The angular distance between two points on the sphere is the arc subtended by the plane angle between the lines of sight from the observer to these points, and is therefore an arc of the great circle through the two points. This great circle is unique, and unless otherwise specified angular distances on the celestial sphere are always measured along great circles; but occasionally the length of a small circle arc joining two points is needed. This second kind of angular measure is an arc of a circle lying in some plane that does not pass through the center of the sphere; an indefinite number of different small circles can be passed through the same two points. The third kind of angular measure is the angle of intersection of two great circles, which is the dihedral angle between the planes of the two circles; both planes pass through the observer.

The figure formed on the surface of a sphere by any three great circles that do not all intersect in the same point is a spherical triangle. In general, three great circles divide the sphere into eight areas, each of which is a spherical triangle; however, when any three parts of a triangle are given, in general they define without ambiguity a single one of these areas. Exceptions occur when some of the parts exceed 180°, or when, as in plane geometry, two sides and the angle opposite one of them are given. Under these conditions, two solutions frequently exist; but the case where three angles all less than 180° are given, which in plane geometry is indeterminate, is readily soluble on the sphere.

The principal problem of spherical trigonometry is: given any three parts of a spherical triangle, find one or more of the other three parts. On the celestial sphere the sides and angles of the triangles may be of any magnitude up to 360°. The fundamental general relations among the parts are valid for triangles with sides of any length, and may therefore be applied immediately to the triangles of spherical astronomy; but it is necessary to determine at least the algebraic signs of both the sine and the cosine of each arc or angle that may exceed 180° in order to fix the quadrant. The general spherical triangle is always determinate when, in addition to the three given parts, the algebraic sign of the sine or the cosine of one of the required parts is also given; otherwise it always admits of two solutions. In most problems of spherical astronomy, the conditions of the problem supply the information necessary to make the solution determinate.

For convenience of reference, the fundamental relations among the angles $A$, $B$, $C$ of a spherical triangle, and the sides $a$, $b$, $c$, respectively opposite these angles, are given here without proof:

$$\begin{aligned} \sin a \sin B &= \sin b \sin A, \\ \sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A, \\ \cos a &= \cos b \cos c + \sin b \sin c \cos A; \end{aligned} \tag{1}$$

$$\begin{aligned} \sin B \sin a &= \sin A \sin b, \\ \sin B \cos a &= \cos A \sin C + \sin A \cos C \cos b, \\ \cos B &= \sin A \sin C \cos b - \cos A \cos C; \end{aligned} \tag{2}$$

$$\sin A \cot C = \sin b \cot c - \cos b \cos A. \tag{3}$$

In principle, these formulas collectively are sufficient for obtaining the complete solution of any spherical triangle without restriction on the magnitudes of the parts. However, among the many additional relations that may be derived from them are some which frequently are more advantageous in practical numerical calculations.

In particular, when the given parts are the three sides, the third of Eqs. (1) may be used to find the three angles, but more satisfactory results will be obtained by using

$$s = \tfrac{1}{2}(a + b + c);$$

$$\tan^2 \tfrac{1}{2}A = \frac{\sin(s - b)\sin(s - c)}{\sin s \sin(s - a)}. \tag{4}$$

Likewise, instead of using the third of Eqs. (2) when the three angles are given, the three sides should be found from

$$S = \tfrac{1}{2}(A + B + C),$$

$$\tan^2 \tfrac{1}{2}a = \frac{-\cos S \cos(S - A)}{\cos(S - B)\cos(S - C)}. \tag{5}$$

Similarly, when two sides and an angle opposite one, or two angles and a side opposite one, are the given parts, Eq. (3) may be used with the first and third of Eqs. (1) or (2); but it is much more expeditious to use the first of Eqs. (1) or (2) with Napier's analogies:

$$\tan \tfrac{1}{2}c = \frac{\sin \tfrac{1}{2}(A + B)}{\sin \tfrac{1}{2}(A - B)} \tan \tfrac{1}{2}(a - b),$$

$$\tan \tfrac{1}{2}c = \frac{\cos \tfrac{1}{2}(A + B)}{\cos \tfrac{1}{2}(A - B)} \tan \tfrac{1}{2}(a + b); \tag{6}$$

$$\cot \tfrac{1}{2}C = \frac{\sin \tfrac{1}{2}(a + b)}{\sin \tfrac{1}{2}(a - b)} \tan \tfrac{1}{2}(A - B),$$

$$\cot \tfrac{1}{2}C = \frac{\cos \tfrac{1}{2}(a + b)}{\cos \tfrac{1}{2}(a - b)} \tan \tfrac{1}{2}(A + B). \tag{7}$$

A small angle is more accurately found from its sine than from its cosine; for an angle near 90° the reverse is true. However, an angle is always more accurately determined by its tangent or cotangent than by any other function.

When the vertices of a spherical triangle are the poles of the great circle arcs that form the sides of another triangle, each triangle is called the polar triangle of the other. Given a triangle with sides $a$, $b$, $c$ and angles $A$, $B$, $C$, the sides of the polar triangle are $180° - A$, $180° - B$, $180° - C$, and the angles are $180° - a$, $180° - b$, $180° - c$.

Angular magnitudes are ordinarily expressed either in sexagesimal arc measure, in units of the degree and its subdivisions, or else in circular measure in units of the radian. In some writings the centesimal degree is encountered; it is $\frac{1}{100}$ of a right angle, and is subdivided into 100 minutes, each minute into 100 seconds. In calculations for some special purposes, still other units are used; e.g., the mil, which is $\frac{1}{1600}$ of a right angle, or exactly 0°.05625.

In formulas which are derived by the operations of the calculus, or which involve products of angular magnitudes, it is necessary either to express the angles in radians or to introduce the proper conversion factors; the product of two or more angles is obtained in arc measure by expressing all the factors *except one* in radians before the multiplication. The required conversion factors are obtained from the rigorous relation $360° = 2\pi$ radians:

$$\begin{aligned} 1'' &= 0.00000\,48481\,36811\,09536 \quad \text{radian}, \\ 1' &= 0.00029\,08882\,08666, \\ 1° &= 0.01745\,32925. \end{aligned}$$

Areas on the sphere may be expressed in square degrees or in square radians. A pyramid with a square section, having its vertex at the center of the sphere and edges which intersect in face angles of one degree, intercepts an area on the sphere bounded by the four 1° arcs in which the faces intersect the surface; this area, or any equivalent area regardless of shape, is one square degree. The area intercepted on a sphere of unit radius by a cone of any cross section with vertex at the center of the sphere, or the area intercepted on a sphere of any radius $R$ divided by $R^2$, is called a *solid angle*; the unit of solid angle is the square radian, known as a *steradian*. The area of a sphere is $4\pi R^2$ or $4\pi$ steradians. Since the circumference of a great circle is $2\pi R = 360°$, the area of the entire sphere is

$$4\pi\left(\frac{360}{2\pi}\right)^2 = \frac{129600}{\pi} \quad \text{square degrees,}$$

or very closely 41253 square degrees. A steradian is $32400/\pi^2$ square degrees.

In the practical calculations of spherical astronomy, rigorous formulas frequently are not advantageous, and sometimes are not even practicable; results to the required accuracy can often be obtained much more easily and expeditiously with approximate formulas. A common method of obtaining approximations that are advantageous for numerical calculation is the expansion of rigorous formulas in series, especially by Taylor's theorem,

$$f(x_0 + \Delta x) = f(x_0) + \left(\frac{df}{dx}\right)_0 \Delta x + \frac{1}{2!}\left(\frac{d^2f}{dx^2}\right)_0 (\Delta x)^2 + \ldots .$$

In many calculations, only the first two terms on the right need be retained to give the required accuracy. When $\Delta x$ is a small angle, the value of the factor $(\Delta x)^2/2!$ in the third term may be obtained in arc measure by the above-mentioned rule for multiplying angular magnitudes; e.g., if $\Delta x = 5''$,

$$(\Delta x)^2/2! = 5''(5 \text{ arc } 1'')/2$$
$$= 0''.00006.$$

From the Taylor series for the sine, tangent, and cosine, it is evident that when an angle is very small its sine and tangent are nearly equal, and both are numerically equal to the radian measure of the angle to a high order of approximation, while the cosine is very nearly unity:

| | |
|---|---|
| 1″ | 0.00000 48481 36811 10 |
| sin 1″ | 0.00000 48481 36811 08 |
| tan 1″ | 0.00000 48481 36811 13 |
| 1′ | 0.00029 08882 1 |
| sin 1′ | 0.00029 08882 0 |
| tan 1′ | 0.00029 08882 2 |
| 5′ | 0.00145 44410 |
| sin 5′ | 0.00145 44405 |
| tan 5′ | 0.00145 44421 |
| 1° | 0.01745 33 |
| sin 1° | 0.01745 24 |
| tan 1° | 0.01745 51 |
| cos 1″ | 0.99999 99999 88 |
| cos 1′05″ | 0.99999 9950 |
| cos 10′50″ | 0.99999 50. |

By means of these approximations many calculations may be greatly simplified without appreciable loss of accuracy.

When all three sides of a spherical triangle are so small that their cubes may be neglected, the triangle may be treated as plane, since the fundamental relations (1) reduce to the relations for plane triangles when the approximations $\sin a = a$ and $\cos a = 1 - \frac{1}{2}a^2$, etc., are substituted, neglecting quantities of the third and higher orders.

When one angle $C$ and the opposite side $c$ of a spherical triangle are so small that their squares may be neglected, the angle $A$ and the exterior angle $180° - B$ are nearly equal. From the second of Eqs. (2), with the appropriate cyclical interchange of letters,

$$C \cos b = \sin(A + B)$$
$$= \sin[180° - (A + B)],$$

or

$$180° - B = A + C \cos b.$$

If $a = b$, the angles $A$ and $B$ are very nearly 90°, and from the first of Eqs. (1)

$$\begin{aligned} c &= C \sin a \\ &= C \sin b; \end{aligned}$$

this is the great circle distance between the vertices $A$ and $B$.

To the first order, the change produced in any part of a spherical triangle by small changes in the three other parts upon which it depends may be found from the differential formulas obtained by differentiating the fundamental relations among the sides and angles:

$$\begin{aligned} \sin C\, da &= \sin b\, dA + \cos C \sin a\, dB + \sin A \cos b\, dc, \\ \cot a\, da &= \cot A\, dA - \cot B\, dB + \cot b\, db, \\ da &= \cos C\, db + \cos B\, dc + \sin C \sin b\, dA, \\ dA &= \sin C \sin b\, da - \cos c\, dB - \cos b\, dC. \end{aligned} \tag{8}$$