

CHAPTER 4

Parallax

AT different points of observation, the directly observed positions of a celestial body are located at different points on the sphere, since the directions to the body, and consequently its geometric projections on the sphere, are different. The difference between the geometric directions to a celestial body from two points in space, measured by the angle at the body between the straight lines from these two points, is denoted in general by the term *parallax*; this angle is equal to the arc on the celestial sphere between the geometric projections from the two points.

In accordance with the conception of the celestial sphere as mathematically infinite, a displacement of the observer in space does not affect the reference circles on the sphere; the difference between the aspects of the circles of the sphere at different geographic locations is due, not to the difference in the position of the observer in space, but to the different directions of the vertical at different points on the curved surface of the Earth and the consequent displacement of the horizon system of circles over the celestial sphere. Because of parallax, therefore, the directly observed positions of the celestial bodies relative to the reference circles depend upon the point of observation. The equatorial or ecliptic coordinates of a body that is not too distant from the Earth are consequently appreciably different at geographic localities that are widely enough separated; and they also vary at any fixed geographic location during the course of the diurnal circuit of the observer around the axis of rotation of the Earth. Likewise, the horizon coordinates, in addition to being different because of the different directions of the vertical, are further affected by the parallax due directly to the difference of position in space.

In actual usage, the term *parallax* is usually restricted to cases in which the distance between the two points of observation is small compared with the distance from either of the points to the observed object. The difference between the positions in which the body would be seen on the celestial sphere from the two points simultaneously is then small and may be represented by small variations of the coordinates. In practice, *parallax* usually signifies the difference of position as seen from an adopted standard point of reference and a particular point of actual observation. The most

commonly used standard point for objects in the solar system is the center of the Earth; and for the stars, the center of the Sun. When the standard point is the center of the Earth, the angular distance between the geometric projections of the body on the celestial sphere is called the *geocentric parallax* or *diurnal parallax*; when the standard point is the center of the Sun, the parallax is known as the *heliocentric* or *annual parallax*.

The term is not ordinarily used when the difference is so great that it represents essentially an actual transformation from one coordinate system to another, as, e.g., in the case of the heliocentric place of a planet compared with its geocentric place. However, it is sometimes extended to the difference between the directions in which the Earth and the Sun would be seen from a planet, which is called the *annual parallax* of the planet in analogy with the annual parallax of a star.

Geocentric Parallax

The geometric directions to a celestial body from an observer on the surface of the Earth, and from the center of the Earth, lie in a plane that passes through the geocentric zenith. The angle p between these directions, which is the geocentric parallax, is therefore the difference between the observed angular distance from the geocentric zenith z'_g and the geocentric distance z_g from the geocentric zenith,

$$p = z'_g - z_g, \quad (33)$$

where $z'_g \geq z_g$ (see Fig. 10). The angle p is the angle subtended at the body by the radius ρ from the center of the Earth to the observer. The geocentric parallax vanishes at the geocentric zenith, and is a maximum at the geocentric horizon where $z'_g = 90^\circ$.

The displacement p on the celestial sphere by geocentric parallax is entirely a displacement in *geocentric zenith distance*, and therefore *very nearly* in a vertical circle; in accordance with Eq. (33), it depresses a celestial body toward the horizon. However, since the geocentric zenith does not coincide with the geodetic zenith, the topocentric position will in general be displaced both in azimuth and altitude from the geocentric position. Were the Earth spherical, there would be no parallax at all in azimuth, excepting effects of deflections of the vertical; the actual parallax in azimuth due to the flattening of the Earth is at most very slight and vanishes in the meridian.

To determine exact values of the effects of geocentric parallax, the geographic location of the observer must be expressed in terms of geodetic, not astronomical, coordinates since the parallax is determined by the geometric position of the observer relative to the center of the Earth.

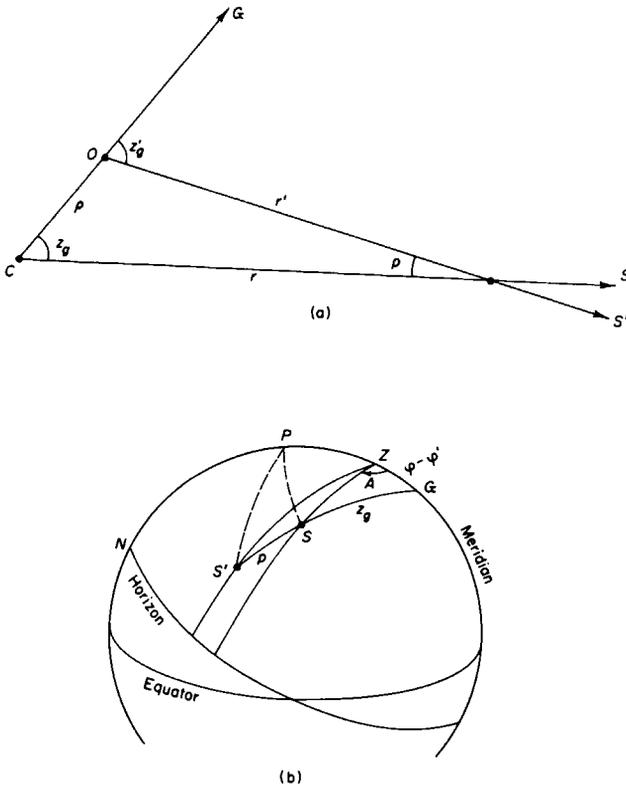


FIG. 10. Geocentric parallax. (a) C , center of Earth; O , observer; G , geocentric zenith; S , geocentric direction; and S' , topocentric direction. (b) Z , geodetic zenith; P , celestial pole; and G , geocentric zenith.

Therefore, when high accuracy is required and the parallax is large, as, e.g., in precise parallax reductions for the Moon, astronomical coordinates should be corrected for local deflections of the vertical, and the elevation above the spheroid should be included in the geodetic coordinates.

From the law of sines for plane triangles,

$$\sin p = \frac{\rho}{r} \sin z'_g,$$

where r is the geocentric distance of the body. The value of p at $z'_g = 90^\circ$ is called the *horizontal parallax*; denoting it by H , we have $\sin H = \rho/r$, and

$$\sin p = \sin H \sin z'_g.$$

From this equation and Eq. (33), $\sin p = \sin H \sin(p + z_g)$; expanding and dividing by $\cos p$ gives

$$\tan p = \frac{\sin H \sin z_g}{1 - \sin H \cos z_g}$$

with the solution

$$p = (H - \frac{1}{6}H^3) \sin z_g + \frac{1}{2}H^2 \sin 2z_g + \frac{1}{3}H^3 \sin 3z_g + \dots$$

The value of H when ρ is the equatorial radius a of the standard spheroid, i.e., at geocentric latitude $\varphi' = 0^\circ$, is the *equatorial horizontal parallax* H_0 ; its value is given by $\sin H_0 = a/r$. The equatorial horizontal parallax of a body at its mean geocentric distance r_0 is called the mean equatorial horizontal parallax, π :

$$\sin \pi = a/r_0,$$

$$\pi = \sin \pi + \frac{1}{3} \sin^3 \pi + \dots,$$

$$\sin p = \frac{\rho}{a} \frac{r_0}{r} \sin \pi \sin z'_g.$$

Were the Earth spherical, the horizontal parallax would be the angular semidiameter that the Earth would appear to have if viewed from the body, since the direction to the body from any point on the Earth where $p = H$ would then be tangent to the surface of the Earth; but because of the spheroidal form of the Earth, a tangent to the surface does not in general coincide with the direction $z'_g = 90^\circ$ at the point of tangency, and the difference depends both on the latitude of the point and on the azimuth of the tangent. Hence the angle subtended by ρ at an external point when $z'_g = 90^\circ$ is not strictly the same as the angle subtended by the radius of the Earth at this external point.

However, the equatorial horizontal parallax is the apparent equatorial semidiameter of the elliptical disk of the Earth that would be observed from a body which is at declination 0° when this body is on the geodetic horizon, at the east or west point, of an observer at the geodetic equator; the geocentric zenith there coincides with the geodetic zenith, and the two tangents from the body to the extremities of an equatorial diameter of the disk of the Earth are perpendicular to the radii of the circular equator and are directed east and west. The only diametral section of the Earth that is circular is the equatorial section.

The geocentric parallax is in general so small that rarely are rigorous formulas required, but they are sometimes needed for the Moon. The equatorial horizontal parallax of the Moon averages about 1° , and in precise calculations the exact trigonometrical formulas must be used. The Moon is

so near that the elevation of the observer above sea level must be taken into account; e.g., the altitude of the Lick Observatory subtends an angle of $0''.7$ at the Moon.

Ordinarily it is sufficiently accurate to put $p = \sin p$, and neglect the parallax in azimuth. Then $z'_g = z' - (\varphi - \varphi') \cos A$, where z' is the topocentric zenith distance from the geodetic zenith, and A is the angle in azimuth measured westward from the meridian; disregarding deflections of the vertical, z' is the observed zenith distance from the astronomical zenith. To this approximation, the *parallax in altitude*, in terms of the equatorial horizontal parallax H_0 , expressed in seconds of arc, is

$$p'' = \frac{\rho}{a} H_0'' \sin[z' - (\varphi - \varphi') \cos A] \quad (34)$$

to be added to the observed altitude in order to obtain the geocentric altitude above the astronomical horizon. Since $H/H_0 = \rho/a$, and

$$\rho/a = 1 - f \sin^2 \varphi + \frac{5}{8} f^2 \sin^2 2\varphi + \dots,$$

the horizontal parallax at any latitude is

$$H = H_0(1 - f \sin^2 \varphi + \frac{5}{8} f^2 \sin^2 2\varphi + \dots) \quad (35)$$

where f is the flattening of the Earth. Neglecting the flattening in addition to the other approximations gives $p'' = H'' \sin z'$.

The horizontal parallax H at latitude φ is sometimes called the *reduced parallax* for this latitude. The difference $H_0 - H = [1 - (\rho/a)]H_0$ is known as the reduction of the equatorial parallax, or sometimes as the augmentation of the horizontal parallax, for the latitude; it is the correction required to H for the flattening of the Earth in order to obtain the equatorial horizontal parallax H_0 . From Eq. (35),

$$H_0 - H = H_0[\frac{1}{2}f(1 - \cos 2\varphi) - \frac{5}{16}f^2(1 - \cos 4\varphi) + \dots].$$

For the Moon, this reduction is of the order of $+6''(1 - \cos 2\varphi)$, ranging from $0''$ at the equator to about $6''$ at latitude 45° and $12''$ at the poles.

Parallax in Azimuth and Altitude

Rigorous formulas for the components of the parallactic displacement (33) in both altitude and azimuth may be derived from the coordinates of the body in the topocentric and geocentric polar coordinate systems which have the plane of the horizon as their fundamental planes.

The plane of the horizon at a point on the surface of the Earth, the plane of the meridian, and the plane of the prime vertical form a rectangular topocentric coordinate system. With the plane of the horizon as the XY -plane,

the positive X -axis directed toward the south, and the Y -axis toward the west, and the positive Z -axis directed toward the zenith, the topocentric coordinates of a celestial body are

$$x' = r' \sin z' \cos A', \quad y' = r' \sin z' \sin A', \quad z' = r' \cos z',$$

in which r' is the distance from the observer, z' the observed geometric zenith distance, and A' the azimuth reckoned from *south toward west* up to 360° . Planes through the center of the Earth parallel to the coordinate planes of this topocentric system likewise intersect the celestial sphere in the observer's horizon, meridian, and prime vertical; in this geocentric rectangular system, the coordinates of the body are

$$x = r \sin z \cos A, \quad y = r \sin z \sin A, \quad z = r \cos z,$$

where r is the geocentric distance, z the geocentric zenith distance from the *geodetic* zenith (not the geocentric zenith), and A the geocentric azimuth from south toward west.

The observer is at the origin of the topocentric system, and his coordinates in the geocentric system are $\rho \sin(\varphi - \varphi')$, 0 , $\rho \cos(\varphi - \varphi')$. Therefore, the rigorous relations between the topocentric and the geocentric coordinates in the horizon system are

$$\begin{aligned} r' \sin z' \cos A' &= r \sin z \cos A - \rho \sin(\varphi - \varphi'), \\ r' \sin z' \sin A' &= r \sin z \sin A, \\ r' \cos z' &= r \cos z - \rho \cos(\varphi - \varphi'). \end{aligned}$$

To express these equations in terms of the differences between the geocentric and the topocentric coordinates, multiply the first by $\sin A$, the second by $\cos A$, and subtract, then multiply the first by $\cos A$, the second by $\sin A$, and add, obtaining for the determination of $A' - A$ from A and r the equations

$$\begin{aligned} r' \sin z' \sin(A' - A) &= \rho \sin(\varphi - \varphi') \sin A, \\ r' \sin z' \cos(A' - A) &= r \sin z - \rho \sin(\varphi - \varphi') \cos A. \end{aligned}$$

Multiplying the first of these equations by $\sin \frac{1}{2}(A' - A)$, the second by $\cos \frac{1}{2}(A' - A)$, adding, and introducing the auxiliary

$$\tan \gamma' = \tan(\varphi - \varphi') \frac{\cos \frac{1}{2}(A' + A)}{\cos \frac{1}{2}(A' - A)},$$

gives

$$\begin{aligned} r' \sin z' &= r \sin z - \rho \cos(\varphi - \varphi') \tan \gamma', \\ r' \cos z' &= r \cos z - \rho \cos(\varphi - \varphi'), \end{aligned}$$

from which, for determining $z' - z$ from z and r ,

$$r' \sin(z' - z) = \rho \cos(\varphi - \varphi') \frac{\sin(z - \gamma')}{\cos \gamma'},$$

$$r' \cos(z' - z) = r - \rho \cos(\varphi - \varphi') \frac{\cos(z - \gamma')}{\cos \gamma'}.$$

Obvious modifications of the procedure by which these equations have been derived give, for determining $A' - A$ and $z' - z$ from A' , z' , r , the formulas

$$\sin(z' - z) = \frac{\rho \cos(\varphi - \varphi') \sin(z' - \gamma')}{r \cos \gamma'},$$

$$\sin(A' - A) = \frac{\rho \sin(\varphi - \varphi') \sin A'}{r \sin z},$$

in which the approximate value $\gamma' = (\varphi - \varphi') \cos A'$ may be used to obtain a first approximation to $z' - z$.

Multiplying the first of the equations for $z' - z$ in terms of z and r by $\sin \frac{1}{2}(z' - z)$, the second by $\cos \frac{1}{2}(z' - z)$, and adding, gives for determining r' ,

$$r' = r - \frac{\rho \cos(\varphi - \varphi') \cos\{\frac{1}{2}(z' + z) - \gamma'\}}{\cos \gamma' \cos \frac{1}{2}(z' - z)}$$

or, in another form, by multiplying the equation for $r' \sin z'$ by $\cos \gamma'$, the equation for $r' \cos z'$ by $\sin \gamma'$, and subtracting,

$$r' = r \frac{\sin(z - \gamma')}{\sin(z' - \gamma')}. \quad (36)$$

Also, by squaring and adding the two equations for $z' - z$,

$$(r'/r)^2 = 1 - 2n \cos(z - \gamma') + n^2,$$

where

$$n = \frac{\rho \cos(\varphi - \varphi')}{r \cos \gamma'}.$$

With the equatorial radius a of the Earth as the unit of distance, the substitution of $1/r = \sin H_0$ in these equations gives

$$\tan(A' - A) = \frac{m \sin A}{1 - m \cos A}, \quad \tan(z' - z) = \frac{n \sin(z - \gamma')}{1 - n \cos(z - \gamma')}, \quad (37)$$

in which

$$m = \frac{\rho \sin H_0 \sin(\varphi - \varphi')}{\sin z}, \quad n = \frac{\rho \sin H_0 \cos(\varphi - \varphi')}{\cos \gamma'};$$

A', z' denote the topocentric values, and A, A' are reckoned from *south through west*.

Developed in series, these expressions become

$$\gamma' = (\varphi - \varphi') \cos A - \frac{(\varphi - \varphi')^2 \rho \sin H_0 \sin^2 A}{2 \sin z} \sin 1'' + \dots,$$

$$A' - A = (1/\sin 1'')m \sin A + \frac{1}{2}m^2 \sin 2A + \frac{1}{3}m^3 \sin 3A + \dots,$$

$$z' - z = (1/\sin 1'')n \sin(z - \gamma') + \frac{1}{2}n^2 \sin 2(z - \gamma')$$

$$+ \frac{1}{3}n^3 \sin 3(z - \gamma') + \dots,$$

and therefore to the first order

$$\gamma' = (\varphi - \varphi') \cos A,$$

$$A' - A = \rho H_0 \sin(\varphi - \varphi') \sin A \operatorname{cosec} z,$$

$$z' - z = \rho H_0 \sin(z - \gamma').$$

Similarly, in terms of A' and z' ,

$$A' - A = \rho H_0 \sin(\varphi - \varphi') \sin A' \operatorname{cosec} z',$$

$$z' - z = \rho H_0 \sin(z' - \gamma');$$

cf. Eq. (34).

Parallax in Right Ascension and Declination

The effects of geocentric parallax on the equatorial coordinates may be obtained from the relations between geocentric and topocentric coordinates in the equator system by the same procedure as used for the horizon system. The rigorous equations that connect the geocentric coordinates (r, α, δ) with the topocentric coordinates (r', α', δ') referred to an origin at a point (ρ, φ') on the surface of the Earth are

$$r' \cos \delta' \cos \alpha' = r \cos \delta \cos \alpha - \rho \cos \varphi' \cos \tau,$$

$$r' \cos \delta' \sin \alpha' = r \cos \delta \sin \alpha - \rho \cos \varphi' \sin \tau, \tag{38}$$

$$r' \sin \delta' = r \sin \delta - \rho \sin \varphi',$$

in which τ is the local sidereal time. Putting $r'/r = f$ and $1/r = \sin H_0$, i.e., taking the equatorial radius a of the Earth as the unit of distance, we have from the first two equations

$$f \cos \delta' \sin(\alpha' - \alpha) = \rho \sin H_0 \cos \varphi' \sin(\alpha - \tau),$$

$$f \cos \delta' \cos(\alpha' - \alpha) = \cos \delta - \rho \sin H_0 \cos \varphi' \cos(\alpha - \tau),$$

which give

$$\begin{aligned}\tan(\alpha - \alpha') &= \frac{\rho \cos \varphi' \sin H_0 \sin h}{\cos \delta - \rho \cos \varphi' \sin H_0 \cos h} \\ &= \frac{(\rho \cos \varphi' \sin H_0 \sec \delta) \sin h}{1 - (\rho \cos \varphi' \sin H_0 \sec \delta) \cos h},\end{aligned}\quad (39)$$

where $h = \tau - \alpha$ is the local hour angle; and with

$$\tan \gamma = \frac{\tan \varphi' \cos \frac{1}{2}(\alpha - \alpha')}{\cos[h + \frac{1}{2}(\alpha - \alpha')]}$$

we have

$$\begin{aligned}f \sin \delta' &= \sin \delta - \rho \sin H_0 \sin \varphi', \\ f \cos \delta' &= \cos \delta - \rho \sin H_0 \sin \varphi' \cot \gamma,\end{aligned}$$

whence

$$\tan(\delta - \delta') = \frac{(\rho \sin H_0 \sin \varphi' \operatorname{cosec} \gamma) \sin(\gamma - \delta)}{1 - (\rho \sin H_0 \sin \varphi' \operatorname{cosec} \gamma) \cos(\gamma - \delta)},\quad (40)$$

and

$$\frac{r'}{r} = \frac{\sin(\delta - \delta')}{\sin(\delta' - \gamma)}.\quad (41)$$

Equations (39), (40), and (41) give the topocentric coordinates rigorously when the geocentric coordinates r , α , δ are known. When the observed coordinates α' , δ' are to be reduced to the geocentric values α , δ it is better to use the forms

$$\begin{aligned}\sin(\alpha - \alpha') &= \rho \sin H_0 \cos \varphi' \sec \delta \sin(\tau - \alpha'), \\ \tan \gamma' &= \tan \varphi' \cos \frac{1}{2}(\alpha - \alpha') \sec[h + \frac{1}{2}(\alpha - \alpha')], \\ \sin(\delta - \delta') &= \rho \sin H_0 \sin \varphi' \operatorname{cosec} \gamma' \sin(\gamma' - \delta'),\end{aligned}\quad (42)$$

but successive approximations are necessary in practice, because $\alpha - \alpha'$ must first be computed by using δ' in place of δ .

Equations (39) and (40) may also be obtained directly from the triangles formed on the celestial sphere by the geocentric zenith, the celestial pole, and the geocentric and topocentric positions of the body. These rigorous expressions have been put in various other forms to facilitate numerical calculations, and many different approximations have been used in practice. They may be developed in series by putting, for brevity,

$$\psi = \frac{\rho \cos \varphi' \sin H_0}{\cos \delta}, \quad \chi = \frac{\rho \sin \varphi' \sin H_0}{\sin \gamma},$$

giving

$$\tan(\alpha - \alpha') = \frac{\psi \sin h}{1 - \psi \cos h}, \quad \tan(\delta - \delta') = \frac{\chi \sin(\gamma - \delta)}{1 - \chi \cos(\gamma - \delta)},$$

which are standard trigonometric equations with the solutions

$$\alpha - \alpha' = \psi \sin h + \frac{1}{2}\psi^2 \sin 2h + \frac{1}{3}\psi^3 \sin 3h + \dots,$$

$$\delta - \delta' = \chi \sin(\gamma - \delta) + \frac{1}{2}\chi^2 \sin 2(\gamma - \delta) + \frac{1}{3}\chi^3 \sin 3(\gamma - \delta) + \dots.$$

At upper transit, h and $\alpha - \alpha'$ vanish, whence $\gamma = \gamma' = \varphi'$, and Eqs. (40) and (42) become

$$\tan(\delta - \delta') = \frac{\rho \sin H_0 \sin(\varphi' - \delta)}{1 - \sin H_0 \cos(\varphi' - \delta)},$$

$$\sin(\delta - \delta') = \rho \sin H_0 \sin(\varphi' - \delta').$$

Equations (38) may be written in the form

$$x' - x = -\rho \cos \varphi' \cos \tau,$$

$$y' - y = -\rho \cos \varphi' \sin \tau,$$

$$z' - z = -\rho \sin \varphi';$$

ordinarily the left members are so small compared with r that the expressions on the right may be substituted for dx, dy, dz in the differential formulas (18) for $d\alpha, d\delta, dr$, and we then obtain

$$\alpha - \alpha' = \frac{\rho \cos \varphi' \sin(\tau - \alpha)}{r \cos \delta},$$

$$\delta - \delta' = \frac{\rho}{r} \{-\sin \delta \cos \varphi' \cos(\tau - \alpha) + \cos \delta \sin \varphi'\}, \quad (43)$$

$$r - r' = \rho \{\cos \delta \cos \varphi' \cos(\tau - \alpha) + \sin \delta \sin \varphi'\},$$

which are readily seen to be approximations to the rigorous formulas. In Eqs. (43), ρ and r must be expressed in the same unit of length. In practice ρ is commonly measured in units of the equatorial radius a of the Earth, and r in astronomical units (au); we must then replace ρ by $\rho \pi_{\odot} \sin 1''$, where π_{\odot} is the constant of solar parallax in seconds of arc, defined by the relation

$$1 \text{ au} = a/\sin \pi_{\odot}.$$

From this relation

$$\rho \text{ equatorial radii} = \rho \pi_{\odot} \sin 1'' \text{ au}.$$

Equivalently, the factor $\pi_{\odot} \sin 1''$ may be regarded as converting $1/r$ from astronomical units to equatorial radii, since

$$r \text{ au} = r/\pi_{\odot} \sin 1'' \text{ equatorial radii}.$$

The values of $\alpha - \alpha'$ and $\delta - \delta'$ have been expressed as series in powers of the approximate values (43) by A. D. Maxwell.*

* A. D. Maxwell, *Astr. Jour.* **42**, No. 973 (1932).

For practical application, these equations may be written in more convenient form. For example, putting

$$\begin{aligned}\rho\pi_{\odot} \sin 1'' \cos \varphi' \sin(\tau - \alpha) &= A, \\ -\rho\pi_{\odot} \sin 1'' \cos \varphi' \cos(\tau - \alpha) &= B, \\ \rho\pi_{\odot} \sin 1'' \sin \varphi' &= C,\end{aligned}$$

we have

$$\begin{aligned}\alpha - \alpha' &= A/r \cos \delta, \\ \delta - \delta' &= (B/r) \sin \delta + (C/r) \cos \delta, \\ r - r' &= -B \cos \delta + C \sin \delta,\end{aligned}$$

in which at any particular observatory A and B may be tabulated with the hour angle $\tau - \alpha$ as argument, and C is a constant. Another form, convenient when such tables are not available, may be obtained by introducing auxiliaries g and G defined by

$$\begin{aligned}\sin \varphi' &= g \sin G, \\ \cos \varphi' \cos(\tau - \alpha) &= g \cos G,\end{aligned}$$

and putting

$$\begin{aligned}\frac{1}{15}\rho\pi_{\odot} \sin 1'' \cos \varphi' &= E, \\ \rho\pi_{\odot} \sin 1'' \sin \varphi' &= D.\end{aligned}$$

Then

$$\tan G = \tan \varphi' / \cos(\tau - \alpha), \quad G < 180^\circ$$

$$\begin{aligned}\pi_{\alpha}'' &= \alpha - \alpha' \\ &= \frac{E \sin(\tau - \alpha)}{r \cos \delta},\end{aligned}$$

$$\begin{aligned}\pi_{\delta}'' &= \delta - \delta' \\ &= \frac{D \sin(G - \delta)}{r \sin G},\end{aligned}$$

$$r - r' = D \frac{\cos(G - \delta)}{\sin G};$$

$$r\pi_{\alpha} = 0''.587\rho \cos \varphi' \sin h \sec \delta,$$

$$r\pi_{\delta} = 8''.80\rho \sin \varphi' \cos \delta - 8''.80\rho \cos \varphi' \cos h \sin \delta,$$

in which D and E are constants for any particular observatory and are called *parallax constants*. The quantities $r\pi_{\alpha}$, $r\pi_{\delta}$ are called *parallax factors*. Observers often publish parallax factors with their observations, rather than correct the observations for parallax; but these factors may either be the components of the parallax in right ascension and declination for a distance

of one astronomical unit, as here defined, in which case they need only be divided by the actual distance, or they may be the components corresponding to a parallax of 1", in which case they are to be multiplied by the actual horizontal parallax.

At upper meridian transit, $\tau - \alpha = 0^\circ$, and

$$\begin{aligned} \alpha - \alpha' &= 0, \\ \delta - \delta' &= \frac{\rho\pi_\odot \sin 1''}{r} \sin(\varphi' - \delta), \\ r - r' &= \rho\pi_\odot \sin 1'' \cos(\varphi' - \delta); \end{aligned}$$

at lower meridian transit, $\tau - \alpha = 180^\circ$.

When the geocentric distance of a celestial body is not known, as in the case of a newly discovered object, the effects of parallax may be eliminated by referring its position in space to the topocentric coordinate system with origin at the observer. For this purpose, the topocentric coordinates of the Sun may be found from its geocentric rectangular coordinates, X, Y, Z by Eqs. (38); with sufficient accuracy they are, denoting $15 E$ by F ,

$$\begin{aligned} X' &= X - F \cos \tau, \\ Y' &= Y - F \sin \tau, \\ Z' &= Z - D. \end{aligned}$$

Parallax in Celestial Longitude and Latitude

The effect of geocentric parallax on ecliptic coordinates may be derived directly from the expressions for parallax in right ascension and declination. The sidereal time τ is the right ascension of the observer's meridian, and therefore of the geocentric zenith; and the geocentric latitude φ' is the declination of the geocentric zenith. Replacing the equatorial coordinates (τ, φ') of the geocentric zenith by its longitude and latitude, and substituting (λ, β) for (α, δ) , in any of the different forms of the expressions for parallax in right ascension and declination, gives the corresponding expressions for parallax in λ and β .

The celestial longitude and latitude of the geocentric zenith may be obtained from the formulas that have previously been developed for the relations of the ecliptic system to the horizon system, taking into account the difference between the geocentric and geodetic zeniths if necessary to obtain the required accuracy.

In the ecliptic system, the effects of parallax may be eliminated when the distance is unknown, by referring positions in space to the point where the line from the observer to a celestial body intersects the plane of the ecliptic, a method introduced by Gauss. From this point, known as the *locus fictus*,

the celestial longitude λ' and latitude β' of the body are the same as from the point of observation. The rectangular coordinates of the observer referred to the locus fictus as origin are

$$(r - r') \cos \lambda' \cos \beta', \quad (r - r') \sin \lambda' \cos \beta', \quad (r - r') \sin \beta',$$

where r is the distance of the body from the locus fictus, and r' its distance from the observer; from these, and the rectangular geocentric coordinates of the observer and the Sun in the ecliptic system, the position of the locus fictus in the plane of the ecliptic relative to the Sun may be found. However, the method cannot be applied when the celestial latitude of the body is too near 0° .

Differential Parallax

To reduce the observed angular distance s' between two celestial objects to the value s that would be observed from the center of the Earth, we have from the triangle formed on the celestial sphere by the zenith and the observed positions at zenith distances z'_1, z'_2 ,

$$\cos s' = \cos z'_1 \cos z'_2 + \sin z'_1 \sin z'_2 \cos \Delta A,$$

where ΔA is the difference in azimuth; and from the triangle formed by the zenith and the geocentric positions, neglecting the flattening of the Earth,

$$\cos s = \cos z_1 \cos z_2 + \sin z_1 \sin z_2 \cos \Delta A.$$

Therefore,

$$\begin{aligned} \cos s &= \cos(z_1 - z_2) + \sin z_1 \sin z_2 (\cos \Delta A - 1) \\ &= \cos(z_1 - z_2) + \sin z_1 \sin z_2 \left\{ \frac{\cos s' - \cos z'_1 \cos z'_2}{\sin z'_1 \sin z'_2} - 1 \right\} \\ &= \cos(z_1 - z_2) - \{ \cos(z'_1 - z'_2) - \cos s' \} \frac{\sin z_1 \sin z_2}{\sin z'_1 \sin z'_2}. \end{aligned}$$

In most practical applications, this expression may be simplified by approximations, according to circumstances.

For objects which are very close together on the celestial sphere, the differential parallax in right ascension and declination may be obtained by differentiating Eqs. (43); and with the expressions obtained, the effect of parallax on the angular distance Δ and position angle p_1 may be found by differentiating the equations

$$\begin{aligned} \tan p_1 &= \frac{(\alpha_2 - \alpha_1) \cos \delta}{\delta_2 - \delta_1}, \\ \Delta &= (\alpha_2 - \alpha_1) \cos \delta \sin p_1 + (\delta_2 - \delta_1) \cos p_1, \end{aligned}$$

which are obtained from the expressions for differential right ascension and declination in Chapter 2.

By applying these corrections for the differential parallax due to the difference of position on the celestial sphere, the geocentric relative positions are obtained; but if the objects are at different geocentric distances, their relative positions will be different at different times because of the dependence of parallax upon distance. Stellar parallaxes are so small that in practice the differential annual parallax due to differences of distance is all that can be actually measured.

Annual Parallax

Only for objects within the solar system are the effects of geocentric parallax appreciable to observation; but the displacement of the observer in space with the orbital motion of the Earth is sufficiently great to cause measurable parallactic displacements of the nearer stars. The change in the observed geometric position of a star due to the orbital motion of the Earth is termed *heliocentric parallax* or *annual parallax*.

The heliocentric parallax Π is the angle subtended at a star by the semi-major axis a of the orbit of the Earth; $\sin \Pi = a/r$, where r is the distance of the star from the Sun. At any orbital position of the Earth (Fig. 11) the geocentric position of the star is displaced from the heliocentric position toward the geocentric position of the Sun, in the plane of the Sun, the Earth, and the star, and therefore along the great circle on the celestial sphere through the geocentric position of the Sun and the heliocentric position of the star, by an amount $\theta - \theta'$ given by

$$\begin{aligned}\sin(\theta - \theta') &= \frac{R}{r} \sin \theta' \\ &= \frac{R}{a} \sin \Pi \sin \theta',\end{aligned}$$

which, because of the very small magnitude of even the largest of these displacements, may be written

$$\theta - \theta' = \Pi \sin \theta.$$

The displacement of a star by annual parallax is therefore a maximum when the star is 90° from the Sun.

Rigorous equations are never required for determining the effects of annual parallax on the coordinates of a star. The approximate formulas that are needed in practice may readily be obtained in the same way as Eqs. (43).

4. Parallax

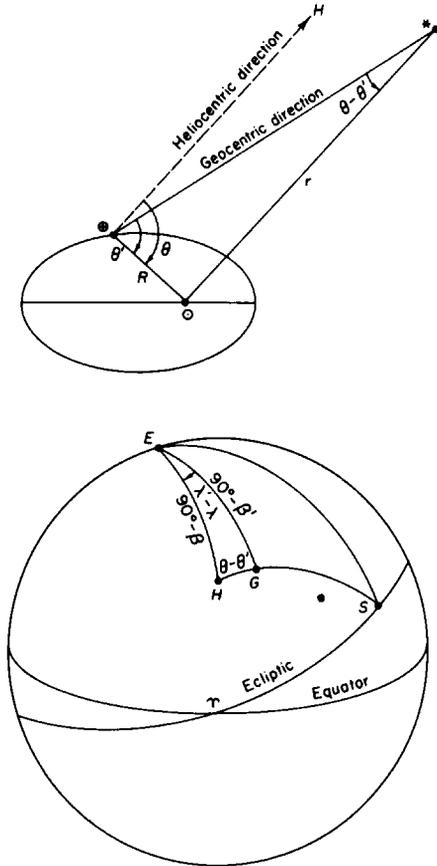


FIG. 11. Annual parallax.

Denoting the rectangular geocentric coordinates of a star by x', y', z' , and the heliocentric coordinates by x, y, z , and neglecting the latitude of the Sun, it is evident that in the ecliptic system

$$\begin{aligned} x' - x &= R \cos \odot, \\ y' - y &= R \sin \odot, \\ z' - z &= 0, \end{aligned}$$

and in the equatorial system,

$$\begin{aligned} x' - x &= R \cos \odot, \\ y' - y &= R \sin \odot \cos \epsilon, \\ z' - z &= R \sin \odot \sin \epsilon, \end{aligned}$$

where \odot is the geocentric longitude of the Sun, and the right-hand members may be substituted for dx, dy, dz in Eqs. (18).

Evidently, on the *heliocentric* celestial sphere the annual parallax is related to the heliocentric positions of the star, the Earth, and the pole of the ecliptic, in the same way as the geocentric parallax on the *geocentric* sphere is related to the geocentric positions of a celestial object, the zenith, and the celestial pole. The effects of annual parallax may therefore be obtained immediately from the expressions for geocentric parallax, or directly from the triangles on the geocentric sphere, by substituting the Sun for the zenith, the pole of the ecliptic for the celestial pole, $\odot - \lambda$ for h , and θ for z , with $\varphi' = 0$, and λ, β , in place of α, δ , but reversing the direction of the displacement.

Annual Parallax in Geocentric Longitude and Latitude

In the ecliptic system, the displacement of the geocentric position (λ', β') from the heliocentric position (λ, β) by annual parallax is represented by

$$\begin{aligned}(\lambda - \lambda') &= +\Pi \sin(\lambda - \odot) \sec \beta, \\(\beta - \beta') &= +\Pi \cos(\lambda - \odot) \sin \beta.\end{aligned}$$

Eliminating \odot gives the locus of the geocentric position on the celestial sphere during each revolution of the Earth around the Sun,

$$\frac{(\Delta\lambda \cos \beta)^2}{\Pi^2} + \frac{(\Delta\beta)^2}{\Pi^2 \sin^2 \beta} = 1,$$

the equation of an ellipse with semimajor axis Π parallel to the ecliptic. This curve is called the *parallactic ellipse*; for a star on the ecliptic, it reduces to a straight line.

Annual Parallax in Geocentric Right Ascension and Declination

In the equatorial system, the displacement of the geocentric position (α', δ') from the heliocentric position (α, δ) by annual parallax is represented by

$$\begin{aligned}\alpha - \alpha' &= \Pi \{ \sin \alpha \cos \odot - \cos \epsilon \cos \alpha \sin \odot \} \sec \delta, \\ \delta - \delta' &= \Pi \{ (\cos \epsilon \sin \alpha \sin \delta - \sin \epsilon \cos \delta) \sin \odot + \cos \alpha \sin \delta \cos \odot \}.\end{aligned}$$

In terms of the rectangular geocentric equatorial coordinates of the Sun, X, Y, Z , these expressions become

$$\begin{aligned}\alpha - \alpha' &= \Pi(X \sin \alpha - Y \cos \alpha) \sec \delta, \\ \delta - \delta' &= \Pi(X \cos \alpha \sin \delta + Y \sin \alpha \sin \delta - Z \cos \delta).\end{aligned}$$

The factor $X \sin \alpha - Y \cos \alpha$ in $\alpha - \alpha'$ is called the *parallax factor* in right ascension.

These formulas may be expressed in terms of auxiliaries which, for the purpose of calculating annual parallaxes, may be considered constant for many years, by putting

$$\begin{aligned} -\cos \epsilon \cos \alpha &= m \sin M, \\ -\sin \alpha &= m \cos M, \\ \sin \alpha \sin \delta \cos \epsilon - \cos \delta \sin \epsilon &= n \sin N, \\ -\cos \alpha \sin \delta &= n \cos N, \end{aligned}$$

with which the reduction from the heliocentric position to the geocentric position becomes

$$\begin{aligned} \alpha' - \alpha &= \Pi m \cos(M + \odot) \sec \delta, \\ \delta' - \delta &= \Pi n \cos(N + \odot). \end{aligned}$$