CHAPTER 5

Astronomical Refraction

DURING the passage of light through the atmosphere of the Earth, the continual variation of air density along the path causes a continual change of the direction of propagation; in addition, scattering and selective absorption by the gases of the atmosphere and by the dust particles in the air cause attenuation and change of spectral composition.

The irregularity and variability of atmospheric refraction, and its dependence on the color characteristics of the light, make the accurate determination of this indispensable correction to astronomical observations exceedingly difficult. In a general way, as a ray of light travels through the atmosphere it is bent downward by refraction, the total deviation varying approximately as the tangent of the zenith distance; the effect is therefore to displace the position of a celestial object upward, making all bodies appear higher in the sky than they actually are geometrically, by an amount that averages about 34′ at the horizon and decreases very rapidly with increasing altitude. That is, the general effect of atmospheric refraction is normally to increase the altitude of a celestial body without altering the azimuth, but since the amount depends upon the state of the atmosphere at every point on the path of the ray at the time, its determination is a difficult problem; and because of continual variations and irregularities throughout the atmosphere, the refraction is subject to irregular fluctuations and anomalies, especially near the horizon, that cannot be determined by theory.

On the hypothesis that the atmosphere is horizontally stratified, i.e., that the index of refraction μ is radially symmetric, $\mu = \mu(r)$ where r is distance from the center of the Earth; and from the law of refraction

$$\mu r \sin \zeta = \text{const.} \tag{44}$$

along the path of the ray, where ζ is the angle which the direction of the ray makes with r, and is therefore equivalent to the angle of incidence on each stratum (see Fig. 12). The angle which the direction of the ray at any point P makes with the radius vector, r_0, to the observer is

$$z = \theta + \zeta,$$

where θ is the geocentric zenith distance of P from the zenith of the observer.
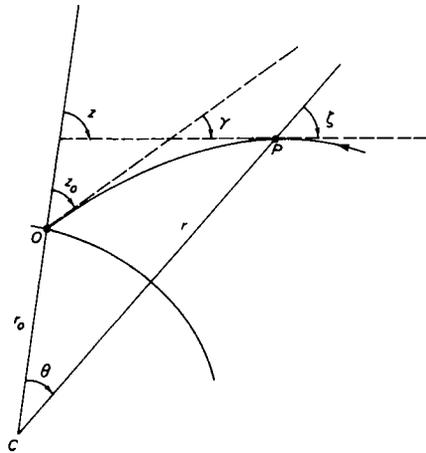
FIG. 12. Astronomical refraction.

At the observer, where $r = r_0$, $\mu = \mu_0$, $\theta = 0$, the apparent zenith distance is $z_0 = \zeta_0$.

The constant in Eq. (44) is equal to

$$\mu_0 r_0 \sin \zeta_0 \equiv k,$$

and hence

$$\tan \zeta = \frac{k}{(\mu^2 r^2 - k^2)^{1/2}} \,. \tag{45}$$

The change in the direction of the ray, i.e., the total amount of refraction, from any point P on the path, to the observer is

$$\gamma = z - z_0,$$

and therefore

$$dy = dz$$
$$= d\theta + d\zeta. \tag{46}$$

Some writers have erroneously put $dy = d\zeta$; note that r is not constant.

Differentiating Eq. (44) and dividing by $r\mu \sin \zeta$, we have

$$d\zeta = - \tan \zeta \frac{d(r\mu)}{r\mu} \,;$$

and from differential geometry

$$d\theta = \tan \zeta \frac{dr}{r} \,. \tag{47}$$

Therefore, by Eqs. (46) and (45),

$$dy = -\tan\zeta\,\frac{d\mu}{\mu}$$

$$= -\frac{k}{(r^2\mu^2 - k^2)^{1/2}}\frac{d\mu}{\mu} \qquad (48)$$

$$= +\tan\zeta\,d\!\left(\log\frac{\mu_0}{\mu}\right);$$

and by Eqs. (47) and (45),

$$d\theta = \frac{k}{(r^2\mu^2 - k^2)^{1/2}}\frac{dr}{r}. \qquad (49)$$

These equations are rigorous. The integration of Eq. (49) gives the path of the ray, and the integration of Eq. (48) gives the astronomical refraction

$$\delta z = \int_1^{\mu_0}\left\{\left(\frac{r\mu}{r_0\mu_0}\cosec z_0\right)^2 - 1\right\}^{-1/2}\frac{d\mu}{\mu}. \qquad (50)$$

The integrands may be expressed in various different forms that can be developed in series and integrated term by term; but the integration requires the function $\mu(r)$ to be known. The index of refraction for a given wave length depends almost entirely on density, since the composition of the atmosphere up to great heights is virtually uniform and constant except for variations of the relatively small water vapor content. From the distribution of density with height, and laboratory determinations of the relation of μ to density, the function $\mu(r)$ may be derived. However, it is not to be expected that this function is of a form that will enable a rigorous analytical theory of refraction to be constructed; either an analytical approximation or a numerical integration will be necessary.

Several different empirical laws have been formulated that represent more or less closely the relation of μ to the density ρ; among them are the Dale-Gladstone law,

$$\mu - 1 = c\rho,$$

and the Clausius-Mosotti equation,

$$\frac{\mu^2 - 1}{\mu^2 + 2} = c\rho,$$

where c is a constant.

Approximate Refraction

A first approximation may be obtained by putting

$$r/r_0 = 1 + s,$$

and expanding the integrand of Eq. (50) into a series: the quantity s is only about 0.01 at a height of 40 miles, beyond which the atmosphere is ineffective in producing appreciable refraction; neglecting s^2,

$$(r^2\mu^2 - k^2)^{-1/2} = \frac{1}{r_0} \{(\mu^2 - \mu_0^2 \sin^2 z_0) + 2s\mu^2\}^{-1/2},$$

which may be expanded by the binomial theorem for values of z_0 that are small enough to keep $\mu^2 - \mu_0^2 \sin^2 z_0$ large compared with $2\mu^2 s$. Under this condition, by Eq. (48).

$$\delta z = \mu_0 \sin z_0 \int_1^{\mu_0} \frac{d\mu}{\mu(\mu^2 - \mu_0^2 \sin^2 z_0)^{1/2}}$$

$$-\mu_0 \sin z_0 \int_1^{\mu_0} \frac{s\mu \, d\mu}{(\mu^2 - \mu_0^2 \sin^2 z_0)^{3/2}} + \cdots \tag{51}$$

Retaining only the first term of the development, we have

$$\delta z = \sin^{-1}(\mu_0 \sin z_0) - z_0;$$

putting $\mu_0 = 1 + x$ and developing in powers of x by Maclaurin's theorem, neglecting powers higher than the first,

$$\delta z = (\mu_0 - 1) \tan z_0. \tag{52}$$

To this approximation, the refraction is independent of the structure of the atmosphere, and is entirely determined by the local conditions at the observer. Physically, this approximation is equivalent to neglecting the curvature of the Earth; and the result may be obtained directly on this basis instead of from the rigorous formula: then CP is parallel to OZ (Fig. 13) and

$$\zeta = z$$
$$= z_0 + \gamma. \tag{53}$$

From the law of refraction, when a ray passes through a parallel-stratified medium the final direction is the same as if the entire medium had the density of the last stratum; i.e., the direction at O is the same as if the entire atmosphere had the same index of refraction μ_0 as at O. Hence,

$$\sin \zeta = \mu_0 \sin z_0$$

or, by Eq. (53), taking $\cos \gamma = 1$, $\sin \gamma = \gamma$,

$$\gamma = (\mu_0 - 1) \tan z_0$$
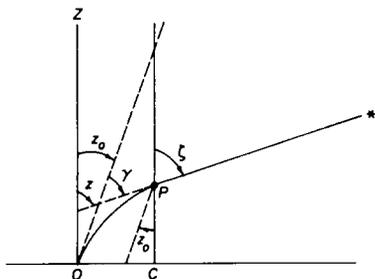
as in Eq. (52).

FIG. 13. Approximate refraction.

Adopting 1.0002927 for the index of refraction at the standard conditions 0°C and 760 mm and the Dale-Gladstone law, we have from this formula and the ideal gas law, in terms of the temperature t in degrees Centigrade and the height of the barometer B in millimeters at the observer,

$$\delta z = \frac{273 \times 0.0002927 \times 206264''.8}{760} \frac{B}{273 + t} \tan z_0$$

$$= 21''.7 \frac{B}{273 + t} \tan z_0. \tag{54}$$

For the standard conditions, this gives $\delta z = 60''.4 \tan z_0$.

This approximation is very close at small zenith distances, because near the zenith the curvature of the atmospheric strata has less effect on the path of the ray, as there is less total change in the direction of r; but it becomes useless at altitudes below about 30°. The change in the direction of the normal due to the curvature alters the direction of the ray continuously in the same sense, unlike what would occur were the strata plane; the longer the path through the atmosphere, the greater the effect. At zenith distances sufficiently greater than 90° for the ray to clear the surface of the Earth and pass completely through the atmosphere, the decrease in density along the latter part of the path is unable to compensate the effect of the increase in density along the preceding part of the path.

The approximation obtained by retaining the first two terms of Eq. (51) is of the form

$$\delta z = A \tan z_0 + B \tan^3 z_0, \tag{55}$$

in which the constants A and B may either be expressed in terms of physical constants as in the preceding approximation or, preferably, determined empirically from observations; this expression may also be independently established directly.* In terms of the physical quantities upon which the coefficients depend,

$$\delta z = (\mu_0 - 1)(1 - H_0) \tan z_0 - (\mu_0 - 1)[H_0 - \tfrac{1}{2}(\mu_0 - 1)] \tan^3 z_0$$

in which H_0 is the ratio of the height of the homogeneous atmosphere to the radius of the Earth. This expression is a common element of all expressions for refraction, and is the fundamental refraction formula for many practical purposes; but it is not sufficiently precise for astronomical purposes at very low altitudes.

* See Lord Rayleigh, *Phil. Mag.* **36**, 141 (1893), or W. J. Humphreys, "Physics of the Air," 3 ed., pp. 458–462. McGraw-Hill, New York, 1940 [reprinted by Dover, New York, 1964].

With $H_0 = (7.990 \times 10^5 \text{ cm})/(6.3709 \times 10^8 \text{ cm})$ and $\mu_0 - 1 = 0.0002927$,

$$\delta z = 60''.29 \tan z_0 - 0''.06688 \tan^3 z_0.$$

The index of refraction varies from color to color, and star images are actually vertical spectra; the dispersion, although very small, systematically affects precise astronomical observations. The values for the approximate angular distance between the red and blue-green images of a star are listed in the accompanying tabulation.

z	Distance
30°	0''.35
45°	0''.60
60°	1''.04
75°	2''.24

The systematic effect on astronomical observations depends upon the spectral type of the star.

Theories of Astronomical Refraction

The second approximation Eq. (55) is valid for zenith distances up to about 75°. Analytical theories to higher approximations, some of them theoretically valid to zenith distances greater than 90°, have been constructed on the basis of many elaborate investigations, beginning in the seventeenth century, depending upon different adopted variations of atmospheric density with height.

The variation of density depends primarily upon temperature distribution with height, but also to some extent upon the distribution of water vapor. The density distribution remained imperfectly known until very late in modern times. Astronomical refraction and the nature of its effects were recognized by Cleomedes in the first century A.D., and it was mentioned by Ptolemy in his work on optics. Tables of refraction were constructed by Tycho Brahe, and were later improved by Kepler. However, none of these writers had any knowledge of either the correct physical law of refraction or the actual nature of the atmosphere. Little was known of the structure of the atmosphere even by the seventeenth century; but the exact distribution of the temperature and other characteristics does not become of critical importance for determining the amount of refraction until comparatively great zenith distances are reached. To a zenith distance of about 80°, it is almost immaterial what hypothesis is adopted for the rate of decrease of

atmospheric density with height. The refraction tables constructed in the seventeenth and eighteenth centuries by Cassini, Bouguer, Newton, and others were based on widely different and highly artificial density distributions which differ greatly from any possible actual distribution; but they all gave practically the same values of refraction to a distance of nearly 80° from the zenith.

In the series expansions the coefficients of the first two terms are almost completely determined by the atmospheric conditions at the point of observation alone; they are affected slightly by the decrease of gravity with height, but this and other refinements are of little practical importance. The terms beyond the second depend upon the distribution of density with height; but for zenith distances up to nearly 75° these terms are so small, for any reasonable law of density distribution, that the refraction is practically independent of the structure of the atmosphere. Beyond this limit the third and higher terms become increasingly important; up to zenith distances somewhat beyond 80° they have nearly the same values for any likely density distribution, but the coefficients are increasingly sensitive to the exact distribution, and beyond the zenith distances where their magnitudes become appreciable the amount of refraction depends more and more upon the structure of the atmosphere. Different theories consequently give increasingly discrepant results as the horizon is approached.

At very low altitudes, no theory can be relied upon to give accurate values of the refraction on individual occasions. Near the horizon, theory can at most represent only average refractions. In constructing tables of refraction for low altitudes, dependence has in general been placed upon observations rather than upon theory, but on individual occasions, especially under extreme or abnormal atmospheric conditions, any tables regardless of their basis may be expected to depart more or less widely from the actual values of the refraction.

The developments in detail of the different theories of refraction that have been the basis for tables used in precise astronomical work may be found in the original publications; it is impracticable to reproduce the lengthy and complex mathematical calculations here. The early theories are now of only historical interest; comprehensive accounts of the principal ones are given by Newcomb* and Chauvenet.† These theories were unsatisfactory, both because the hypothetical models of the atmosphere upon which they were based do not sufficiently approximate the actual atmosphere and because the series expansions used in them are not valid near the horizon. With the

* S. Newcomb, "Compendium of Spherical Astronomy." Macmillan, New York, 1906 [reprinted by Dover, New York, 1960].

† W. Chauvenet, "Manual of Spherical and Practical Astronomy," Vol. :, 5 ed. Lippincott, Philadelphia, Pennsylvania, 1863 [reprinted by Dover, New York, 1960].

accumulated meteorological data on the upper atmosphere now available, theories of refraction may be based on empirical density distributions that more closely approximate the actual structure of the atmosphere; and analytical developments that are theoretically valid at all zenith distances may also be used.

The most important of the earlier refraction tables were the tables constructed by Bessel for the *Tabulae Regiomontanae* (1830). They were based on a theory developed by Kramp, and on Bradley's observations; at altitudes below 5° they were constructed entirely from observations. These, and later tables based upon them, were in general use until in 1870 the Pulkovo Observatory tables were published, based principally on theoretical investigations by Gyldén.

The Pulkovo tables in successive editions, and other tables derived from them, have remained in widespread use. Meanwhile, many tables have been constructed from more recent theoretical developments, particularly the refraction theories of J. E. Willis* and B. Garfinkel.† Garfinkel's theory is mathematically valid to zenith distances below the astronomical horizon, but the numerical evaluation of the refraction from the theoretical expressions is difficult and laborious.

Refraction Tables

For the purpose of constructing tables of refraction, the adopted theoretical formula is expressed in terms of selected standard atmospheric conditions at the point of observation. The refraction for these standard conditions, which is known as the *mean refraction*, is tabulated with the observed zenith distance or altitude as argument; and auxiliary tables are constructed for corrections to be applied to the mean refraction to obtain the value for the actual conditions at the time of observation. For example, the approximate formula (54) may be written as the product of the mean refraction $R_0 = 60''.4 \tan z_0$ and a correcting factor for temperature and pressure,

$$\delta z = R_0 \frac{0.3593B}{273 + t}.$$

The Pulkovo tables provide corrections for temperature, pressure, humidity, the latitude and elevation of the observer, and the color of the star; but in practice some of these corrections are often omitted. Refraction depends upon the hygrometric state of the atmosphere because of the effects of water

* J. E. Willis, A determination of astronomical refraction from physical data. *Trans. Amer. Geophys. Union* **22**, 324–336 (1941).

† B. Garfinkel, An investigation in the theory of astronomical refraction. *Astr. Jour.* **50**, 169–179 (1944).

vapor on both the density and the refractive power. A standard value is incorporated in the tables, and a further correction has very commonly been neglected; but in precise meridian observations the neglect of the correction may cause appreciable systematic seasonal errors, as shown by Scott.* A correction for latitude and elevation is needed in principle because the density distribution with height depends upon local gravity, but the effect is very small.

The constants in the theoretical refraction formulas are preferably evaluated from observations rather than from physical theory. For this purpose, the principal method has been the observation of zenith distances of circumpolar stars at both upper and lower culmination; but many other methods have also been used, especially for determining the amount of refraction near the horizon.

A circumpolar star at declination δ, which at upper culmination is north of the zenith at north latitude φ, is at geometric zenith distance $\delta - \varphi$ when at upper culmination, and $180° - \delta - \varphi$ when at lower culmination. The amount of refraction δz is the correction $z - z'$, always positive, to the observed zenith distance z', required to obtain the geometric zenith distance z; denoting it by R, the observed zenith distance at upper culmination is

$$z_1' = (\delta - \varphi) - R_1,$$

and at lower culmination,

$$z_2' = (180° - \delta - \varphi) - R_2,$$

and therefore

$$\tfrac{1}{2}(R_1 + R_2) = (90° - \varphi) - \tfrac{1}{2}(z_1' + z_2').$$

As many observations must be taken as are necessary to obtain a sufficient number of independent simultaneous equations to determine the constants in the theoretical expression for the left-hand member. In practice, because of the inevitable errors of observation and other uncertainties, a large number of observations of different stars are necessary to obtain accurate values of the constants.

Among other methods that may be used is the measurement of zenith distances of stars which are on or near the celestial equator, from a point of observation at or near $\varphi = 0°$; when $\delta = \varphi = 0$, the geometric zenith distance z is equal to the hour angle h, and R may be determined from the apparent zenith distance z' and the time at which z' is reached. The refraction constants may also be obtained by observing the meridian zenith distances of the same star at a north latitude φ_1 and a south latitude φ_2; e.g., taking

* F. P. Scott, A source of systematic error in meridian declinations. *Astr. Jour.* **60,** 93–95 (1955).

$R = k \tan z'$, we have

$$k = \frac{(\varphi_1 + \varphi_2) - (z_1' + z_2')}{\tan z_1' + \tan z_2'}.$$

Similarly, k may be obtained from the observed zenith distances of the Sun at the two solstices at any latitude φ. Still another method, due to Loewy, is by micrometric measurements of the change in the relative position of two stars during the diurnal motion, made by placing a prism with two silvered surfaces in front of the objective of an equatorial, bringing images of the stars into the field of view by rays which make a strictly constant angle with each other.

Among the observations that have been used to determine the amount of refraction near the horizon are: the comparison of observed altitudes of stars with altitudes computed from the right ascension and declination; the comparison of altitudes of the Sun, observed from ships at sea, with computed altitudes based on geographic positions obtained by observing stars at moderate altitudes; the comparison of the observed distances between the brighter stars of the Pleiades cluster at low altitudes with the observed distances at higher altitudes, and similarly for the distances between selected markings on the disk of the Moon.*

Geodetic Refraction

The limiting case of astronomical refraction is the refraction of a ray which grazes the visible horizon. When the ray comes from a point actually on the surface of the Earth at the visible horizon, or from any other source within the atmosphere, so that the entire path between source and observer is through the atmosphere, the refraction is known as terrestrial refraction or geodetic refraction.

Geodetic refraction is highly complex and variable, depending upon time of day, season of year, geographic region, character of terrain, state of weather, and other factors. In practice, none of the theoretical, statistical, or empirical rules that have been formulated for it have been satisfactory. Only with an adequate knowledge of temperature, pressure, and humidity at all points along the path of the ray can the amount of refraction be accurately and reliably determined; and the tendency in geodesy has been to depend as little as possible upon procedures which require the determination of the refraction.

* See, e.g., G. M. Clemence, Astronomical refraction at great zenith distances. *Astr. Jour.* **56**, 123–124 (1951); K. Aa. Strand, Astronomical refraction at large zenith distances. *Astr. Jour.* **57**, 166 (1952); I. Greenbaum, Observed and theoretical values of astronomical refraction at low altitudes. *Astr. Jour.* **59**, 17–19 (1954); D. H. Sadler and W. A. Scott, Refraction at low altitudes. *Jour. Inst. Navig.* **13**, 139–147 (1960).

Refraction Corrections to Astronomical Observations

In addition to the variations and irregularities of refraction near the horizon, irregular deviations from the theoretical tabular values frequently occur at all altitudes, and the refraction does not in general completely vanish at the zenith, principally because of the continually varying departure of the atmosphere from the state of concentric homogeneity that is assumed in deriving the theoretical expressions. Because of differences in weather conditions over adjoining regions, horizontal gradients of pressure and temperature usually exist; and moreover turbulence and irregular motions are widely prevalent throughout the atmosphere. Consequently, local anomalies and irregular fluctuations of varying degrees of rapidity frequently occur.

These irregular and varying departures from theory are of great importance in precise astronomical observations, and set definite limits on the accuracy that can be obtained in many types of measurements. Without observational data on the distribution of humidity and density with height along the path of a ray at the time of an observation, and an evaluation of the refraction integral by exact numerical integration, a precise value of the refraction for an individual observation cannot be reliably determined. The values of the refraction determined from theory will in general deviate more or less from the actual, and will represent only the mean of a large number.

In precise astronomical observations, the inevitable errors in the refraction corrections are partially eliminated in the adjustment of the observations, where a solution for a correction to the constant of refraction is always included in the discussion of a long series of observations. This gives the average error for the particular observations included in the discussion and will be different for each set of observations. However, especially in the neighborhood of the horizon, a reliable result requires the mean to be obtained from a very great number of observations, taken at very different states of the atmosphere, over a long interval of time, and there is evidence that the meteorological effects are partly systematic, peculiar to each locality, so that they are not completely eliminated in the adjustment of the observations.

In cases where individual observations or limited numbers of observations, obtained when there is opportunity, must be used for practical purposes, as in navigation or surveying, the error that may exist in the refraction corrections is not in general important; but discretion is necessary, according to the purpose, especially with observations too near the horizon or under abnormal atmospheric conditions. In navigation, by observing three or four stars well distributed in azimuth and within a limited range of altitude, the irregular

variations of refraction that may exist in azimuth have no effect on the fix; and only the *variation* of the error in the refraction with altitude is involved, not any constant actual error that may exist in the tabular refraction. Even very near the horizon, the consequent error of the fix is only of the order of 1″.

The small-scale rapidly variable irregularities in the atmosphere produce fluctuations of refraction which cause the unsteadiness of star images that characterizes the quality of astronomical "seeing"; they also cause the rapid variations in the brightness of the stars known as scintillation or "twinkling." These and other physical effects which the atmosphere produces in addition to the systematic geometric effects on the apparent positions of celestial objects likewise impose limits on the accuracy of astrometric measurements, but are not strictly within the limits of spherical astronomy.

Corrections to Celestial Coordinates

Normal refraction does not affect the azimuth of a celestial object. The observed altitude is greater than the geometric altitude; but the observed azimuth is the same as the actual geometric azimuth. However, the observed azimuth is in general different from what the geometric azimuth would be at a *geometric* altitude equal to the observed altitude; consequently, the hour angle and the declination are in general affected by refraction. The effects of refraction on celestial coordinates are comparatively small, and for most purposes may be expressed by differential formulas.

The actual geometric zenith distance of an object located at a point M in space is the angle Z which the straight line OM from the observer to the object makes with the vertical (Fig. 14). It is not rigorously equal to the sum of the apparent zenith distance z_0 and the total amount of refraction δz; the value of Z differs from $z = z_0 + \delta z$ by the amount of the parallax p of the object between the observer O and the point O' where the direction of the ray at the limit of the atmosphere meets the vertical. Rigorously, $Z = z_0 + R$, where $R = \delta z - p$. However, in consequence of the small magnitude of the refraction and the great distances of celestial bodies, the height h of the point O' is so small that in general the difference between R and δz is negligible, and no distinction between Z and z is necessary. For objects very close to the Earth, such as artificial satellites, the difference is important; but the only celestial body for which it can become appreciable is the Moon at low altitudes.*

For example, the effect of p is sometimes of significance in eclipses of the Sun and lunar occultations, where two objects at greatly different distances from the Earth are observed in apparent contact. When the limb of the Moon M is in apparent contact with the limb of the Sun S at the observer O,

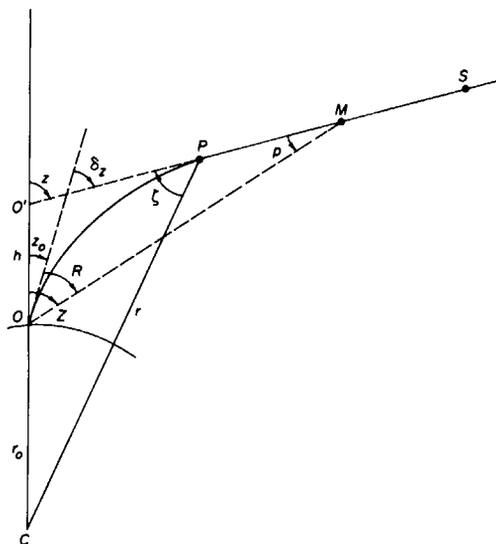* Cf. P. Herget, *Astr. Jour.* **64,** 334 (1959).

FIG. 14. Effect of refraction on zenith distance.

they are in geometric contact at O'. This may be taken into account by using $r_0 + h$ instead of r in computing the local circumstances of eclipses and occultations; but in general it is necessary to do so only for altitudes less than about $10°$. By Eq. (44), at the point P where the ray enters the atmosphere and $\mu = 1$,

$$r \sin \zeta = \mu_0 r_0 \sin z_0;$$

in the triangle CPO',

$$\frac{r_0 + h}{r} = \frac{\sin \zeta}{\sin(z_0 + \delta z)},$$

and therefore

$$1 + \frac{h}{r_0} = \frac{\mu_0 \sin z_0}{\sin(z_0 + \delta z)},$$

from which h may be determined.

From Eqs. (28) and (29), with φ and A constant,

$$d\delta = \mp \cos q \, dz,$$

according to whether φ is positive or negative; and from Eqs. (26) and (27) with φ constant, and the expression for $d\delta$,

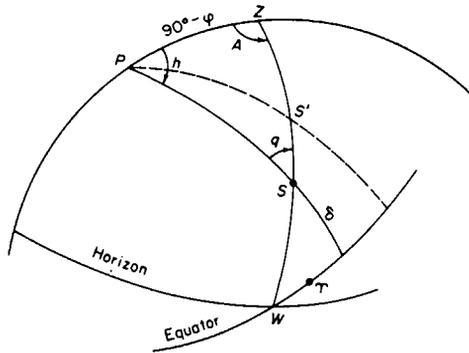$$\cos \delta \, dh = \pm \sin q \, dz,$$

Fig. 15. Refraction in right ascension and declination: S, geometric position and S' position affected by refraction.

according as h is less than or greater than 12^h (see Fig. 15). Therefore, denoting the refraction in zenith distance by

$$z - z_{obs} = R,$$

we have for the corrections to the observed declination and right ascension to remove the effects of refraction

$$\delta - \delta_{obs} = \begin{cases} -R \cos q, & \varphi > 0 \\ +R \cos q, & \varphi < 0 \end{cases}$$

$$\alpha - \alpha_{obs} = \begin{cases} -R \sec \delta \sin q, & 0^h < h < 12^h \\ +R \sec \delta \sin q, & 12^h < h < 24^h \end{cases} \tag{56}$$

since $d\alpha = -dh$. In right ascension, the refraction vanishes in the meridian; and in declination, at $q = 90°$ when by Eq. (27) the hour angle is given by $\cos h = \tan \varphi \cot \delta$, a value that can be reached above the horizon by only circumpolar objects.

From Eqs. (28) and (29), the effect of refraction on the parallactic angle is

$$q - q_{obs} = \mp R \sin q \tan \delta$$

according to whether φ is positive or negative.

Differential Refraction

In astrometric observations, corrections for refraction are often required to the measured differences of the coordinates of two objects that are relatively close together; in general these corrections may be determined by

differential formulas, in which $\pm R$ may be put equal to $k \tan z$. Denoting the differences of the geometric coordinates z, α, δ by

$$\Delta z = z_2 - z_1, \qquad \Delta\alpha = \alpha_2 - \alpha_1, \qquad \Delta\delta = \delta_2 - \delta_1,$$

and the differences of the observed coordinates z', α', δ' by

$$\Delta z' = z_2' - z_1', \qquad \Delta\alpha' = \alpha_2' - \alpha_1', \qquad \Delta\delta' = \delta_2' - \delta_1',$$

we have, to the first order,

$$\begin{aligned}
\Delta z &= \Delta z' + (R_2 - R_1) \\
&= \Delta z' + \frac{\partial R}{\partial z'}\Delta z' \\
&= \Delta z' + k \sec^2 z' \, \Delta z';
\end{aligned}$$

and similarly

$$\Delta\alpha = \Delta\alpha' + \frac{\partial(\alpha - \alpha')}{\partial\alpha}\Delta\alpha + \frac{\partial(\alpha - \alpha')}{\partial\delta}\Delta\delta,$$

$$\Delta\delta = \Delta\delta' + \frac{\partial(\delta - \delta')}{\partial\alpha}\Delta\alpha + \frac{\partial(\delta - \delta')}{\partial\delta}\Delta\delta,$$

in which, from Eqs. (56), with $\partial/\partial\alpha = -\partial/\partial h$,

$$\frac{\partial(\alpha - \alpha')}{\partial\alpha} = k(1 - \tan z \cos q \tan \delta + \tan^2 z \sin^2 q),$$

$$\frac{\partial(\alpha - \alpha')}{\partial\delta} = k \sec \delta \, (\tan^2 z \sin q \cos q - \tan z \sin q \tan \delta),$$

$$\frac{\partial(\delta - \delta')}{\partial\alpha} = k(\tan^2 z \sin q \cos q \cos \delta + \tan z \sin q \sin \delta),$$

$$\frac{\partial(\delta - \delta')}{\partial\delta} = k(1 + \tan^2 z \cos^2 q).$$

Refraction in Angular Distance and Position Angle

With $R = k \tan z$, we obtain to the first order, by differentiation of the triangle ZS_1S_2 with the angle at the zenith constant (Fig. 16) for the displacements in angular distance Δ and position angle p_1 by refraction,

$$d\Delta = -k[\cos P \tan z_2 + \cos(p_1 - q) \tan z_1],$$
$$d(p_1 - q) = +k \operatorname{cosec} \Delta[\sin(p_1 - q) \cos \Delta \tan z_1 - \sin P \tan z_2].$$

FIG. 16. Refraction in apparent distance and position angle.

Constructing a perpendicular from Z to the great circle through S_1 and S_2, intersecting this circle at an angular distance from S_1 of $S_1R = s$, we have from the right triangles ZS_1R and ZS_2R,

$$\tan s = + \cos(p_1 - q) \tan z_1,$$

$$\tan(s - \Delta) = - \cos P \tan z_2;$$

and the expression for $d\Delta$ becomes

$$d\Delta = -k[\tan s - \tan(s - \Delta)]$$

$$= -k \tan \Delta \frac{1 + \tan^2 s}{1 + \tan \Delta \tan s}$$

$$= -k \tan \Delta \frac{1 + \tan^2 z_1 \cos^2(p_1 - q)}{1 + \tan \Delta \tan z_1 \cos(p_1 - q)} .$$

From the triangle ZRS_2, with the identity $\sin P = \tan P \cos P$, we obtain $\sin P \tan z_2 = (\tan ZR)/\cos(s - \Delta)$, in which, from the same relations in the triangle ZRS_1,

$$\tan ZR = \sin(p_1 - q) \tan z_1 \cos s;$$

and with this expression and the refraction in the parallactic angle, $dq = k \tan \delta \sin q \tan z_1$, we obtain from the expression for $d(p_1 - q)$,

$$dp_1 = k\left\{\sin(p_1 - q) \tan z_1 \frac{\tan s - \tan \Delta}{1 + \tan s \tan \Delta} + \tan \delta \sin q \tan z_1\right\}.$$

In the case of double stars, if the components are close together and of the same color, both stars will be affected in practically the same way and there

will be no relative change in the apparent position angle or distance between them as their zenith distance changes with the diurnal motion. If, however, the two stars differ widely in color, the length of their atmospheric spectra will depend on their colors, and, if observed with a reflector which brings all colors into focus at the same distance from the mirror, their observed position angles and distances will vary as the star moves across the sky from east to west.

For example, Mira Ceti is so distinctly red in color that when we look at its atmospheric color band at great zenith distances the blue end is very faint as compared with the red and we should naturally set the micrometer wire near the red where the light seems of greatest intensity. The companion, at a distance of 0″.7 southeast of Mira, is a very different kind of star. Its blue light is intense and refraction extends its image in the different colors of the atmospheric spectrum to a total length of about one second of arc when on the meridian at Mount Wilson at a zenith distance of 37°. The blue is apparently above the red. The result is that the blue of the companion star appears almost east of Mira and a spectrograph with slit lying east-west records the spectra of both stars side by side when the telescope is guided upon Mira. If the observation is made at a considerable eastern hour angle the line joining the two stars becomes perpendicular to the horizon and the blue light of the companion is nearly superposed upon the red light of Mira, while in the west the greatest separation of the spectral images is obtained when the stars are on a horizontal line at right angles to the dispersion.

In terms of $d\Delta$ and dp_1, the differential refractions in right ascension and declination are, by differentiation of Eq. (13),

$$d(\alpha_2 - \alpha_1) = \sec \delta_2 [\sin p_1 \, d\Delta + \Delta \cos p_1 \, dp_1],$$

$$d(\delta_2 - \delta_1) = \cos p_1 \, d\Delta - \Delta \sin p_1 \, dp_1.$$

From the measured apparent angular distance s' between two widely separated celestial objects and their apparent altitudes, the geometric angular distance s may be rigorously determined with the exact formulas of spherical trigonometry by calculating the angle at the zenith between the vertical circles through the observed positions, and then computing s from this angle, which is not changed by normal refraction, and the altitudes corrected for refraction; the half-angle formulas are convenient for this purpose. In practice, however, exact formulas are not ordinarily necessary. The differential expression previously given for $d\Delta$ is usually sufficient, but for this purpose is more convenient in the form

$$ds = \cos C \, dz_2 + \cos B \, dz_1,$$

in which cos C and cos B are determined by

$$\cos z_1 = \cos s \cos z_2 + \sin s \sin z_2 \cos C,$$
$$\cos z_2 = \cos s \cos z_1 + \sin s \sin z_1 \cos B;$$

and therefore

$$(s - s') \sin s = R_2\{\cos z_1 \operatorname{cosec} z_2 - \cos s \cot z_2\}$$
$$+ R_1\{\cos z_2 \operatorname{cosec} z_1 - \cos s \cot z_1\},$$

where R denotes the refraction in zenith distance, $z - z'$.