CHAPTER 12

Variations of Ecliptic and Equatorial Coordinates from Precession and Nutation

THE variations that occur in the coordinates of a point on the celestial sphere, as the fundamental reference circles move among the stars, depend in a complex manner upon the varying position of the point relative to the circles, especially in the equatorial system. The variations of the coordinates are different from point to point on the sphere, and from time to time at the same point. The instantaneous rates at which the coordinates are changing at any moment must therefore be carefully distinguished from the accumulated amounts of the changes produced by the integrated effects over any extended interval of time.

At any instant, from the instantaneous motion of the *ecliptic alone*, which is without effect on the position of the equator, the rate of variation of the right ascension at a point on the mean equator of date regarded as fixed is $d\alpha/dt = -da/dt$, due to the motion of the equinox along the equator; the declination is not affected. The rates of variation of the longitude and latitude at any point S on the sphere due to the rotational motion of the ecliptic, obtained by differentiating the triangle SKP (Fig. 47), with the right angle at P and the side SK constant, and with $-\kappa\, dt$ for the differential of the angle at K, are

$$d\lambda/dt = \kappa \cos(\lambda - \Pi) \tan \beta \qquad d\beta/dt = -\kappa \sin(\lambda - \Pi)$$
$$= -\kappa \cos(\lambda + N) \tan \beta, \qquad = +\kappa \sin(\lambda + N);$$

and in addition, the consequent motion of the equinox along the ecliptic

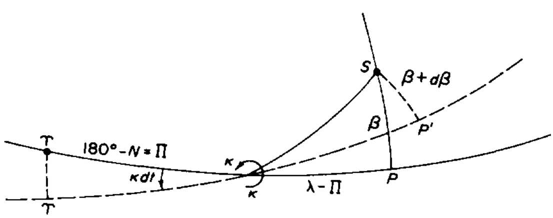

FIG. 47. Rotation of the ecliptic.

causes a further variation of λ at the rate $(-da/dt)\cos\epsilon$ where ϵ denotes the mean obliquity of date.

Concurrently, from the instantaneous lunisolar motion of the *mean equator alone*, which independently takes place without affecting the ecliptic, the rate of variation of the instantaneous longitude λ on the ecliptic *of date* regarded as fixed is the rate of the *lunisolar* precession on the *fixed ecliptic of date* (i.e., the rate of the lunisolar precession $d\Psi_1/dt$ referred to the given instant as an arbitrary epoch) which is due to the consequent motion of the equinox along the ecliptic; the latitude β is not affected. Adding these variations to the preceding rates of variation produced by the motion of the ecliptic gives for the instantaneous rates of the precessional variations of the coordinates in the ecliptic system

$$\begin{aligned}\frac{d\lambda}{dt} &= \frac{d\Psi}{dt} - \kappa\cos(\lambda + N)\tan\beta,\\ \frac{d\beta}{dt} &= +\kappa\sin(\lambda + N),\end{aligned} \tag{148}$$

where $d\Psi/dt$ is the rate of general precession in longitude on the ecliptic of date; i.e., $(d\Psi_1/dt) - (da/dt)\cos\epsilon$ referred to date as arbitrary epoch by (129).

The rates of the precessional variations of the instantaneous right ascension and declination from the motion of the *mean equator alone*—which are due both to the lunisolar motion of the mean equator itself, and to the consequent motion of the equinox along the equator—may be obtained from the differential relations between the coordinates in the equatorial system and the coordinates in the ecliptic system, by putting $d\lambda = d\Psi_1$, $d\beta = 0$, and $d\epsilon = 0$, where Ψ_1 is the lunisolar precession on the fixed ecliptic of date. Adding the further variation $-da/dt$ of α due to the motion of the ecliptic, the instantaneous rates of the precessional variations of the coordinates in the equatorial system are, by (14) and (11),

$$\begin{aligned}\frac{d\alpha}{dt} &= \frac{\partial\alpha}{\partial\lambda}\frac{d\lambda}{dt} - \frac{da}{dt}\\ &= \left(\cos\epsilon\frac{d\Psi_1}{dt} - \frac{da}{dt}\right) + \left(\sin\epsilon\frac{d\Psi_1}{dt}\right)\sin\alpha\tan\delta\\ &= m + n\sin\alpha\tan\delta,\\ \frac{d\delta}{dt} &= \frac{\partial\delta}{\partial\lambda}\frac{d\lambda}{dt}\\ &= \left(\sin\epsilon\frac{d\Psi_1}{dt}\right)\cos\alpha\\ &= n\cos\alpha,\end{aligned} \tag{149}$$

where the epoch is continuously at the arbitrary given instant.

These expressions may also be obtained directly from the representation of the lunisolar motion of the equator by a rotation around the line of solstices at the rate n, in the same way as the variation of λ and β were obtained from the rotation of the ecliptic. Differentiating the appropriate triangle gives for the variations that are immediately due to the rotation,

$$d\alpha/dt = n\cos(\alpha - 90°)\tan\delta \qquad d\delta/dt = -n\sin(\alpha - 90°)$$
$$= n\sin\alpha\tan\delta, \qquad = +n\cos\alpha.$$

In addition, the consequent motion of the equinox along the equator causes a further variation of α at the rate $(d\Psi_1/dt)\cos\epsilon$, which combines with the variation $-da/dt$ from the independent motion of the ecliptic to form the general precession in right ascension m. The general precession in right ascension rotates the hour circles westward around the moving pole, and increases right ascensions at rate m; it leaves declinations unchanged, but the accompanying motion of the pole produces both a further motion of the hour circles which introduces the other term of the variation of right ascension, and also a motion of the parallels of declination which causes the whole of the variation of declination.

In the *equatorial coordinate system*, therefore, the variation of declination is entirely due to the lunisolar precession and nutation of the equator. The variation of right ascension depends upon the motions of both the ecliptic and the equator, and is the resultant of the general precession in right ascension and the nutation in right ascension. In the *ecliptic coordinate system*, the variation of celestial latitude is entirely due to the secular motion of the ecliptic; the variation of celestial longitude depends upon the motions of both the ecliptic and the equator, and is the resultant of the general precession in longitude and the nutation in longitude.

Over even very long intervals of time, the change in latitude is comparatively slight, and the change in longitude is nearly proportional to the time; the *difference* in longitude of two stars remains almost constant. Hence a very easy calculation suffices to obtain a closely approximate reduction, and the coordinates at the two epochs are related in a simple way, in contrast to the complexity of the variations in equatorial coordinates and the lengthy trigonometric computations required to determine the changes over long intervals.

The following tabulation shows the positions of Altair in the catalogs of Ptolemy and Flamsteed, expressed both in equatorial coordinates and by the differences in longitude and latitude from Spica:

Epoch	Longitude from Spica	Latitude from Spica	Right ascension	Declination
A.D. 138	97°10′	29°10′	274°52′	5°43′
A.D. 1690	97°52′	29°19′	293°54′	8°05′

The difference in equatorial coordinates is so great and depends in such a complex way upon the location of the star that no star is identifiable from its right ascension and declination at two epochs without trigonometric calculation; but notwithstanding fairly large probable errors in Ptolemy's catalog, the ecliptic coordinates, especially when referred to another given star, offer no difficulty.

Reduction of Ecliptic Coordinates for Precession

The coordinates (λ_1, β_1), either heliocentric or geocentric, referred to the mean equinox and ecliptic of any particular date t_1, may be rigorously reduced to the mean equinox and ecliptic of any other date t_2, where t_1 and t_2 are reckoned from a fundamental epoch t_0, by means of the relations

$$\begin{aligned}\cos\beta_2\cos(\lambda_2-\Lambda_2) &= \cos\beta_1\cos(\lambda_1-\Pi_1),\\ \cos\beta_2\sin(\lambda_2-\Lambda_2) &= \cos\beta_1\sin(\lambda_1-\Pi_1)\cos\pi_1+\sin\beta_1\sin\pi_1, \\ \sin\beta_2 &= \sin\beta_1\cos\pi_1-\cos\beta_1\sin(\lambda_1-\Pi_1)\sin\pi_1,\end{aligned} \tag{150}$$

in which

$$\Lambda_2 = \Pi_1 + \Upsilon + \tfrac{1}{2}p_1q_1(t_2-t_1)^2,$$

where π_1, Π_1, denote the inclination and node of the ecliptic at t_2 on the ecliptic of t_1, obtained from (137) and (138), Π_1 being reckoned from the mean equinox Υ^1 of t_1, and Υ is the amount of general precession in longitude from t_1 to t_2; to the third order, by (145),

$$\Upsilon = (\Upsilon''-\Upsilon') + \tfrac{1}{2}(\Pi_1''-\Pi_1')\pi_1''\pi_1' - \tfrac{1}{12}(\Pi_1''-\Pi_1')^3 + p_1q_1t_1(t_2-t_1),$$

where Υ', π_1', Π_1' are the values at t_1, and Υ'', π_1'', Π_1'', the values at t_2, referred to the fundamental epoch t_0.

The relations (150) may be obtained either from the triangle SZ_1Z_2 (Fig. 48), or by successive rotations of the rectangular ecliptic coordinate system around the Z_1-axis through the angle Π_1 eastward, around the line of intersection of the two positions of the ecliptic through the angle π_1, and around the Z_2-axis westward through the angle Λ_2. They may be adapted to numerical computation by writing them in the form

$$\begin{aligned}\cos\beta_2\cos(\lambda_2-\Lambda_2) &= \cos\beta_1\cos(\lambda_1-\Pi_1),\\ \cos\beta_2\sin(\lambda_2-\Lambda_2) &= \cos\beta_1\sin(\lambda_1-\Pi_1)\\ &\quad+\sin\pi_1[\sin\beta_1-\cos\beta_1\sin(\lambda_1-\Pi_1)\tan\tfrac{1}{2}\pi_1],\\ \sin\beta_2 &= -\sin\beta_1+2\cos^2\tfrac{1}{2}\pi_1[\sin\beta_1-\cos\beta_1\sin(\lambda_1-\Pi_1)\tan\tfrac{1}{2}\pi_1].\end{aligned}$$

Multiplying the first by $\cos(\lambda_1-\Pi_1)$, the second by $\sin(\lambda_1-\Pi_1)$, adding,

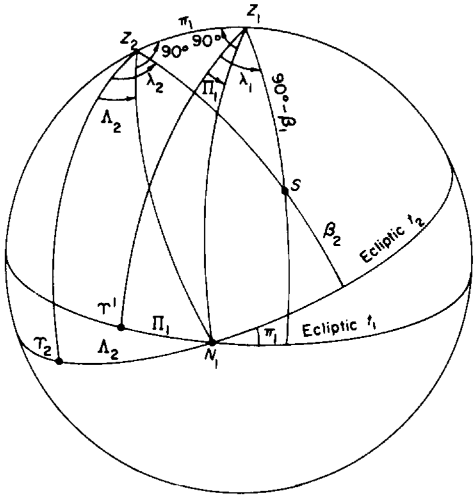

FIG. 48. Precessional variations of celestial longitude and latitude.

and putting for brevity,

$$\Delta\lambda = \lambda_2 - \lambda_1 - (\Lambda_2 - \Pi_1),$$
$$q = \sin \pi_1\{\tan \beta_1 - \sin(\lambda_1 - \Pi_1) \tan \tfrac{1}{2}\pi_1\},$$

gives

$$\cos \beta_2 \cos \Delta\lambda = \cos \beta_1\{1 + q \sin(\lambda_1 - \Pi_1)\};$$

while multiplying the first by $-\sin(\lambda_1 - \Pi_1)$, the second by $\cos(\lambda_1 - \Pi_1)$, and adding, gives

$$\cos \beta_2 \sin \Delta\lambda = q \cos \beta_1 \cos(\lambda_1 - \Pi_1).$$

From these last two equations, we immediately obtain for the determination of λ_2,

$$\tan \Delta\lambda = \frac{q \cos(\lambda_1 - \Pi_1)}{1 + q \sin(\lambda_1 - \Pi_1)}, \tag{151}$$
$$\lambda_2 = \lambda_1 + \Upsilon + \Delta\lambda + \tfrac{1}{2}p_1 q_1(t_2 - t_1)^2;$$

while for the determination of β_2, by multiplying the first by $\cos \frac{1}{2} \Delta\lambda$, the second by $\sin \frac{1}{2} \Delta\lambda$, and adding, obtaining

$$2 \sin \tfrac{1}{2}(\beta_2 + \beta_1) \sin \tfrac{1}{2}(\beta_2 - \beta_1) = -q \cos \beta_1 \frac{\sin(\lambda_1 - \Pi_1 + \frac{1}{2} \Delta\lambda)}{\cos \frac{1}{2} \Delta\lambda},$$

and by writing the equation for $\sin \beta_2$ in the form

$$2 \sin \tfrac{1}{2}(\beta_2 + \beta_1) \cos \tfrac{1}{2}(\beta_2 - \beta_1) = q \cos \beta_1 \cot \tfrac{1}{2}\pi_1,$$

we have by division,

$$\tan \tfrac{1}{2}(\beta_2 - \beta_1) = -\frac{\sin(\lambda_1 - \Pi_1 + \frac{1}{2}\Delta\lambda)}{\cos \frac{1}{2}\Delta\lambda} \tan \tfrac{1}{2}\pi_1, \tag{152}$$

or, with ample accuracy,

$$\beta_2 = \beta_1 - \pi_1 \sin(\lambda_1 - \Pi_1 + \tfrac{1}{2}\Delta\lambda) \sec \tfrac{1}{2}\Delta\lambda.$$

In practice, expressions to this high order of accuracy are required only when the interval $t_2 - t_1$ is very long; ordinarily, approximations of much simpler form are sufficient. For a short interval, the products of the length of the interval by the values of $d\lambda/dt$ and $d\beta/dt$ at the midpoint of the interval may be used; a form which is adapted to the construction of tables for facilitating the computation of reductions is

$$\Delta\lambda = a - b\cos(\lambda_1 + c)\tan\beta_1,$$
$$\Delta\beta = b\sin(\lambda_1 + c),$$

in which the coefficients a, b, c are

$$a = \bar{p}(t_2 - t_1),$$
$$b = \bar{\kappa}(t_2 - t_1),$$
$$c = \bar{N} + \tfrac{1}{2}a,$$

where $\bar{p}$, $\bar{\kappa}$, $\bar{N}$ denote the values of $d\Psi/dt$, κ, and $N = 180° - \Pi$, midway from t_1 to t_2.

These expressions are essentially equivalent to the first and second-order terms of the Taylor's series

$$\lambda(t_2) = \lambda(t_1) + \left(\frac{\partial\lambda}{\partial t}\right)_1 \Delta t + \frac{1}{2}\left(\frac{d^2\lambda}{dt^2}\right)_1 (\Delta t)^2 + \cdots,$$

and similarly for β; with the values of p, κ, and N for the beginning of the interval Δt, only the first-order terms would be represented.

Reduction of Ecliptic Coordinates for Nutation

Since nutation is due entirely to the motion of the equator, it affects the ecliptic coordinates only through its effect on the position of the equinox; that is, celestial latitudes are unchanged, and the celestial longitude referred to the true equinox of date is

$$\lambda = \lambda_0 + \Delta\Psi$$

where λ_0 is the longitude referred to the mean equinox of date and $\Delta\Psi$ is the nutation in longitude.

Precessional Variations of Ecliptic Rectangular Coordinates

The coordinates x_1, y_1, z_1 of any body in a rectangular system that has the xy-plane in the plane of the ecliptic of an epoch t_1, with the z-axis directed toward the north pole of the ecliptic and the x-axis toward the vernal equinox, may be transformed to the coordinates x_2, y_2, z_2 referred to the ecliptic and equinox of any other time t_2, by the usual formulas from analytic geometry

$$\begin{aligned} x_2 &= \cos(x_2, x_1)x_1 + \cos(x_2, y_1)y_1 + \cos(x_2, z_1)z_1, \\ y_2 &= \cos(y_2, x_1)x_1 + \cos(y_2, y_1)y_1 + \cos(y_2, z_1)z_1, \\ z_2 &= \cos(z_2, x_1)x_1 + \cos(z_2, y_1)y_1 + \cos(z_2, z_1)z_1, \end{aligned}$$

Taking account of only the secular motion of the ecliptic and the general precession, we have from the appropriate spherical triangles (Fig. 49) by the cosine law, neglecting the term $\frac{1}{2}p_1q_1t^2$ in the precession,

$$\begin{aligned} \cos(x_2, x_1) &= \cos \Pi_1 \cos(\Pi_1 + \Upsilon) + \sin \Pi_1 \sin(\Pi_1 + \Upsilon) \cos \pi_1, \\ \cos(x_2, y_1) &= \sin \Pi_1 \cos(\Pi_1 + \Upsilon) - \cos \Pi_1 \sin(\Pi_1 + \Upsilon) \cos \pi_1, \\ \cos(x_2, z_1) &= -\sin(\Pi_1 + \Upsilon) \sin \pi_1; \end{aligned}$$

$$\begin{aligned} \cos(y_2, x_1) &= \cos \Pi_1 \sin(\Pi_1 + \Upsilon) - \sin \Pi_1 \cos(\Pi_1 + \Upsilon) \cos \pi_1, \\ \cos(y_2, y_1) &= \sin \Pi_1 \sin(\Pi_1 + \Upsilon) + \cos \Pi_1 \cos(\Pi_1 + \Upsilon) \cos \pi_1, \\ \cos(y_2, z_1) &= \cos(\Pi_1 + \Upsilon) \sin \pi_1; \end{aligned}$$

$$\begin{aligned} \cos(z_2, x_1) &= \sin \Pi_1 \sin \pi_1, \\ \cos(z_2, y_1) &= -\cos \Pi_1 \sin \pi_1, \\ \cos(z_2, z_1) &= \cos \pi_1; \end{aligned}$$

whence, putting $\cos \pi_1 = 1 - 2 \sin^2\frac{1}{2}\pi_1$ and similarly for $\cos \Upsilon$,

$$\begin{aligned} x_2 = x_1 &- 2\{\sin^2\tfrac{1}{2}\Upsilon + \sin \Pi_1 \sin(\Pi_1 + \Upsilon) \sin^2\tfrac{1}{2}\pi_1\}x_1 \\ &+ \{2 \cos \Pi_1 \sin(\Pi_1 + \Upsilon) \sin^2\tfrac{1}{2}\pi_1 - \sin \Upsilon\}y_1 \\ &- \{\sin(\Pi_1 + \Upsilon) \sin \pi_1\}z_1, \\ y_2 = y_1 &+ \{2 \sin \Pi_1 \cos(\Pi_1 + \Upsilon) \sin^2\tfrac{1}{2}\pi_1 + \sin \Upsilon\}x_1 \\ &- 2\{\sin^2\tfrac{1}{2}\Upsilon + \cos \Pi_1 \cos(\Pi_1 + \Upsilon) \sin^2\tfrac{1}{2}\pi_1\}y_1 \\ &+ \{\cos(\Pi_1 + \Upsilon) \sin \pi_1\}z_1, \\ z_2 = z_1 &+ (\sin \Pi_1 \sin \pi_1)x_1 \\ &- (\cos \Pi_1 \sin \pi_1)y_1 \\ &- (2 \sin^2\tfrac{1}{2}\pi_1)z_1, \end{aligned}$$

in which Υ is the general precession from t_1 to t_2, and Π_1, π_1, are the node and inclination of the ecliptic at t_2 on the ecliptic at t_1.

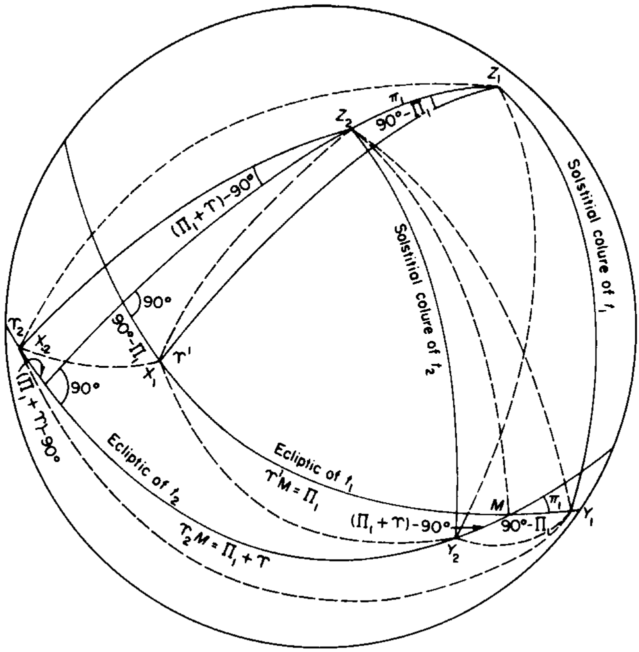

FIG. 49. Variations of ecliptic rectangular coordinates.

Since

$$x_1 = \cos(x_1, x_2)x_2 + \cos(x_1, y_2)y_2 + \cos(x_1, z_2)z_2,$$
$$y_1 = \cos(y_1, x_2)x_2 + \cos(y_1, y_2)y_2 + \cos(y_1, z_2)z_2,$$
$$z_1 = \cos(z_1, x_2)x_2 + \cos(z_1, y_2)y_2 + \cos(z_1, z_2)z_2,$$

the inverse transformation may be made with the same coefficients; its matrix is the conjugate of the matrix of the preceding transformation (i.e., the rows and columns are interchanged) in accordance with the fact that the general precession from t_2 to t_1 is $-\Upsilon$ and the node of the ecliptic of t_1 on the ecliptic of t_2 reckoned from Υ_2 is $180° + (\Pi_1 + \Upsilon)$, whence, e.g., replacing Υ in $\cos(x_2, y_1)$ by $-\Upsilon$ and Π_1 by $180° + (\Pi_1 + \Upsilon)$ we obtain $\cos \Pi_1 \sin(\Pi_1 + \Upsilon) - \sin \Pi_1 \cos(\Pi_1 + \Upsilon) \cos \pi_1 = \cos(y_2, x_1)$, etc.

Precessional Variations of the Elements of a Plane Relative to the Plane of the Ecliptic

The inclination i of a plane in space to the plane of the ecliptic at any particular time t, and the longitude of its ascending node Ω, reckoned from the mean equinox, depend upon the epoch t because of the precessional motions of the ecliptic and the equinox.

From Napier's analogies in the spherical triangle formed by the intersections of the celestial sphere with a plane P and the positions of the plane of the ecliptic at two different times t_1, t_2, (Fig. 50) neglecting the term $\frac{1}{2}p_1q_1t^2$ in the general precession,

$$\tan \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) - \Delta\omega] = \frac{\cos \frac{1}{2}(i_1 + \pi_1)}{\cos \frac{1}{2}(i_1 - \pi_1)} \tan \tfrac{1}{2}(\Omega_1 - \Pi_1),$$

$$\tan \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) + \Delta\omega] = \frac{\sin \frac{1}{2}(i_1 + \pi_1)}{\sin \frac{1}{2}(i_1 - \pi_1)} \tan \tfrac{1}{2}(\Omega_1 - \Pi_1),$$

$$\tan \tfrac{1}{2}(i - i_1) = -\tan \tfrac{1}{2}\pi_1 \frac{\cos \frac{1}{2}(\Omega_1 + \Omega - 2\Pi_1 - \Upsilon)}{\cos \frac{1}{2}(\Omega_1 - \Omega + \Upsilon)},$$

in which π_1, Π_1 are the inclination and node of the ecliptic at t_2 on the ecliptic of t_1, Π_1 being reckoned from the mean equinox of t_1; Υ is the amount of general precession in longitude from t_1 to t_2; i_1, Ω_1, are the inclination and node of P on the ecliptic of t_1, and i, Ω on the ecliptic of t_2, each node being reckoned from the mean equinox of the corresponding date; and $\Delta\omega$ is the arc of P intercepted between the two positions of the ecliptic.

Gauss's equations may also be used:

$$\cos \tfrac{1}{2}i \cos \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) - \Delta\omega] = \cos \tfrac{1}{2}(\Omega_1 - \Pi_1) \cos \tfrac{1}{2}(i_1 - \pi_1),$$
$$\sin \tfrac{1}{2}i \cos \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) + \Delta\omega] = \cos \tfrac{1}{2}(\Omega_1 - \Pi_1) \sin \tfrac{1}{2}(i_1 - \pi_1),$$
$$\cos \tfrac{1}{2}i \sin \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) - \Delta\omega] = \sin \tfrac{1}{2}(\Omega_1 - \Pi_1) \cos \tfrac{1}{2}(i_1 + \pi_1),$$
$$\sin \tfrac{1}{2}i \sin \tfrac{1}{2}[\Omega - (\Pi_1 + \Upsilon) + \Delta\omega] = \sin \tfrac{1}{2}(\Omega_1 - \Pi_1) \sin \tfrac{1}{2}(i_1 + \pi_1).$$

These rigorous formulas are rarely ever required. Approximations to any desired order of accuracy may be obtained from them by means of expansions in series or by other methods of approximation.

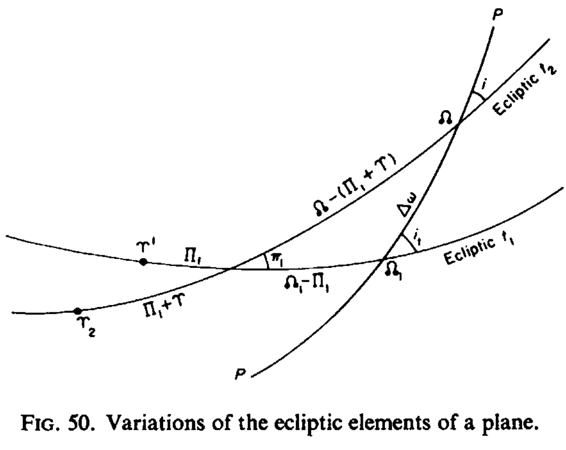

FIG. 50. Variations of the ecliptic elements of a plane.

In the spherical triangle formed by the intersections of the celestial sphere with the plane P and the two positions of the plane of the ecliptic,

$$\frac{\sin \Delta\omega}{\sin \pi_1} = \frac{\sin(\Omega_1 - \Pi_1)}{\sin i},$$

$$\begin{aligned}\sin \pi_1 \cos(\Omega_1 - \Pi_1) &= \sin i_1 \cos i - \cos i_1 \sin i \cos \Delta\omega \\ &= \sin(i_1 - i) + 2 \sin i \cos i_1 \sin^2 \tfrac{1}{2} \Delta\omega, \\ \sin \Delta\omega \cos i &= \sin[\Omega - (\Pi_1 + \Upsilon)] \cos(\Omega_1 - \Pi_1) \\ &\quad - \cos[\Omega - (\Pi_1 + \Upsilon)] \sin(\Omega_1 - \Pi_1) \cos \pi_1 \\ &= \sin(\Omega - \Omega_1 - \Upsilon) \\ &\quad + 2 \cos[\Omega - (\Pi_1 + \Upsilon)] \sin(\Omega_1 - \Pi_1) \sin^2 \tfrac{1}{2}\pi_1;\end{aligned}$$

and substituting the first equation into the third,

$$\begin{aligned}\sin[\Omega - (\Omega_1 + \Upsilon)] &= \sin \pi_1 \sin(\Omega_1 - \Pi_1) \cot i \\ &\quad - 2 \cos [\Omega - (\Pi_1 + \Upsilon)] \sin(\Omega_1 - \Pi_1) \sin^2 \tfrac{1}{2}\pi_1.\end{aligned}$$

The motion of the ecliptic is so slight that to a high order of accuracy over very long intervals of time these equations may be written in the approximate forms

$$\Delta\omega = \frac{\sin(\Omega_1 - \Pi_1)}{\sin i_1} \pi_1,$$

$$i = i_1 - \pi_1 \cos(\Omega_1 - \Pi_1) + (\tfrac{1}{2} \Delta\omega)^2 \sin 1'' \sin 2i_1,$$

$$\Omega = \Omega_1 + \Upsilon + \pi_1 \sin(\Omega_1 - \Pi_1) \cot i - (\tfrac{1}{2}\pi_1)^2 \sin 1'' \sin 2(\Omega_1 - \Pi_1),$$

from which $\Delta\omega$, i, and Ω may be computed in succession.

Ordinarily, the second-order terms in the expressions for i and Ω may be neglected. The equations then become practically equivalent to the formulas which are obtained from the relations among the differentials of the elements. At any instant t, the ecliptic is rotating at a rate $\kappa = d\pi_1/dt$ around an axis in longitude $\Pi_\kappa = 180° - N$ from the mean equinox of date; and the mean equinox is moving westward along the ecliptic at a speed $p = d\Upsilon/dt$. By differentiating the triangle (Fig. 51), we find that during an interval dt,

$$\begin{aligned}di &= -\cos(\Omega - \Pi_\kappa)\, d\pi_1, \\ d\Omega &= d\Upsilon + \sin(\Omega - \Pi_\kappa) \cot i\, d\pi_1, \\ d\omega &= \sin(\Omega - \Pi_\kappa) \operatorname{cosec} i\, d\pi_1,\end{aligned}$$

in which $d\pi_1 = \kappa\, dt$. By the mean value theorem of the integral calculus, over the interval from t_1 to t_2

$$i = i_1 - [\kappa \cos(\Omega - \Pi_\kappa)]_{t=t'}(t_2 - t_1), \qquad t_1 < t' < t_2$$

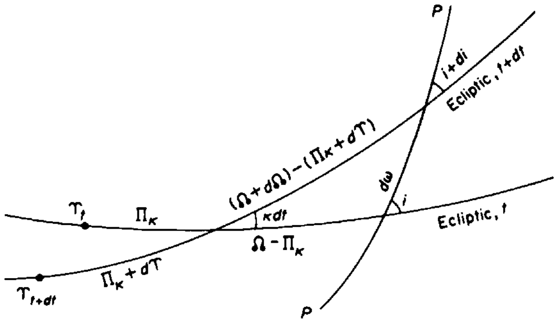

FIG. 51. Approximate variations of the ecliptic elements of a plane.

and similarly for ☊ and $\Delta\omega$; and by taking $t' = \frac{1}{2}(t_1 + t_2)$, i.e., midway between t_1 and t_2, only second or higher order errors are introduced. In practice, therefore, the equations that are usually used are

$$i = i_1 - \kappa \cos(\Omega - \Pi_\kappa) \cdot (t_2 - t_1)$$
$$\Omega = \Omega_1 + \{p + \kappa \sin(\Omega - \Pi_\kappa) \cot i\}(t_2 - t_1)$$
$$\Delta\omega = \kappa \sin(\Omega - \Pi_\kappa) \operatorname{cosec} i \cdot (t_2 - t_1)$$

in which for ☊, i, κ, Π_κ, p their values at time $\frac{1}{2}(t_1 + t_2)$ are to be used; when ☊ and i at this mid-time are not known, either estimated values (e.g., for the node, Ω_1 increased by half the precession in longitude over the interval from t_1 to t_2), or the initial values may be used, and the resulting first approximation to the required values then used to obtain a second approximation if necessary in order to secure the required accuracy.

Reduction of Equatorial Coordinates for Precession

From the triangle SP_0P (Fig. 52) the rigorous formulas for the reduction of the equatorial coordinates α_0, δ_0 of S, referred to the mean equinox and equator of t_0, to the values α, δ referred to the mean equinox and equator of t, are

$$\cos\delta \sin A' = \cos\delta_0 \sin A,$$
$$\cos\delta \cos A' = \cos J \cos\delta_0 \cos A - \sin J \sin\delta_0,$$
$$\sin\delta = \sin J \cos\delta_0 \cos A + \cos J \sin\delta_0,$$

in which $A = \alpha_0 + \zeta_0$, $A' = \alpha - z$, and J denotes the inclination of the mean equator at t to the mean equator at t_0. The required values of the reduction constants ζ_0, z, J are the values at time t referred to t_0 as epoch. If t_0 is not the fundamental epoch, they must be obtained from the general expressions in

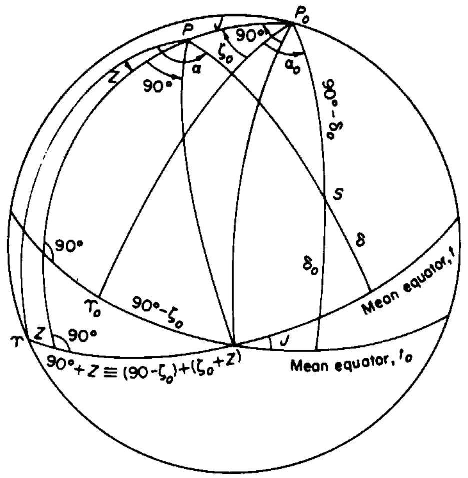

FIG. 52. Precessional variations of equatorial coordinates.

powers of the interval $t - t_0$ where t_0 and t are reckoned from the fundamental epoch.

These formulas evidently may be obtained immediately from the rigorous equations for the reduction of ecliptic coordinates by replacing (λ, β) by (α, δ) and

$$\begin{array}{ll} \Lambda_2 & \text{by } 90° + z \\ \Upsilon & \text{by } \mu = \zeta_0 + z \\ \pi_1 & \text{by } J \\ \Pi_1 & \text{by } 90° - \zeta_0 \end{array}$$

and these same replacements in the equations (151) and (152) into which the rigorous formulas were developed for practical numerical computation give

$$\begin{aligned} q &= \sin J[\tan \delta_0 + \cos(\alpha_0 + \zeta_0) \tan \tfrac{1}{2}J], \\ \tan[(\alpha - \alpha_0) - \mu] &= \frac{q \sin(\alpha_0 + \zeta_0)}{1 - q \cos(\alpha_0 + \zeta_0)}, \\ \mu &= \zeta_0 + z, \\ \tan \tfrac{1}{2}(\delta - \delta_0) &= \frac{\tan \tfrac{1}{2}J \cos\{(\alpha_0 + \zeta_0) + \tfrac{1}{2}[(\alpha - \alpha_0) - \mu]\}}{\cos \tfrac{1}{2}[(\alpha - \alpha_0) - \mu]}. \end{aligned} \tag{153}$$

These rigorous formulas are required only for stars near the pole when, because of either the great length of the interval $t - t_0$ or the very high declination, the change in position is an appreciable fraction of the polar distance.

Even with rigorous formulas, however, only approximate coordinates can be determined for remote dates, because the values of the reduction constants ζ_0, J, z can be found only by means of series in powers of the time, which cannot be relied upon to give accurate results for more than a few centuries on either side of the epoch.

At times which precede the epoch, the inclination J of the equator of date to the equator of epoch is reckoned negative; geometrically, the *descending* node is then at right ascension $90° - \zeta_0$, and the pole is at right ascension $180° - \zeta_0$, but this convention of algebraic sign avoids the necessity of taking explicit account of this, since ζ_0 and z are also negative. However, if desired, the initial and terminal times may be interchanged by replacing ζ_0 by $-z$, z by $-\zeta_0$, and J by $-J$.

The rigorous expressions may be expanded in series, either directly or by Maclaurin's theorem; and from these expansions, approximations of the form and accuracy required for any particular purpose may be obtained. To the second order, the left member of the second equation is $(\alpha - \alpha_0) - \mu$; developing the right member in powers of q, and expanding q in powers of J, we obtain, to the second order,

$$\alpha = \alpha_0 + (z + \zeta_0) + J\sin(\alpha_0 + \zeta_0)\tan\delta_0 + \tfrac{1}{4}J^2\sin 2(\alpha_0 + \zeta_0) + \tfrac{1}{2}J^2\sin 2(\alpha_0 + \zeta_0)\tan^2\delta_0,$$

$$\delta = \delta_0 + J\cos(\alpha_0 + \zeta_0) - \tfrac{1}{2}J^2\sin^2(\alpha_0 + \zeta_0)\tan\delta_0.$$

These forms may be expressed in terms of the mean obliquity of epoch ϵ^0 and powers and products of the lunisolar precession Ψ_1 in longitude and $\Delta\epsilon^0$ in obliquity on the fixed ecliptic of t_0, by substituting the development (127) of μ and the developments

$$J\cos\zeta_0 = \Psi_1\sin\epsilon^0 + \Psi_1\Delta\epsilon^0\cos\epsilon^0 - \tfrac{1}{6}\Psi_1^3\sin\epsilon^0 - \tfrac{1}{2}\Psi_1(\Delta\epsilon^0)^2\sin\epsilon^0 + \tfrac{1}{6}J^2\Psi_1\sin\epsilon^0,$$

$$J\sin\zeta_0 = -\Delta\epsilon^0 + \tfrac{1}{2}\Psi_1^2\sin\epsilon^0\cos\epsilon^0 + \tfrac{1}{6}(\Delta\epsilon^0)^3 + \tfrac{1}{2}\Psi_1^2\Delta\epsilon^0\cos^2\epsilon^0 - \tfrac{1}{6}J^2\Delta\epsilon^0,$$

which follow from the second and third of (125). The first-order terms are

$$\alpha = \alpha_0 - a + \Psi_1\cos\epsilon^0 + \Psi_1\sin\epsilon^0\sin\alpha_0\tan\delta_0 - \Delta\epsilon^0\cos\alpha_0\tan\delta_0,$$

$$\delta = \delta_0 + \Psi_1\sin\epsilon^0\cos\alpha_0 + \Delta\epsilon^0\sin\alpha_0.$$

Approximate formulas for practical use when the rigorous expressions are not required may be obtained by developing the coordinates in powers of the time by Maclaurin's theorem,

$$\alpha(t) = \alpha(0) + \left(\frac{d\alpha}{dt}\right)_0 t + \frac{1}{2}\left(\frac{d^2\alpha}{dt^2}\right)_0 t^2 + \cdots,$$

and similarly for δ. From the expressions (149) that have already been derived for the rates of precession in right ascension and declination,

$$\frac{d\alpha}{dt} = m + n \sin\alpha \tan\delta,$$

$$\frac{d\delta}{dt} = n \cos\alpha;$$

the higher order terms of the series may be found by differentiation. The second derivatives, which represent the secular variations of α and δ, are

$$\frac{d^2\alpha}{dt^2} = n^2 \sin 2\alpha\{\tfrac{1}{2} + \tan^2\delta\}$$

$$+ mn \tan\delta \cos\alpha + \frac{dm}{dt} + \frac{dn}{dt} \tan\delta \sin\alpha,$$

$$\frac{d^2\delta}{dt^2} = -n^2 \sin^2\alpha \tan\delta - mn \sin\alpha + \frac{dn}{dt} \cos\alpha;$$

but the expressions for the successive derivatives of higher order rapidly become too complex for practical application, and in practice various approximations are used, depending on the interval of time over which the reduction extends.

The expressions for the successive derivatives in the Maclaurin series, or the expansions of α and δ in terms of z, ζ_0, J, may be developed in powers of the time by substituting the power series in the time for m, n, J, etc.; and by this means, α and δ may be represented completely by series in powers of the time with numerical coefficients. The principal value of this representation is for the construction of tables to facilitate the computation of a large number of reductions; but the use of tables for this purpose has now been mostly superseded in practice since high-speed computing machinery has become available.

The reduction constants ζ_0, z, and J, which express the relative positions of the mean equator of date and mean equator of epoch, also determine the positions of the mean pole of date and mean pole of epoch relative to each other. At time t_2 reckoned from an epoch t_1, the coordinates of the mean celestial pole of date referred to the mean equinox and equator of the epoch t_1 are

$$\alpha_2 = 360° - \zeta_0, \qquad \delta_2 = 90° - J,$$

and the coordinates of the mean pole of the epoch referred to the mean equinox and equator of date are

$$\alpha_1 = 180° + z, \qquad \delta_1 = 90° - J,$$

where ζ_0, z, and J are the values at time t_2 referred to the epoch t_1. Expressing the reduction constants in terms of the coordinates of the pole, in the triangle formed by the two mean equators and the ecliptic of t_1 we have

$$\begin{aligned}
\sin\delta_2 &= \cos\epsilon^0 \cos\Theta_1 + \sin\epsilon^0 \sin\Theta_1 \cos\Psi'_1,\\
\cos\delta_2 \sin\alpha_2 &= \cos\epsilon^0 \sin\Theta_1 \cos\Psi'_1 - \sin\epsilon^0 \cos\Theta_1,\\
\cos\delta_2 \cos\alpha_2 &= \sin\Theta_1 \sin\Psi'_1;\\
\sin\delta_1 &= \cos\epsilon^0 \cos\Theta_1 + \sin\epsilon^0 \sin\Theta_1 \cos\Psi'_1,\\
\cos\delta_1 \sin(\alpha_1 + a) &= \sin\epsilon^0 \cos\Theta_1 \cos\Psi'_1 - \cos\epsilon^0 \sin\Theta_1,\\
\cos\delta_1 \cos(\alpha_1 + a) &= -\sin\epsilon^0 \sin\Psi'_1,
\end{aligned}$$

where ϵ^0 denotes the mean obliquity at t_1, Θ_1 is the inclination of the equator of t_2 to the ecliptic of t_1, and Ψ'_1 is the lunisolar precession in longitude, a the planetary precession, referred to the epoch t_1.

When t_2 precedes the epoch t_1, the reduction constants are negative. Algebraically, the pole of date is then at a declination $\delta_2 = (90° - J) > 90°$ in right ascension $\alpha_2 = 360° - \zeta_0$, which is to be interpreted geometrically as declination $(180° - \delta_2) < 90°$ in right ascension $\alpha_2 \pm 180°$; similarly for the position of the pole of epoch relative to the pole of date.

Rectangular Coordinates of Close Circumpolar Stars

The coordinates of close circumpolars may conveniently be reduced for precession by first transforming them to rectangular coordinates. The equatorial rectangular coordinates are given by the direction cosines

$$\begin{aligned}
x &= \cos\alpha \cos\delta,\\
y &= \sin\alpha \cos\delta,\\
z &= \sin\delta.
\end{aligned}$$

Differentiating with respect to t and substituting (149) gives for the instantaneous rates of precession

$$\frac{dx}{dt} = -my - nz,$$

$$\frac{dy}{dt} = +mx,$$

$$\frac{dz}{dt} = +nx.$$

After reducing the rectangular coordinates by means of these variations, they may be transformed back to α and δ.

Reduction of Equatorial Coordinates for Nutation

Evidently, a rigorous trigonometric reduction from the right ascension and declination α_0, δ_0, referred to the mean equinox and equator of date to the coordinates α, δ, referred to the true equinox and equator of date may be obtained by replacing, in the reduction for precession, J by the arc g_0 from the mean pole P_0 to the actual pole P (Fig. 53), ζ_0 by the angle G_0 between the mean equinoctial colure and the great circle through the two poles, and z by the angle G' at the true pole between the true equinoctial colure and the

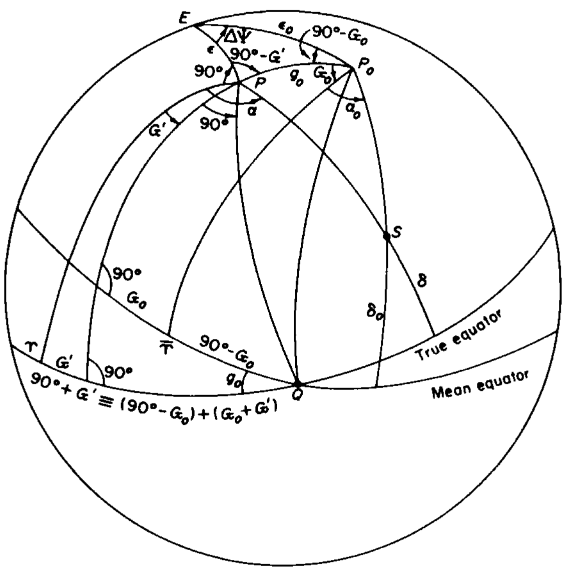

FIG. 53. Variations of equatorial coordinates due to nutation.

great circle through P_0, P:

$$\cos\delta\sin(\alpha - G') = \cos\delta_0\sin(\alpha_0 + G_0),$$
$$\cos\delta\cos(\alpha - G') = \cos g_0\cos\delta_0\cos(\alpha_0 + G_0) - \sin g_0\sin\delta_0,$$
$$\sin\delta = \sin g_0\cos\delta_0\cos(\alpha_0 + G_0) + \cos g_0\sin\delta_0,$$

where g_0, G_0, G' may be found in terms of the nutations $\Delta\Psi$, $\Delta\epsilon$, and the mean obliquity ϵ_0, either by (146) or from the triangle formed by P_0, P, and the pole E of the ecliptic of date, in which $\epsilon = \epsilon_0 + \Delta\epsilon$.

Since in the triangle EP_0P the angle $\Delta\Psi$ is always less than 20″ and the arc P_0P never exceeds the maximum value of $\Delta\epsilon$, about 10″, rigorous equations are not required for the determination of g_0, G_0, G'. A reduction that for all practical purposes is virtually rigorous may be obtained by replacing J, ζ_0, and z in the form (153) of the reduction for precession by the first-order approximations to g_0, G_0, and G' determined from the first-order terms of the developments (147) of $g_0\sin G_0$, $g_0\cos G_0$, and $G' + G_0$ in powers and products of $\Delta\Psi$ and $\Delta\epsilon$. Denoting by g_n, G_n, f_n, the approximations to g_0, G_0, and $G' + G_0$ which these first-order terms define, we have

$$g_n\sin G_n = -\Delta\epsilon,$$
$$g_n\cos G_n = \Delta\Psi\sin\epsilon_0,$$
$$f_n = \Delta\Psi\cos\epsilon_0;$$

and the reduction for nutation becomes

$$A_n = \alpha_0 + G_n,$$
$$p_n = g_n\tan\delta_0,$$
$$\tan(A'_n - A_n) = \frac{p_n\sin A_n}{1 - p_n\cos A_n},$$
$$\alpha = \alpha_0 + (A'_n - A_n) + f_n,$$
$$\delta = \delta_0 + g_n\cos\tfrac{1}{2}(A'_n + A_n)\sec\tfrac{1}{2}(A'_n - A_n).$$

Ordinarily, instead of this practically exact reduction, the expressions obtained from a development by Taylor's theorem are sufficient. Since nutation is caused entirely by the motion of the equator, the right ascension and declination referred to the true equinox and equator of date are functions of the quantities $\epsilon_0 + \Delta\epsilon$ and $\Psi + \Delta\Psi$ which fix the position of the equator relative to the ecliptic of date; and by Taylor's theorem,

$$\begin{aligned}\alpha &= \alpha(\Psi + \Delta\Psi, \epsilon_0 + \Delta\epsilon)\\ &= \alpha_0(\Psi, \epsilon_0) + \left(\frac{\partial\alpha}{\partial\Psi}\right)_0\Delta\Psi + \left(\frac{\partial\alpha}{\partial\epsilon}\right)_0\Delta\epsilon\\ &\quad + \frac{1}{2}\left(\frac{\partial^2\alpha}{\partial\Psi^2}\right)_0(\Delta\Psi)^2 + \left(\frac{\partial^2\alpha}{\partial\Psi\,\partial\epsilon}\right)_0\Delta\Psi\,\Delta\epsilon + \frac{1}{2}\left(\frac{\partial^2\alpha}{\partial\epsilon^2}\right)_0(\Delta\epsilon)^2 + \cdots,\end{aligned}$$

and similarly for δ, in which $\alpha_0(\Upsilon, \epsilon_0)$, $\delta_0(\Upsilon, \epsilon_0)$ are the mean right ascension and declination. Since $\Delta\Upsilon$ is the variation $\Delta\lambda$ in celestial longitudes, and $\Delta\beta = 0$, we have from the formulas for the relations between differential variations in the ecliptic and equatorial coordinate systems,

$$\frac{\partial\alpha}{\partial\Upsilon} = \frac{\partial\alpha}{\partial\lambda} = \cos\epsilon + \sin\epsilon\sin\alpha\tan\delta,$$

$$\frac{\partial\alpha}{\partial\epsilon} = -\cos\alpha\tan\delta,$$

and similarly

$$\frac{\partial\delta}{\partial\Upsilon} = \cos\alpha\sin\epsilon, \qquad \frac{\partial\delta}{\partial\epsilon} = \sin\alpha,$$

from which, by further differentiation, the higher derivatives may be obtained.

Equivalent expressions, to any desired order, may also be obtained from the direct expansion of the exact equations, which may be found by making the same substitutions as before in the expansion of the reduction for precession, with the further replacement of Υ_1 by $\Delta\Upsilon$, ϵ^0 by ϵ_0, and $\Delta\epsilon^0$ by $\Delta\epsilon$, and putting $a = 0$.

To the first order,

$$\begin{aligned} \alpha - \alpha_0 &= (\cos\epsilon_0 + \sin\epsilon_0\sin\alpha_0\tan\delta_0)\,\Delta\Upsilon - \cos\alpha_0\tan\delta_0\,\Delta\epsilon, \\ \delta - \delta_0 &= \sin\epsilon_0\cos\alpha_0\,\Delta\Upsilon + \sin\alpha_0\,\Delta\epsilon, \end{aligned} \tag{154}$$

which in terms of g_n, G_n, f_n become

$$\begin{aligned} \alpha - \alpha_0 &= f_n + g_n\sin(G_n + \alpha_0)\tan\delta_0, \\ \delta - \delta_0 &= g_n\cos(G_n + \alpha_0). \end{aligned} \tag{155}$$

From these expressions, with $\delta_0 = 0°$, the equation of the equinoxes is $\Delta\Upsilon\cos\epsilon_0$; cf. (147).

Besselian Reduction for Precession and Nutation

In practice, a position (α, δ) referred to the true equinox and equator of any particular date is usually found by first determining the mean place (α_0, δ_0) at the beginning of the Besselian year in which the date lies, then adding the further precession to date and the nutation at date. Over the fraction τ of a year, the first-order reductions for precession and nutation are, by (149) and (154),

$$\begin{aligned} \alpha - \alpha_0 &= \tau(m + n\sin\alpha_0\tan\delta_0), \\ \delta - \delta_0 &= \tau n\cos\alpha_0, \end{aligned} \tag{156}$$

for the precession to date, and

$$\begin{aligned}\alpha - \alpha_0 &= (\cos \epsilon_0 + \sin \epsilon_0 \sin \alpha_0 \tan \delta_0)\, \Delta\Psi - \cos \alpha_0 \tan \delta_0\, \Delta\epsilon,\\ \delta - \delta_0 &= \sin \epsilon_0 \cos \alpha_0\, \Delta\Psi + \sin \alpha_0\, \Delta\epsilon,\end{aligned} \qquad (157)$$

for the nutation at date.

These reductions may be combined and expressed in a form which facilitates the calculations. Denoting the annual rate of lunisolar precession on the fixed ecliptic of the beginning of the year by ψ' and the annual rate of planetary precession in right ascension by λ', we have for $\tau = 0$, from (131),

$$m = \psi' \cos \epsilon_0 - \lambda', \qquad n = \psi' \sin \epsilon_0;$$

and (157) may be written

$$\alpha - \alpha_0 = \left\{\frac{m + \lambda'}{\psi'} + \frac{n}{\psi'} \sin \alpha_0 \tan \delta_0\right\} \Delta\Psi - \Delta\epsilon \cos \alpha_0 \tan \delta_0,$$

$$\delta - \delta_0 = \Delta\Psi \frac{n}{\psi'} \cos \alpha_0 + \Delta\epsilon \sin \alpha_0.$$

The sum of Eqs. (156) and (157) then becomes

$$\begin{aligned}\alpha - \alpha_0 &= n\left\{\tau + \frac{\Delta\Psi}{\psi'}\right\}\left(\frac{m}{n} + \sin \alpha_0 \tan \delta_0\right)\\ &\quad + \lambda' \frac{\Delta\Psi}{\psi'} - \Delta\epsilon \cos \alpha_0 \tan \delta_0,\\ \delta - \delta_0 &= n\left\{\tau + \frac{\Delta\Psi}{\psi'}\right\} \cos \alpha_0 + \Delta\epsilon \sin \alpha_0;\end{aligned}$$

and putting

$$\begin{aligned}A &= n\left(\tau + \frac{\Delta\Psi}{\psi'}\right) = n\tau + \Delta\Psi \sin \epsilon_0,\\ B &= -\Delta\epsilon,\\ E &= \lambda' \frac{\Delta\Psi}{\psi'},\\ a &= \frac{m}{n} + \sin \alpha_0 \tan \delta_0,\\ b &= \cos \alpha_0 \tan \delta_0,\\ a' &= +\cos \alpha_0,\\ b' &= -\sin \alpha_0,\end{aligned} \qquad (158)$$

we have

$$\begin{aligned}\alpha - \alpha_0 &= Aa + Bb + E,\\ \delta - \delta_0 &= Aa' + Bb'.\end{aligned} \qquad (159)$$

This form of the reduction was originally introduced by Bessel, and in combination with the similar form of reduction for stellar aberration, Eq. (71), became a standard method for the practical calculation of apparent places of stars; but from time to time, different practices have been followed in the generally adopted notation and methods of obtaining the factors. The factors a, b, a', b', which depend upon the position are so nearly constant for any particular star that the same values can be used for each star over an extended interval of time, and these factors are therefore known as the *Besselian star constants*. The factors A, B, E, which depend upon the motions of the reference circles, are known as *Besselian star numbers* or *day numbers*, and are tabulated for each day in the national ephemerides. The terms in τ give the effects of precession, and the other terms give the effects of the nutation in longitude and in obliquity. Formerly, in many publications, the letters A and B were used to designate the aberration day numbers, while C, D, and E denoted the day numbers for precession and nutation. In the national ephemerides, the present practice became the standard after 1915, but *before 1960* the notation was

$$A = \tau + \frac{\Delta\Psi}{\psi'},$$
$$a = m + n \sin \alpha_0 \tan \delta_0,$$
$$a' = n \cos \alpha_0.$$

By computing g and G from the A and B defined in (158),

$$g \sin G = B,$$
$$g \cos G = A,$$

the precession from the beginning of the year to date is included in g, G along with the nutation; and putting

$$f = \frac{m}{n} A + E,$$

we have from (159)

$$\begin{aligned} \alpha - \alpha_0 &= f + g \sin(G + \alpha_0) \tan \delta_0, \\ \delta - \delta_0 &= g \cos(G + \alpha_0). \end{aligned} \tag{160}$$

Omitting τ from A, we obtain, either with A, B, E or with f, g, G the nutation alone; and Eqs. (160) then reduce to Eqs. (157), since $f = m\tau + \Delta\Psi \cos \epsilon_0$.

The quantities f, g, G, which are known as *independent day numbers*, are more convenient than A, B, E when only positions of a star at a few isolated dates are wanted, since the calculation of the star constants in (159) is avoided; but when an extended ephemeris is required, the Besselian day numbers are more advantageous.

Before 1960, in terms of the quantity *then* denoted by A,

$$g \cos G = nA,$$
$$f = mA + E.$$

The Second-Order Reduction for Precession and Nutation

An extension of the Besselian first-order reduction to the second order, expressed in terms of either the Besselian day numbers or the independent day numbers, may be constructed from the second-order terms of the Taylor series for the precession and nutation, or from direct expansions of the rigorous formulas.

When second-order terms are included, the reduction from the mean equinox and equator to the true equinox and equator must be distinguished from the reverse reduction. Moreover, the corrections for precession to date must be applied to the initial coordinates α_0, δ_0 referred to the mean equinox and equator, *before* the correction for nutation is determined, because the expressions that have been obtained dynamically for the nutation are in terms of quantities referred to the mean equinox of date, and explicitly represent a displacement of the reference circles from their mean positions; but α_0 and δ_0 may be the coordinates either of the geometric position on the celestial sphere or of the position displaced by aberration.

To determine the reduction for precession from the beginning of the year to date, the beginning of the year is taken as the epoch, and the time is measured by the fraction τ of the year elapsed since this epoch. With the right ascension and declination referred to the mean equinox and equator of the beginning of the year denoted by α_0, δ_0, the reduction to the values α_1, δ_1 referred to the mean equinox and equator of date may be obtained immediately from the Maclaurin series for precession in right ascension and declination by putting $t = \tau$, and substituting the values of the derivatives for $\tau = 0$; to the second order

$$\begin{aligned}\alpha_1 - \alpha_0 = {} & +\{m_0 + n_0 \sin\alpha_0 \tan\delta_0\}\tau \\ & + \Big\{\tfrac{1}{4}n_0^2 \sin 2\alpha_0 + \tfrac{1}{2}m_0 n_0 \cos\alpha_0 \tan\delta_0 \\ & + \tfrac{1}{2}n_0^2 \sin 2\alpha_0 \tan^2\delta_0 + \frac{1}{2}\left(\frac{dm}{dt}\right)_0 + \tfrac{1}{2}\sin\alpha_0 \tan\delta_0\left(\frac{dn}{dt}\right)_0\Big\}\tau^2,\end{aligned}$$

$$\begin{aligned}\delta_1 - \delta_0 = {} & +(n_0 \cos\alpha_0)\tau - \frac{1}{2}\Big\{n_0^2 \sin^2\alpha_0 \tan\delta_0 \\ & + m_0 n_0 \sin\alpha_0 - \cos\alpha_0\left(\frac{dn}{dt}\right)_0\Big\}\tau^2,\end{aligned}$$

where m_0, n_0 are the annual rates of general precession in right ascension and precession in declination at the beginning of the year. The same expressions may be obtained from the expansions of the rigorous equations in powers and products of Ψ_1 and $\Delta\epsilon^0$ by substituting for Ψ_1, $\Delta\epsilon^0$, and a the Maclaurin series

$$\Psi_1(\tau) = \Psi_1(0) + \left(\frac{d\Psi_1}{dt}\right)_0 \tau + \cdots, \text{ etc.,}$$

and expressing the results in terms of m_0 and n_0; the displacements Ψ_1, a, and $\Delta\epsilon^0$ vanish at $\tau = 0$, since this is the epoch.

The nutation at date, which is added to α_1, δ_1, to obtain the right ascension and declination α, δ referred to the true equinox and equator of date, may likewise be obtained either from the Taylor series

$$\begin{aligned} \alpha &= \alpha(\Psi + \Delta\Psi, \epsilon_0 + \Delta\epsilon) \\ &= \alpha_1 + \left(\frac{\partial\alpha}{\partial\Psi}\right)_1 \Delta\Psi + \left(\frac{\partial\alpha}{\partial\epsilon}\right)_1 \Delta\epsilon + \cdots, \text{ etc.,} \end{aligned}$$

or from the expressions obtained by direct expansion of the exact equations,

$$\begin{aligned} \alpha - \alpha_1 &= (G' + G_0) + g_0 \sin(\alpha_1 + G_0) \tan\delta_1 \\ &\quad + \tfrac{1}{4} g_0^2 \sin 2(\alpha_1 + G_0) + \tfrac{1}{2} g_0^2 \sin 2(\alpha_1 + G_0) \tan^2\delta_1 + \cdots, \\ \delta - \delta_1 &= g_0 \cos(\alpha_1 + G_0) - \tfrac{1}{2} g_0^2 \sin^2(\alpha_1 + G_0) \tan\delta_1 + \cdots. \end{aligned}$$

From the definitions of the day numbers, and the expansions (147) of $G' + G_0$, $g_0 \sin G_0$, $g_0 \cos G_0$ in powers and products of $\Delta\Psi$ and $\Delta\epsilon$,

$$\begin{aligned} G' + G_0 &= \left(\frac{m}{n} A' + E\right) + \tfrac{1}{2} A'B + \frac{A'^2}{12}\left(\frac{m}{n} A' + E\right) \\ &= f - m\tau - \tfrac{1}{2} n\tau g \sin G + \tfrac{1}{4} g^2 \sin 2G, \\ g_0 \sin G_0 &= B + \tfrac{1}{2} A'\left(\frac{m}{n} A' + E\right) - \tfrac{1}{6} B^3 - \tfrac{1}{2} B\left(\frac{m}{n} A' + E\right)^2 + \tfrac{1}{6} g_0^3 \sin G_0 \\ &= g \sin G + \tfrac{1}{2} fg \cos G - \tfrac{1}{2} m\tau g \cos G - \tfrac{1}{2} n\tau(f - m\tau), \\ g_0 \cos G_0 &= A' - B\left(\frac{m}{n} A' + E\right) - \tfrac{1}{2} A'B^2 - \tfrac{1}{6} A'^3 \operatorname{cosec}^2\epsilon_0 + \tfrac{1}{6} g_0^3 \cos G_0 \\ &= g \cos G - fg \sin G + m\tau g \sin G - n\tau, \end{aligned}$$

in which $A' = A - n\tau$; similarly, the successive derivatives in the Taylor series may be expressed in terms of m, n and either the Besselian or the independent day numbers. To the second order, the equation of the equinoxes is

$$G' + G_0 = \Delta\Psi \cos \epsilon_0 - \tfrac{1}{2}\Delta\Psi\, \Delta\epsilon \sin \epsilon_0.$$

Substituting the expressions for α_1, δ_1, and putting

$$m = m_0 + \left(\frac{dm}{dt}\right)_0 \tau + \cdots, \qquad n = n_0 + \left(\frac{dn}{dt}\right)_0 \tau + \cdots,$$

we obtain for the reduction from α_0, δ_0 to α, δ as far as the second order, in terms of the independent day numbers,

$$\begin{aligned}\alpha = \alpha_0 &+ f + g \sin(G + \alpha_0) \tan \delta_0 \\ &+ \tfrac{1}{4}g^2 \sin 2(G + \alpha_0) + \tfrac{1}{4}g^2 \sin 2G \\ &+ \tfrac{1}{2} fg\{\cos G \cos \alpha_0 - 2 \sin G \sin \alpha_0\} \tan \delta_0 \\ &+ \tfrac{1}{2}g^2 \sin 2(G + \alpha_0) \tan^2\delta_0, \end{aligned} \qquad (161)$$

$$\begin{aligned}\delta = \delta_0 &+ g \cos(G + \alpha_0) \\ &- fg\{\tfrac{1}{2} \cos G \sin \alpha_0 + \sin G \cos \alpha_0\} \\ &- \tfrac{1}{4}g^2\{1 - \cos 2(G + \alpha_0)\} \tan \delta_0.\end{aligned}$$

Differential Precession and Nutation

Since the values of the reductions $\alpha - \alpha_0$ and $\delta - \delta_0$ depend upon the values of the right ascension and declination, the reductions for two different objects will not be the same, and consequently the differences $\Delta\alpha$ and $\Delta\delta$ of the coordinates of the two objects referred to the true equator and equinox of date are not equal to the differences $\Delta\alpha_0$ and $\Delta\delta_0$ of the coordinates referred to the mean equinox and equator of the beginning of the year. To the first order

$$\Delta\alpha = \Delta\alpha_0 + \frac{\partial(\alpha - \alpha_0)}{\partial\alpha}\Delta\alpha + \frac{\partial(\alpha - \alpha_0)}{\partial\delta}\Delta\delta,$$

$$\Delta\delta = \Delta\delta_0 + \frac{\partial(\delta - \delta_0)}{\partial\alpha}\Delta\alpha + \frac{\partial(\delta - \delta_0)}{\partial\delta}\Delta\delta;$$

and differentiating (159) gives

$$\begin{aligned}\Delta\alpha &= \Delta\alpha_0 + (A \cos \alpha - B \sin \alpha) \tan \delta\, \Delta\alpha \\ &\qquad + (A \sin \alpha + B \cos \alpha) \sec^2\delta\, \Delta\delta \\ &= \Delta\alpha_0 + g \cos(G + \alpha) \tan \delta\, \Delta\alpha + g \sin(G + \alpha) \sec^2\delta\, \Delta\delta, \\ \Delta\delta &= \Delta\delta_0 - (A \sin \alpha + B \cos \alpha)\, \Delta\alpha \\ &= \Delta\delta_0 - g \sin(G + \alpha)\, \Delta\alpha,\end{aligned}$$

where g is in radians, and α and δ on the right are the values midway from the beginning of the year to date.

Precession and Nutation in Position Angle

The variation of the position angle p of the arc s from a point P_1 to a nearby point P_2 is entirely due to the lunisolar motion of the celestial pole, and is the negative of the variation of the angle η at P_1 in the triangle formed by P_1, the celestial pole and the pole of the ecliptic. By differentiation of this triangle, with β constant,

$$\cos \delta \, d\eta = \cos \alpha \, d\epsilon - \sin \epsilon \sin \alpha \, d\lambda,$$

and to the first order,

$$\Delta p = \{\psi' \sin \epsilon \sin \alpha - \cos \alpha \, \Delta\epsilon'\} \sec \delta,$$

where ψ' and $\Delta\epsilon'$ denote the total amounts of lunisolar precession and nutation in longitude and in obliquity.

From the precession alone, neglecting the lunisolar precession in obliquity,

$$dp = n \sin \alpha \sec \delta.$$

In terms of the independent star numbers, the expression for Δp becomes

$$\Delta p = g \sin(G + \alpha) \sec \delta,$$

which may also be obtained by differentiating $s_0 \cos p_0 = \Delta\delta_0$, taking for the derivative of the right-hand member the expression for $\Delta\delta - \Delta\delta_0$, and in the derivative of the left-hand member substituting $s_0 \sin p_0 = \cos \delta \, \Delta\alpha$.

The position angle of the ninth magnitude companion of Polaris, at a distance of 18″, has increased by 14° since 1780, but 10° of this variation is due to the change in the coordinate system from precession.

Variations of Equatorial Rectangular Coordinates from Precession and Nutation

The coordinate axes of a rectangular system that has its xy-plane in the plane of the mean equator, with the x-axis directed toward the mean equinox, are continually changing their directions in space because of the precessional motions of the mean equator and the mean equinox. At any time t_2 the angles that these axes make with the directions which they had at any other time t_1 are given by

$$\begin{aligned}
\cos(x_2, x_1) &= -\sin\zeta_0 \sin z + \cos\zeta_0 \cos z \cos J,\\
\cos(x_2, y_1) &= -\cos\zeta_0 \sin z - \sin\zeta_0 \cos z \cos J,\\
\cos(x_2, z_1) &= -\cos z \sin J,\\
\cos(y_2, x_1) &= +\sin\zeta_0 \cos z + \cos\zeta_0 \sin z \cos J,\\
\cos(y_2, y_1) &= +\cos\zeta_0 \cos z - \sin\zeta_0 \sin z \cos J,\\
\cos(y_2, z_1) &= -\sin z \sin J,\\
\cos(z_2, x_1) &= +\cos\zeta_0 \sin J,\\
\cos(z_2, y_1) &= -\sin\zeta_0 \sin J,\\
\cos(z_2, z_1) &= +\cos J;
\end{aligned}$$

these relations may be derived either from the corresponding equations for ecliptic coordinates by replacing π_1 by J, Π_1 by $90° - \zeta_0$, and ♈ by $\zeta_0 + z$, or else directly from the appropriate triangles on the sphere.

Hence, with $\cos J = 1 - 2\sin^2\tfrac{1}{2}J$ and similarly for $\cos(\zeta_0 + z)$, the equations for transforming the equatorial rectangular coordinates (x_1, y_1, z_1) of any body referred to the mean equinox and equator of t_1, to the coordinates (x_2, y_2, z_2) referred to the mean equinox and equator of t_2 are

$$\begin{aligned}
x_2 = x_1 &- 2\{\sin^2\tfrac{1}{2}(\zeta_0 + z) + \cos\zeta_0 \cos z \sin^2\tfrac{1}{2}J\}x_1\\
&+ \{2\sin\zeta_0 \cos z \sin^2\tfrac{1}{2}J - \sin(\zeta_0 + z)\}y_1\\
&- \{\cos z \sin J\}z_1,\\
y_2 = y_1 &+ \{\sin(\zeta_0 + z) - 2\cos\zeta_0 \sin z \sin^2\tfrac{1}{2}J\}x_1\\
&- 2\{\sin^2\tfrac{1}{2}(\zeta_0 + z) - \sin\zeta_0 \sin z \sin^2\tfrac{1}{2}J\}y_1\\
&- \{\sin z \sin J\}z_1,\\
z_2 = z_1 &+ (\cos\zeta_0 \sin J)x_1\\
&- (\sin\zeta_0 \sin J)y_1\\
&- (2\sin^2\tfrac{1}{2}J)z_1,
\end{aligned}$$

in which the values of ζ_0, z, J at t_2 must be obtained from the general expressions for these quantities in terms of the time $t_2 - t_1$ reckoned from the arbitrary epoch t_1, where t_1 is reckoned from an adopted fundamental epoch t_0. The inverse transformation may be made with the same coefficients by interchanging the rows and columns of the matrix of the transformation.

The further corrections for nutation that are required to obtain rectangular coordinates referred to the true equinox and equator may be derived by differentiating the formulas (17) for the equatorial coordinates x, y, z with respect to ecliptic longitude λ and obliquity ϵ, and expressing the results in terms of x, y, z, ϵ, $\Delta\epsilon$, $\Delta\Upsilon$. To the first order,

$$\begin{aligned} dx &= -r \cos \beta \sin \lambda \, d\lambda \\ &= -r(\sin \epsilon \sin \delta + \cos \epsilon \cos \delta \sin \alpha) \, d\lambda \\ &= -(y \cos \epsilon + z \sin \epsilon) \, \Delta\Upsilon, \end{aligned}$$

and similarly

$$dy = +x \cos \epsilon \, \Delta\Upsilon - z \, \Delta\epsilon,$$

$$dz = +x \sin \epsilon \, \Delta\Upsilon + y \, \Delta\epsilon.$$

Precessional Variations of the Elements That Determine the Position of a Plane in Space Relative to the Plane of the Equator

When the plane of the celestial equator at any particular epoch t is adopted as a fundamental reference plane for the purpose of characterizing the position of a plane P in space by means of the node Ω and inclination i of P on the equator, the variations of Ω and i with t because of the precessional motions of the equator and the equinox may evidently be obtained from the corresponding relations for the ecliptic by replacing in the rigorous equations the inclination π_1 and node Π_1 of the ecliptic at t_2 on the ecliptic of t_1 by the inclination J and node $90° - \zeta_0$ of the equator at t_2 on the equator of t_1 (the right ascension $90° - \zeta_0$ of the node being reckoned from the mean equinox of t_1), and the amount Υ of general precession in longitude from t_1 to t_2 by the amount of general precession in right ascension (see Fig. 54).

In the differential relations, which are sufficiently accurate in practice for most purposes, the rate of rotation κ of the ecliptic about an axis in longitude

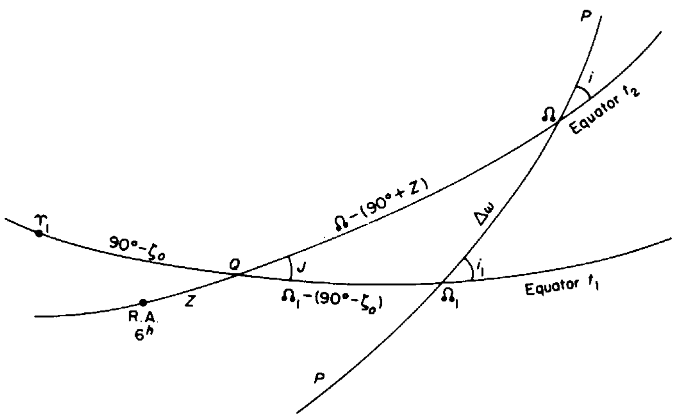

FIG. 54. Precessional variations of the equatorial elements of a plane.

Π_κ from the mean equinox of date is replaced by the rate of rotation n of the equator around an axis in right ascension 6^h from the mean equinox of date, and the rate p of general precession in longitude is replaced by the rate m of general precession in right ascension. Hence,

$$\Delta i = -n \sin \Omega \cdot (t_2 - t_1)$$
$$\Delta \Omega = +\{m - n \cos \Omega \cot i\}(t_2 - t_1)$$
$$\Delta \omega = -n \cos \Omega \operatorname{cosec} i \cdot (t_2 - t_1)$$

in which for Ω, i, n, m their values at time $\frac{1}{2}(t_1 + t_2)$ are used.