Oscillations of a Quantum Electron Gas in a Uniform Magnetic Field

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I. Introduction

Recent experimental efforts to extract information on the electronic structure of metals have involved such techniques as the measurement of electron cyclotron resonance, or of ultrasonic attenuation in the presence of a magnetic field. As an aid in the theoretical understanding of the behavior of metallic electrons in a magnetic field, we would like to describe some aspects of the much simpler problem of an electron gas in a uniform magnetic field. Although the classical version of this problem has been extensively treated (*1*, *2*, *3*, *4*), a quantummechanical analysis is necessary if one wishes to apply the results to metallic electrons. The case of ultrasonic attenuation of an electron gas in a uniform magnetic field has been examined in a semiclassical approximation by M. H. Cohen *et al.* (*5*) and later in the quantummechanical random-phase approximation by Quinn and Rodriguez (*6*).

From a different point of view, RPA derivations of the spectrum of density oscillations have been given by P. S. Zyryanov (*7*) and M. Stephen (*8*). The resulting dispersion relation turns out to have a rather complicated structure, and a considerable amount of additional analysis is required to arrive at a clear picture of what the resonances predicted by the RPA are like. We would like to describe the results of such an analysis in the limit of long wavelengths, giving the location of the resonant frequency, a discussion of their damping, and a semiclassical

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picture of the collective motions involved. We shall only state results, referring the reader to reference (*9*) for the mathematical details.

II. RPA Calculation of the Resonances

We can find the resonant frequencies by calculating the response of an electron gas in a uniform magnetic field, initially in thermal equilibrium, to a weak external potential, $U(\mathbf{r}, t)$. In the random phase approximation, coulomb interactions between electrons are replaced by a self-consistent single particle potential,

$$V(\mathbf{r}, t) = \int d\mathbf{r}' \frac{e^2}{|\mathbf{r} - \mathbf{r}'|} [\varrho(\mathbf{r}', t) - \varrho^0] ,$$

where $\varrho(\mathbf{r}, t)$ is the density of particles, and $\varrho^0 = N/V$ represents a uniform background of positive charge. The single-particle density matrix φ therefore obeys the equation of motion:

$$i\dot{\varphi} = [H_0 + V + U, \varphi] , \tag{1}$$

where H_0 is the Hamiltonian for a single electron in a magnetic field. We look for a solution to Eq. (1) of the form

$$\varphi = \varphi^0 + \varphi^1 ,$$

where φ^0 is the equilibrium single-particle density matrix, and φ^1 is the small change induced by U. Since the self-consistent field vanishes in thermal equilibrium, φ^0 is just the density matrix for noninteracting electrons in a uniform magnetic field. Linearizing Eq. (1) in U and φ^1 gives:

$$i\dot{\varphi} = (H_0, \varphi^1) + (V + U, \varphi^0) . \tag{2}$$

Equation (2) is to be solved subject to the boundary condition that φ^1 vanishes at times before the perturbing field has been turned on.

Equation (2) can be solved exactly for the one-particle density matrix, in terms of which one can express the Fourier transform of the change in the density of particles induced by U. Details of the calculation can be found in ref. (7) or (9). The result (in units where $\hbar = 1$) is

$$\varrho^1(\mathbf{k}, \omega) = L(\mathbf{k}, \omega)\, U(\mathbf{k}, \omega)\,, \tag{3}$$

where L, the linear response function, has the form

$$L(\mathbf{k}, \omega) = L^0(\mathbf{k}, \omega) \Big/ \left[1 - \frac{4\pi e^2}{k^2} L^0(\mathbf{k}, \omega) \right], \tag{4}$$

$$L^0(\mathbf{k}, \omega) = m\omega_c \sum_{nn'} \int \frac{dp}{(2\pi)^2} \frac{f_n(p - k_{\parallel}/2) - f_{n'}(p + k_{\parallel}/2)}{\omega - \dfrac{pk_{\parallel}}{m} + (n - n')\,\omega_c} C_{nn'}(k_{\perp}), \tag{5}$$

$k_{\parallel}$ and $k_{\perp}$ are the components of $\mathbf{k}$ parallel and perpendicular to the magnetic field, and ω_c is the cyclotron frequency $\omega_c = eH/mc$. The coefficients $C_{nn'}$ are given by the squared matrix element of the plane wave $\exp(ik_{\perp} x)$ between two eigenstates of a one-dimensional oscillator with mass m and frequency ω_c:

$$C_{nn'}(k_{\perp}) = |\langle n' \,|\, \exp(ik_{\perp} x) \,|\, n \rangle|^2 .$$

For our analysis we shall only need the facts that $C_{nn'}$ vanishes as

$$k_{\perp}^{2\,|n-n'|}$$

as

$$k_{\perp} \to 0\,;$$

$$C_{n,n+1} \approx (n + 1)\, k_{\perp}^2/2m\omega_c\,;$$

and

$$C_{nn} \approx 1 - (n + \tfrac{1}{2})\, k_{\perp}^2/2m\omega_c\,.$$

Finally,

$$f_n(p) = 1 \,/\, (\exp[\beta(p^2/2m + n\omega_c - \mu)] + 1)$$
$$+ 1 \,/\, (\exp[\beta(p^2/2m + (n + 1)\omega_c - \mu)] + 1)\,,$$

is a sum of Fermi functions for the two possible spin orientations.

The long wavelength resonances determined by the poles of $L(\mathbf{k}, \omega)$ fall naturally into two classes. Two resonances persist in the limit of infinite wavelength. They involve motions resulting from a competition between the Lorentz force and the tendency to execute plasma oscillations along the direction of propagation. When k is small but not zero there are additional modes with frequencies close to $\pm n\omega_c$, where n is an integer greater than or equal to 2. These are very weakly excited by long wavelength disturbances and undergo heavy Landau damping unless the direction of propagation is perpendicular to the magnetic field, or unless the temperature is very close to zero.

The frequencies of the first type of resonance are given to lowest order by the poles of L when $k = 0$. These occur at the roots $\omega_\pm$, of

$$1 = \frac{\omega_p^2 \cos^2\theta}{\omega^2} + \frac{\omega_p^2 \sin^2\theta}{\omega^2 - \omega_c^2},$$

where θ is the angle between $\mathbf{k}$ and $\mathbf{H}$ and ω_p is the plasma frequency,

$$\omega_p = 4\pi\varrho^0 e^2/m .$$

The residue of L at the positive roots $\omega_\pm$ are

$$\mathbf{r}_\pm = \pm \frac{\varrho k^2}{2m} \frac{|\omega_\pm|(\omega_\pm^2 - \omega_c^2)}{\omega_p^2(\omega_p^2 - \omega_+^2)},$$

and $-r_\pm$ at the roots $-\omega_\pm$. In general when the linear response function is a sum of first order poles at ω_i with residue r_i, the longitudinal sum rule,

$$\sum \mathbf{r}_i \omega_i = \varrho k^2/m ,$$

is satisfied. In the present case the four poles at $\pm\omega_\pm$ exhaust the sum rule as $k \to 0$.

These resonant frequencies are the same as those found by Gross in the long wavelength limit of the analogous classical problem. They have a rather simple interpretation, being just the normal frequencies of a single electron, moving in the magnetic field $\mathbf{H}$, and in a one-dimensional oscillator well along the direction of $\mathbf{k}$, with frequency ω_p; i.e., they are the eigenvalues of the one-electron equation:

$$-m\omega^2\mathbf{r} = -m\omega_p^2\,\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{r}) - i\frac{\omega e}{c}\mathbf{r}\times\mathbf{H} .$$

Belonging to the eigenvalue $\omega_\pm$ of the one-electron problem, is a normal mode in which the electron moves on an elliptical orbit in velocity space with axes proportional to

$$(\omega_c/\omega_\pm)\,(\hat{\mathbf{H}}\times\hat{\mathbf{k}})$$

and

$$\hat{\mathbf{k}} - (\omega_c/\omega_\pm)^2\,\hat{\mathbf{H}}(\hat{\mathbf{H}}\cdot\hat{\mathbf{k}})\,.$$

The behavior of such a single electron is helpful in visualizing the collective motion of the electron gas when semiclassic conditions hold — i.e., when the wavelength is long enough and the magnetic field weak enough to permit the calculation of a joint single-particle distribution function in position and velocity. Under these conditions the self-consistent field approximation enables one to calculate the time-dependent joint-distribution function by using the linearized Boltzmann-Vlasov equation, in which the initial equilibrium electronic distribution function is given by a Fermi distribution. From the solution of this equation one can show that at a long wavelength resonance, the collective motion is such that the velocity distribution at each point moves without distortion along the same orbit in velocity space as that belonging to the corresponding normal mode of the one-electron problem.

This analogy to the one-electron problem gives a simple picture of the collective motion in several limiting cases:

(a) Propagation parallel to **H** ($\sin\theta \approx 0$):

$$\omega_+^2 = \omega_p^2\left[1 + \frac{\sin^2\theta\,\omega_c^2}{\omega_p^2 - \omega_c^2} + 0(\sin^4\theta)\right],$$

$$\omega_-^2 = \omega_c^2\left[1 - \frac{\sin^2\theta\,\omega_p^2}{\omega_p^2 - \omega_c^2} + 0(\sin^4\theta)\right].$$

As $\sin\theta \to 0$, the mode belonging to ω_+ reduces to a linear plasma oscillation along the direction of **k** and **H**; the other mode reduces to uniform circular motion about this direction with frequency ω_c. When $\sin\theta$ vanishes, the plasmon alone exhausts the sum rule, reflecting the impossibility of exciting a purely transverse motion by a longitudinal perturbation.

(b) Propagation perpendicular to $\mathbf{H}\,(\cos\theta \approx 0)$:

$$\omega_+^2 = \omega_p^2 + \omega_c^2 - \frac{\cos^2\theta\,\omega_p^2\,\omega_c^2}{\omega_p^2 + \omega_c^2} + 0\,(\cos^4\theta)\,,$$

$$\omega_-^2 = \frac{\omega_p^2\,\omega_c^2\cos^2\theta}{\omega_p^2 + \omega_c^2} + 0\,(\cos^4\theta)\,.$$

As $\cos\theta \to 0$ the ω_- mode reduces to uniform translation parallel to **H** and perpendicular to **k**, and its relative contribution to the sum rule vanishes. The dominant mode belongs to ω_+, and involves an elliptical motion in the plane perpendicular to **H**, with axis parallel to **k** being greater than the perpendicular axis by a factor

$$(1 + \omega_p^2/\omega_c^2)^{1/2}\,.$$

(c) Weak magnetic field or high density $(\omega_p^2 \gg \omega_c^2)$:

$$\omega_+^2 = \omega_p^2[1 + \sin^2\theta\,(\omega_c/\omega_p)^2 + 0\,(\omega_c/\omega_p)^4]\,,$$

$$\omega_-^2 = \omega_c^2\cos^2\theta\,[1 + 0\,(\omega_c/\omega_p)^2].$$

The mode ω_- is what one would find by ignoring the component of **H** perpendicular to **k** and applying case (a); this is because the strong coulomb interaction leads to a rigidity against low-frequency long-wavelength longitudinal oscillations, and hence suppresses precession about the component of **H** perpendicular to **k**. The other root is essentially the plasmon; the way in which it is modified by the weak magnetic field is just what one would find by ignoring the component of **H** parallel to **k** and applying case (b). The plasmon again dominates the sum rule, since

$$\mathbf{r}_-/\mathbf{r}_+ \approx (\omega_c/\omega_p)^3\cos\theta\,\sin^2\theta\,.$$

(d) Strong magnetic field or low density $(\omega_c^2 \gg \omega_p^2)$:

$$\omega_+^2 = \omega_c^2[1 + \sin^2\theta\,(\omega_p/\omega_c)^2 + 0\,(\omega_p/\omega_c)^4]\,,$$

$$\omega_-^2 = \omega_p^2\cos^2\theta\,[1 + 0\,(\omega_p/\omega_c)^2].$$

The root ω_- can be regarded as a plasmon in which, because of the strong magnetic field, the particles move parallel to **H** instead of **k**. The other root is essentially the unperturbed cyclotron resonance; the weak coulomb interaction changes the circular orbit about **H** to an elliptical orbit perpendicular to **H** with minor axis perpendicular to **k**. The oblique plasmon still dominates the sum rule, since

$$\mathbf{r}_+/\mathbf{r}_- \approx (\omega_p/\omega_c)\,(\sin^2\theta/\cos\theta).$$

Effects which are not classical first appear in the lowest order **k** dependent terms of the resonant frequencies. Quantum effects arise here because these terms depend on the mean equilibrium energy of an electron in a magnetic field. When orbit quantization becomes important this energy will develop an oscillatory dependence on the magnetic field. These oscillations are, however, small corrections to the much larger contributions to the energy which are independent of and linear in **H**, and should therefore be very hard to observe.

Another effect which arises when $\mathbf{k} \neq 0$ is the familiar Landau damping, which occurs when it is possible for the collective mode to give up its energy and momentum by exciting an electron to a higher state. In the absence of a magnetic field Landau damping of the plasmon is small, because in order for a particle to absorb the energy ω_p and the small momentum **k**, it must have a very large initial momentum, and is therefore unlikely to be available at low temperatures. In a magnetic field, however, only the momentum parallel to **H** must be conserved. A collective state with $\omega \gg \omega_c$, can give up the bulk of its energy by exciting an electron to a level with higher oscillator quantum number n, leaving only a small remainder to be absorbed by a change in the electron's momentum parallel to **H**. The initial momentum parallel to **H** that such a particle must have is therefore enormously reduced, and the probability of its being available correspondingly enhanced. In the limit of small **k** these factors giving the density of available particles more than outweigh the dynamical transition probabilities in determining the size of the lifetime. One can therefore conclude that a magnetic field increases the Landau damping of the plasmon-like modes.

This feature of the Landau damping becomes very important in the case of the weakly excited higher order cyclotron resonances, occurring very close to $\pm n\omega_c$. Because all but a minute part of their energy can be absorbed by excitation of an electron to a higher oscillator level, there will be many electrons available to absorb their momentum parallel to **H** while taking up the very small remaining energy, unless the wave carries no momentum parallel to **H**. Their Landau damping is therefore very large at nonzero temperatures, unless **k** is perpendicular to **H**. Their behavior in the two cases in which they do not undergo severe Landau damping — perpendicular propagation and zero temperature — is quite different.

A. Perpendicular Propagation

When $k_{\parallel}$ is zero and $k_{\perp}$ is small, resonances occur at energies ω_n very close to $n\omega_c$, for $|n| \geq 2$. It can be shown that the amount by which ω_n differs from $n\omega_c$ is proportional to $\mathbf{k}^{2n-1}$; the magnitude of ω_n exceeds $|n\omega_c|$ if $|n\omega_c| > (\omega_p^2 + \omega_c^2)^{1/2}$; otherwise it is less than $|n\omega_c|$. The residue at the nth higher order cyclotron resonance vanishes as k^{2n}, and they are therefore progressively more difficult to excite. The Boltzman-Vlasov equation furnishes a picture of the collective motions associated with these resonances in the semiclassical case. At a given time the local velocity distribution has the following structure: its (spherically symmetric) equilibrium value is enhanced in n directions perpendicular to **H** spaced $2\pi/n$ apart, and diminished in between these directions. As time evolves this distorted velocity distribution rotates about the direction of **H** with frequency ω_c. This behavior can be understood as arising from n separate groups of electrons, all undergoing ordinary cyclotron motions with frequency ω_c, but so arranged that at any given point of space the density goes through a maximum and minimum n times in a single period.

B. Zero Temperature

In a fixed magnetic field, as **k** approaches zero, undamped higher order cyclotron resonances occur in the response function for arbitrary directions of propagation. When **k** is perpendicular to **H** their wavelength dependence is as described above, but for any other direction of propagation their behavior is quite different for sufficiently small **k**. The analysis one goes through to derive the properties of these resonances from Eq. (5) is quite similar to the usual RPA derivations of zero sound. One finds that for small k resonances occur at $\pm\,\omega_n$, where

$$\omega_n = n\omega_c + |\,\mathbf{k}\,V_F \cos\theta\,|, \quad n \geq 2, \quad V_F = (2\mu/m)^{1/2},$$

when $|\,n\omega_c\,|$ exceeds the two long-wavelength resonant frequencies $|\,\omega_\pm\,|$, and otherwise at

$$\omega_n = n\omega_c - |\,\mathbf{k}V_F \cos\theta\,|\,.$$

In a magnetic field the Fermi velocity, v_F is defined only along directions parallel to **H**. The magnitude of the group velocity of these resonances is therefore given by the projection of the Fermi velocity on **k**; the group velocity has a positive component along **H** when $|\,n\omega_c\,| > |\,\omega_\pm\,|$, and a negative component otherwise.

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