Renormalization in Equilibrium Statistical Mechanics

CYRANO DE DOMINICIS

Service de Physique Théorique,
Centre d'Etudes Nucléaires de Saclay (S. et O.), France

I. Introduction

We are concerned with "renormalization" equilibrium statistical mechanics. By renormalization we mean that we wish to express the thermodynamical functions describing a system not in terms of the ("bare") potentials occurring in the Hamiltonian but in terms of observable quantities (or quantities directly related to observables) like the distribution functions. We shall consider successively normal systems and "superfluid" systems; we shall review the present status of the renormalization program and discuss in general terms the salient features of the existing renormalized formulations. As for Section IV of these notes, which should have contained the explicit construction of renormalized formulations of statistical mechanics, the reader is referred to material already published (*1–3*).

II. Normal Systems

A. Expression in Terms of Potentials

Consider a normal system (superfluid systems are considered later) described by the Hamiltonian

$$H = \sum v_1(x_1, x_1')\, a^+_{x_1} a_{x_1'} + \sum v_2(x_1, x_2;\ x_1', x_2')\, a^+_{x_1} a^+_{x_2} a_{x_1'} a_{x_2'}.$$

Here v_1 describes the one-body part of the Hamiltonian (i.e., the kinetic energy minus the chemical potential μ and possibly a one-body external

potential), v_2 is the usual two-body interaction potential. For simplicity, we consider only one type of particles fermions or bosons; $a_x^{\pm}$ is the operator creating one such particle in a state x [x momentum (or position) and spin].

The grand partition function describing the system is defined by

$$Z = \exp(W) = \text{trace } \exp(-\beta H) \tag{1}$$

where $\beta = (\text{Boltzmann constant} \times \text{temperature})^{-1}$. This definition expresses W as a functional of the potentials let $W[v_1, v_2]$. An explicit form of $W[v_1, v_2]$ which is sometimes easier to manipulate than the compact form [Eq. (1)] is furnished by perturbation expansions.

The one- and two-particle distribution functions are defined by

$$G_1(x_1, x_1') = -\frac{\delta}{\delta\beta v_1(x_1, x_1')} W[v_1, v_2] \tag{2}$$

$$G_2(x_1, x_2; x_1', x_2') = -\frac{\delta}{\delta\beta v_2(x_1, x_2; x_1', x'_2)} W[v_1, v_2]. \tag{3}$$

B. Mass Renormalization

Instead of W, consider now the "free energy"

$$F^{(1)} = W[v_1, v_2] + \beta \int v_1(x_1, x_1') G_1(x_1, x_1') \, dx_1, dx_1'. \tag{4}$$

Through Eq. (2) v_1 may be considered as implicitly given in terms of G_1 (and v_2) and through Eqs. (2) and (4) $F^{(1)}$ is given as a functional of G_1 and v_2, which verifies the relation

$$\beta v_1(x_1, x_1') = \frac{\delta}{\delta G_1(x_1, x_1')} F^{(1)}[G_1, v_2]. \tag{5}$$

This relation inverts Eq. (2) and carries out "mass" renormalization since the "bare" potential v_1 being expressed in terms of its conjugate variable, the one-particle distribution function G_1. The problem is reduced to constructing explicitly the right-hand side of Eq. (5).

C. Variational Properties

The properties of the system after mass renormalization are conveniently described by considering the functional

$$\hat{W} = F(\hat{G}_1, \nu_2) - \beta \int \nu_1(x_1, x_1') \, \hat{G}_1(x_1, x_1') \, dx_1, dx_1' \tag{6}$$

where ν_1 has a fixed value. Under variations of $\hat{G}_1$, $\hat{W}$ remains stationary when Eq. (2) is verified, i.e., when the variations are made around the values $\hat{G}_1 = G_1$ and

$$\hat{W} = W = F(G_1, \nu_2) - \beta \int \nu_1(x_1, x_1') \, G_1(x_1, x_1') \, dx_1 \, dx_1'. \tag{7}$$

Besides, around these values the second variation is given by

$$\int \delta\hat{G}_1(x_1, x_1') \, \delta\hat{G}_1(x_2, x_2') \, \frac{\delta^2}{\delta G_1(x_1, x_1') \, \delta G_1(x_2, x_2')} \, F(G_1, \nu_2)$$

where

$$\frac{\delta^2}{\delta G_1(x_1, x_1') \, \delta G_1(x_2, x_2')} \, F[G_1, \nu_2] = \beta \, \frac{\delta \nu_1}{\delta G_1(x_2, x_2')} \, [x_1, x_1'; G_1] \tag{8}$$

where Eq. (5) has been used to write Eq. (8). Now, considered as a matrix in x_1, x_1', and x_2, x_2' the right-hand side of (8) is the inverse matrix of

$$\frac{\delta G_1}{\delta \beta \nu_1(x_2, x_2')} \, [x_1, x_1'; \nu_1] = - \frac{\delta^2}{\delta \beta \nu_1(x_1, x_1') \, \delta \beta \nu_1(x_2, x_2')} \, W[\nu_1, \nu_2]$$

which has the structure of a fluctuation and is immediatly displayed as a negative definite matrix. The second variation is thus always negative and for $\hat{G}_1 = G_1$, the functional reaches its absolute maximum $\hat{W} = W$.

It is interesting to notice that the Jacobian of the transformation $\nu_1 \leftrightarrow G_1$ is given by the right-hand side of Eq. (8), and its zeroes correspond to infinite fluctuations, and possible vanishing of the second variation. A simple example of this phenomenon occurs in homogeneous classical systems, where the vanishing of the second variation signals that system has become unstable.

D. Discussion

This ideal program of mass renormalization is reduced to constructing explicitly the right-hand side of Eq. (5), e.g., as an expansion in powers of G_1 and ν_2. In this sense this program has been achieved long ago for classical systems by Yvon (*4*) who repeatedly emphasized the importance of renormalization in connection with phase transition problems. Substitution of ν_1 as given by Eq. (5) into Eq. (7) leads to the classical virial expansion (in terms of the one-particle distribution function).

For quantum systems the corresponding results are yet unknown. Several formulations have gone some way in that direction in the sense that they partially eliminate ν_1, i.e., the right-hand side of Eq. (5) is expressed in terms of G_1 (and ν_2) but some residual dependence upon ν_1 is left over.

(a) Lee and Yang (*5*) have expressed ν_1 in terms of $e^{\beta\mu} G_1$, ν_2 and some residual dependence upon ν_1. They were the first to exhibit the stationarity and (for Bose systems) the maximum property.

(b) A different formulation (*6*) which gives back term by term the virial expansions in the classical limit, expresses ν_1 in terms of G_1, ν_2, and a residual dependence upon ν_1. The functional appearing in this formulation has an absolute maximum at its stationarity point for all systems. The residual dependence upon ν_1 in the functional $\hat{W}$ is due to the fact that, in this case, one actually considers

$$\begin{aligned} \exp(\hat{W}) &= \text{trace } \exp(-\beta H) \exp(-\beta \hat{H}_1) \\ \hat{H}_1 &= \sum \hat{v}_1(x_1, x_1')\, a^+_{x_1} a_{x_1'} \end{aligned} \tag{9}$$

with

$$\hat{G}_1(x_1, x_1') = -\frac{\delta W}{\delta \beta \hat{v}_1(x_1, x_1')}\,[\nu_1, \nu_2; \nu_1]\,. \tag{10}$$

After elimination of $\hat{v}_1$ (but not of the ν_1 appearing in H) one obtains a $\hat{W}$

$$\hat{W} = F(\nu_1, \nu_2; \hat{G}_1] + \beta \int \hat{v}_1(x_1, x_1'; \hat{G}_1(x_1, x_1')$$

which has the properties described above.

(c) Formally very similar to the "classical-like" formulation (b), a "Landau-like" formulation (*7*) is also possible which expresses ν_1 in

terms of a Γ_1 (distribution function for "quasi particles"), v_2, and a residual dependence upon v_1. In the zero temperature limit this formulation yields the results of Landau (*8*) for Fermi liquids.

(d) It is possible, however, to fulfill the renormalization program if one allows for the introduction of "time" dependent (a "time" varying between 0 and β) v_1 potentials and conjugate distribution functions. The "time" dependent v_1 potentials serve the purpose of generating the conjugate "time" dependent distribution function which for usual systems appears as one-body Green's function

$$G_1(x_1u_1, x_1'u_1') = \frac{\text{trace } T\,[\exp(-\beta H)\, a^+_{x_1}(u_1)\, a_{x_1'}(u_1')]}{\text{trace } \exp(-\beta H)} \qquad (11)$$
$$0 < u_1,\ u_1' < \beta$$

where T stands for the T product operator. In that formulation the Dyson equation (*9*, *10*),

$$[G_1^0]^{-1}(x_1, u_1; x_1', u_1') = [G_1]^{-1}(x_1, u_1; x_1', u_1') + K_1\{x_1, u_1; x_1', u_1'; G_1, v_2\} \qquad (12)$$

where $-K_1$ is the usual mass operator and G_1^0 the one-body Green's function for the system without the interaction v_2, furnishes the equivalent of Eq. (5), thus expressing v_1 in terms of the time-dependent one-body distribution function. More explicitly Eq. (12) writes

$$\frac{\delta F^{(1)}[G_1, v_2]}{\delta G_1(x_1, u_1; x_1', u_1')} = -\delta(u_1 - u_1')\,\delta(x_1 - x_1')\frac{\partial}{\partial u_1} + [G_1]^{-1}(x_1, u_1; x_1', u_1') + K_1\{x_1, u_1; x_1', u_1'; G_1, v_2\} \qquad (13)$$

leading immediately (*10*) to $F^{(1)}[G_1, v_2]$ by functional integration. Here the mass renormalization has thus been performed by paying the price of introducing a time-dependent distribution function which leads to no definite sign for the second functional derivative

$$\frac{\delta^2}{\delta G_1(x_1u_1, x_1'u_1')\,\delta G_1(x_2u_2, x_2'u_2')} F^{(1)}[G_1, v_2].$$

E. Vertex Renormalization

Instead of $F^{(1)}$, consider now the entropy

$$F^{(2)} = W[v_1, v_2] + \beta \int [v_1(x_1x_1')\, G_1(x_1x_1')\, dx_1\, dx_1' + v_2(x_1x_2, x_1'x_2')\, G_2(x_1x_2, x_1'x_2')\, dx_1\, dx_2\, dx_1'\, dx_2'] \tag{14}$$

which considered through Eqs. (2) and (3) as a functional of G_1, G_2 verifies the relations ($j = 1, 2$)

$$\beta v_j = \frac{\delta}{\delta G_j} F^{(2)}[G_1, G_2]. \tag{15}$$

Relations (15) invert Eqs. (2) and (3) and fulfill the renormalization program for systems involving only one- and two-body potentials, since Eq. (15) expresses the "bare" potentials v_1, v_2 in terms of the distribution functions G_1, G_2. Again the functional

$$\hat{W} = F^{(2)}[\hat{G}_1, \hat{G}_2] - \beta \int [v_1(x_1x_1')\, \hat{G}_1(x_1x_1')\, dx_1\, dx_1' + v_2(x_1x_2, x_1'x_2')\, \hat{G}_2(x_1x_2, x_1'x_2')\, dx_1\, dx_2\, dx'_1\, dx_2'] \tag{16}$$

remains stationary when variations $\delta\hat{G}_1$, $\delta\hat{G}_2$ are made around the value $\hat{G}_1 = G_1$, $\hat{G}_2 = G_2$ and at that point $\hat{W}$ is equal to W and is an absolute maximum.

Notice that the stationarity condition upon $\hat{W}$ is identical to a stationary condition upon $F^{(2)}[\hat{G}_1, \hat{G}_2]$ under the constraint that the average energy

$$E = \int [v_1(x_1x_1')\, \hat{G}_1(x_1x_1')\, dx_1\, dx_1' + v_2(x_1x_2, x_1'x_2')\, \hat{G}_2(x_1x_2, x_1'x_2')\, dx_1\, dx_2\, dx_1'\, dx_2']$$

is being kept constant; β appears thus, in Eq. (16), as a Lagrange multiplier.

As far as results are concerned, the full renormalization scheme is on par with the mere mass renormalization. The right-hand side of Eq. (15) is only explicitly known for classical systems (*12*, *13*). For quantum systems, only partial eliminations of v_1 and v_2 have been so far performed

(for example, it has been carried out in the Lee-Yang formulation (*15*) where some residual dependence upon ν_1 and ν_2 is left over).

Again if one is willing to introduce time-dependent distribution functions (11) and

$$G_2(x_1u_1, x_2u_2; x_1'u_1', x_2'u_2') = \frac{\text{trace } T \exp(-\beta H)\, a^+_{x_1}(u_1)\, a^+_{x_2}(u_2)\, a_{x_1'}(u_1')\, a_{x_2'}(u_2')}{\text{trace } \exp(-\beta H)} . \tag{17}$$

the equivalent of Eq. (15) and of the entropy $F^{(2)}$ can be written explicitly (*13*) in terms of $G_1(x_1u_1, x_1'u_1')$ and $G_2(x_1u_1, x_2u_2; x_1'u_1', x_2'u_2')$ and the renormalization performed; but the definiteness in sign of the second functional derivative is lost.

F. Dependence upon the Equilibrium Parameter

The equilibrium parameters β and $\beta\mu$ only occur in the combinations $\beta\nu_1$ and $\beta\nu_2$. In the complete elimination of ν_1 and ν_2, the equilibrium parameters disappear and the entropy $F^{(2)}[G_1, G_2]$ in its explicit form no longer contains β, $\beta\mu$ (nor ν_1 or ν_2). The functional $F^{(2)}[\hat{G}_1, \hat{G}_2]$ is thus only dependent upon the fact that the original Hamiltonian contains two-body forces (no n-body forces, $n \geq 3$); it preserves no memory of the dynamics of the system nor of its equilibrium parameters. Whether the functional $F^{(2)}$ may be given a meaning for systems perturbed out of their equilibrium, in the sense that the Boltzmann H-function does for an approximate form of the entropy, is not known.

It is interesting to note that in the "Landau-like" formulation, quoted in Section II, D (c), the equilibrium parameters also disappear. Namely, in this case, the entropy functional is formally identical with that of a noninteracting system, i.e., for a Fermi system

$$F^{(2)} = -\text{trace}\,\{(1-\hat{\Gamma})\log(1-\hat{\Gamma}) + \hat{\Gamma}\log\hat{\Gamma}\} . \tag{18}$$

The system at equilibrium is then described by looking for stationary solutions of $F^{(2)}$ under the constraint of constant average energy, where the average energy is expressed as

$$E = E\{\hat{\Gamma}; \nu_1, \nu_2\} . \tag{19}$$

$E\{\hat{\Gamma}; \nu_1, \nu_2\}$ is a functional of $\hat{\Gamma}$ given by an infinite series expansion in $\nu_1, \nu_2, \hat{\Gamma}$ where β (and $\beta\mu$) do not appear. At the stationarity point, e.g., in momentum representation, $\hat{\Gamma}(k) = \Gamma(k)$ which may be called the average occupation number for quasi particles.

In all other formulations quoted above $F^{(2)}$ was an infinite expansion in powers of G_1, G_2, the constraint itself being linear in those variables. Here $F^{(2)}$ has an extremely simple form but the constraint equation contains an infinite expansion in $\nu_1, \nu_2, \hat{\Gamma}$.

G. Why Renormalization?

One may ask why it is desirable to carry out the renormalization procedure. Clearly to obtain the thermodynamical functions in terms of observable quantities is certainly a desirable feature in itself.

For an actual calculation one is reduced to take approximate forms of the thermodynamical functions. In perturbation theory, for example, one would truncate $W[\nu_1, \nu_2]$ to a given order in ν_1, ν_2. In the mass renormalized form, keeping $F^{(1)}[G_1, \nu_2]$ [and correspondingly $\delta F^{(1)}/\delta G_1$ in Eq. (5)] up to first order in ν_2 generates the (self-consistent) Hartree-Fock approximation; to obtain the same result in the perturbation expansion would have necessitated the re-summation of an infinite series of terms. Likewise truncation of the functional $F^{(2)}$ [and of Eq. (15) correspondingly] will generate "doubly" self-consistent equations, obviously harder to solve, which embed infinite re-summations of terms of W or $F^{(1)}$ and are presumed to have better convergence properties, and to be better adapted to the description of phase transitions.

However, approximations generated in the above-described way are not necessarily the best for all purposes. These approximations respect the relation implied by the pair of equations (15) but they always violate the relation

$$\frac{\delta^2}{\delta\beta\nu_1(x_1x_1')\ \delta\beta\nu_1(x_2x_2')} W[\nu_1, \nu_2] = G_2(x_1x_2, x_1'x_2') - G_1(x_1x_1')\, G_1(x_2x_2') \tag{20}$$

implied by the Hamiltonian structure. Relation (20) is a functional relation between G_1 and G_2

$$-\frac{\delta}{\delta\beta\nu_1(x_2x_2')} G_1(x_1x_1') = G_2(x_1x_2, x_1'x_2') - G_1(x_1x_1')\, G_1(x_2x_2') \tag{21}$$

which is related to current conservation. On the other hand, there are cases where it is important (to study collective excitations like sound waves, for example) to embed in the approximations chosen the consequences of current conservation (*14*); in such a case, one would not truncate $F^{(2)}$ but rather $F^{(1)}$ and relation (21) would be used to define G_2.

Finally, there are systems where the renormalization procedure becomes a necessity. This is the case of systems where one cannot afford to use expansions in some of the potentials occurring in the Hamiltonian. Systems with hard core interactions are one example. A second example is furnished by superfluid systems.

III. Superfluid Systems

A. Extended Ensemble

The treatment of superfluid systems (like Bose systems at low temperature) by the methods described above lead to serious convergence difficulties. These difficulties are due to the Bose condensation phenomenon, i.e., the macroscopic occupation of an individual state. Extensions of the grand canonical ensemble have been introduced to describe such systems and avoid convergence difficulties.

Lee and Yang (*15*) have proposed an extended ensemble where the density matrix

$$\varrho = \exp(-\beta H) / \text{trace } \exp(-\beta H)$$

is replaced (in order to describe a possible condensation in the state of momentum $k = 0$) by

$$\varrho_y = \exp(-y) \frac{y^{n_0}}{n_0!} \exp(-\beta H) / \text{trace } \exp(-y) \frac{y^{n_0}}{n_0!} \exp(-\beta H).$$

n_0 is the occupation number operator for the state $k = 0$ ($n_0 = a_0^+ a_0$). y is a free parameter (y stands here for the x used by Lee and Yang times the volume) determined by the condition that W_y defined by

$$W_y = \text{trace } \exp(-y) \frac{y^{n_0}}{n_0!} \exp(-\beta H) \tag{23}$$

is stationary under variation of the number y. At the stationary point $\bar{y}$ (and for infinite systems), $W_{\bar{y}}$ is shown to be equal to the usual W and $\bar{y}$ to be the true macroscopic occupation number of the state $k = 0$.

Bogoliubov (*16*) has suggested the introduction of a different ensemble which is for our purposes more convenient to describe superfluid systems or more generally systems which are degenerate in their statistical equilibrium. To describe systems where condensation is supposed to occur in the state of momentum $k = 0$, the Hamiltonian of the system is supplemented by an (infinitesimal) "source" term

$$H_s = \nu_{1/2}\, a_0^+ + \nu_{1/2}^*\, a_0$$

where $\nu_{1/2}$ is the source potential ($\frac{1}{2}$-body potential; in this case, $\nu_{1/2}$ is reduced to a constant). P. C. Martin (*1*) has discussed and emphasized the usefulness of introducing a generalized source term

$$H_s = \int [\nu_{1/2}(x)\, a_x^+ + \nu_{1/2}^*(x)\, a_x]\, dx \tag{24}$$

which serves the purpose of generating equations of motion of a more complex structure describing situations where the translational invariance is violated (vortex lines).

More precisely consider now

$$\exp(W_s) = \text{trace} \exp[-\beta(H + H_s)] \tag{25}$$

W_s being thus a functional $W_s[\nu_{1/2}, \nu_{1/2}^*, \nu_1, \nu_2]$. The conjugate variables associated with the source potentials $\nu_{1/2}$, $\nu_{1/2}^*$ are

$$G_{1/2}(x) = -\frac{\delta}{\delta\beta\nu_{1/2}^*(x)} W[\nu_{1/2}, \nu_{1/2}^*, \nu_1, \nu_2] \tag{26}$$

(and its complex conjugate). $G_{1/2}^*(x)$ plays the role of a renormalized wave function for the condensate. We may now consider

$$F^{(1/2)} = W_s[\nu_{1/2}, \nu_{1/2}^*, \nu_1, \nu_2] + \beta \int [\nu_{1/2}(x)\, G_{1/2}^*(x) + \nu_{1/2}^*(x)\, G_{1/2}(x)]\, dx \tag{27}$$

as a functional of $G_{1/2}$, $G_{1/2}{}^*$ (and v_1, v_2) through Eq. (26). We have again

$$v_{1/2}{}^*(x) = \frac{\delta}{\delta G_{1/2}(x)} F^{(1/2)}[G_{1/2}, G_{1/2}{}^*, v_1, v_2] . \tag{28}$$

This equation inverts Eq. (26). The functional

$$\begin{aligned} \hat{W}_s = F^{(1/2)}[\hat{G}_{1/2}, \hat{G}_{1/2}{}^*, v_1, v_2] - \beta \int [v_{1/2}(x)\, G_{1/2}{}^*(x) \\ + v_{1/2}{}^*(x)\, G_{1/2}(x)]\, dx \end{aligned} \tag{29}$$

where $v_{1/2}$ and $v_{1/2}{}^*$ are kept fixed, is stationary and maximum for variations $\delta\hat{G}_{1/2}$, $\delta\hat{G}_{1/2}{}^*$ around $\hat{G}_{1/2} = G_{1/2}$, $\hat{G}_{1/2}{}^* = G_{1/2}{}^*$.

In the limit of a vanishingly small source term the system is described by the functional

$$W_s \to F^{(1/2)}[G_{1/2}, G_{1/2}{}^*, v_1, v_2] , \tag{30}$$

supplemented by the stationarity equation

$$\frac{\delta F^{(1/2)}}{\delta G_{1/2}(x)} = 0 \tag{31}$$

and a positive definiteness condition. Above the transition temperature Eq. (31) admits only the trivial solution $G_{1/2} = 0$ and $F^{(1/2)}$ reduces to $W[v_1, v_2]$. The same procedure is extended with no difficulty to superfluid Fermi systems.

Clearly the renormalization procedure is here essential, since the bare potential $v_{1/2}$ is allowed to vanish, whereas it is assumed that domains exist where the conjugate (renormalized) quantity $G_{1/2}{}^*$ is nonvanishing.

B. Renormalization

Consider the one-body distribution function in the ensemble with source (eventually the source is always supposed to vanish)

$$\begin{aligned} G_1(x_1, x_1') &= -\frac{\delta}{\delta\beta v_1(x_1 x_1')} W_s \\ &= \text{trace exp}\,[-\beta(H + H_s)]\; a^+_{x_1} a_{x_1'} \,/\, \text{trace exp}\,[-\beta(H + H_s)] \\ &= \langle a^+_{x_1} a_{x_1'} \rangle . \end{aligned}$$

It can now be rewritten as

$$G_1(x_1x_1') = \langle a^+_{x_1}\rangle \langle a_{x_1'}\rangle + \langle (a^+_{x_1} - \langle a^+_{x_1}\rangle)\,(a_{x_1'} - \langle a_{x_1'}\rangle)\rangle$$
$$= G_{1/2}{}^*(x_1)\, G_{1/2}(x_1') + \tilde{G}_1(x_1, x_1')\,. \tag{32}$$

$\tilde{G}_1$ is the cumulant part of G_1 and this splitting of one-body distribution function for superfluid system in configuration space, corresponds to the one first introduced by Penrose and Onsager (*17*); here $\tilde{G}_1$ tends to zero as $(x_1 - x_1') \to \infty$. The total number of particles of the system is given by

$$\int G_1(x_1, x_1)\, dx_1 = \int |\, G_{1/2}(x_1)\,|^2\, dx_1 + \int \tilde{G}_1(x_1, x_1)\, dx_1\,, \tag{33}$$

the first term being the condensate contribution.

Quantities like $\langle a^+_{x_1} a^+_{x_1'}\rangle$, $\langle a_{x_1} a_{x_1'}\rangle$ are now generally nonvanishing and the matrix notation introduced by Nambu (*18*) becomes very useful. One works with a one-body distribution function which (besides being a matrix in x_1, x_1') is now a 2×2 matrix $\mathbf{G}_1(x_1, x_1')$ the elements of which are $\langle a^+_{x_1} a_{x_1'}\rangle$, $\langle a_{x_1} a^+_{x_1'}\rangle$ $\langle a^+_{x_1} a^+_{x_1'}\rangle$ and $\langle a_{x_1} a_{x_1'}\rangle$. The mass renormalization is then carried out in the same way as was done for normal systems. The explicit knowledge of the renormalized functional has reached about the same stage as for normal systems.

(a). Lee and Yang (*15*) have obtained (in their extended ensemble) a W functional of y and $e^{\beta\mu}\tilde{\mathbf{G}}_1$, which, for the Bose systems considered, is stationary and maximum under variations of y and $e^{\beta\mu}\tilde{\mathbf{G}}_1$.

(b, c). The "classical-like" and the "Landau-like" formulations can also be extended to superfluid systems. The former, besides being actually stationary and maximum under variations of $\mathbf{G}_{1/2}$ and $\mathbf{G}_1$, presumably has only an academic interest. The latter leads to a microscopic formulation of the Landau theory of the Bose liquid (or superfluid Fermi liquids) (*6*).

(d). All these formulations, as was described in Section II, D, do not eliminate completely the one-body potential v_1; but this elimination is possible if one again introduces time-dependent one-body distribution functions $\mathbf{G}_1(x_1u_1, x_1'u_1')$. Indeed, as for normal systems the full renormalization program (eliminating completely $v_{1/2}$, v_1, v_2) has been carried

out explicitly for the time-dependent formulation and only for that formulation (*1*, *2*, *3*).

The entropy $F^{(2)}$ has been exhibited as a functional of $\tilde{\mathbf{G}}_1$, $\tilde{\mathbf{G}}_{3/2}$, $\tilde{\mathbf{G}}_2$, respectively, the cumulant parts of the time-dependent matrices built with two, three, and four operators $a_z(u)$ or $a_x^+(u)$ (i.e., 2^j matrices $j = 2, 3, 4$). $F^{(2)}$ turns out to be independent of $\mathbf{G}_{1/2}$. Besides, the remarks made in Section II, F are also valid for superfluid systems.

IV. Explicit Construction of the Functionals

We have so far completed in general terms a review of the present status of "renormalized" equilibrium statistical mechanics. Explicit construction techniques of the functionals described in Sections II and III will not be reproduced here as they are already in the published literature. For the source ensemble treatment of superfluid systems the reader is referred to refs. (*1*, *2*, *3*) where algebraic and diagrammatic techniques respectively, are described.

References

1. P. C. Martin, *J. Math. Phys.* **4**, 208 (1963). P. Hohenberg, Thesis, Harvard Univ., Cambridge, Mass. (1962).
2. C. De Dominicis, *J. Math. Phys.* **4**, 255 (1963).
3. C. De Dominicis and P. C. Martin, *J. Math. Phys.* **5**, 14 and 31 (1963).
4. J. Yvon, *Actualites Sci. Ind.* **203** (1935); *Cahiers Phys.* **28** (1945); *J. Phys. Radium* **10**, 373 (1949).
5. T. D. Lee and C. N. Yang, *Phys. Rev.* **113**, 1165 (1959); **117**, 22 (1960).
6. R. Balian, C. Bloch, and C. De Dominicis, *Nucl. Phys.* **25**, 529 (1961); **27**, 294 (1961).
7. R. Balian and C. De Dominicis, *Compt. Rend.* **250**, 3285, 4111 (1960); and to be published.
8. L. Landau, *Soviet Phys. JETP* **30**, 1058 (1956).
9. A. Abrikosov, L. Gorkov, and I. Dzyaloshinsky, *Soviet Phys. JETP* **36**, 900 (1959); E. Fradkin, *Nucl. Phys.* **12**, 449 (1959); P. C. Martin and J. Schwinger, *Phys. Rev.* **115**, 1342 (1959).
10. J. Luttinger and J. Ward, *Phys. Rev.* **118**, 1417 (1960).

11. Contrary to a statement of ref. 10.
12. T. Morita and K. Hiroike, *Progr. Theoret. Phys. Kyoto* **25**, 537 (1961).
13. C. De Dominicis, *J. Math. Phys.* **3**, 983 (1962).
14. L. Kadanoff and C. Baym, "Quantum Statistical Mechanics," Chapter 10, Benjamin, New York, 1962.
15. T. D. Lee and C. N. Yang, *Phys. Rev.* **117**, 897 (1960).
16. N. Bogoliubov, *Physica* **S26**, 1 (1960).
17. O. Penrose and L. Onsager, *Phys. Rev.* **104**, 576 (1956).
18. Y. Nambu, *Phys. Rev.* **117**, 648 (1960).