Chapter 2

RADIATIVE TRANSFER THEORY, RADIATION MEAN FREE PATHS, CONSERVATION EQUATIONS, AND SIMILARITY PARAMETERS*

Following a concise survey of basic methods involved in the formulation of radiative transfer problems\textsuperscript{1–7} and a brief discussion of the solution of representative, elementary examples, we describe the origin and definition of the Planck and Rosseland mean free paths.\textsuperscript{1–4} Next we discuss the conservation equations for reacting, multicomponent gas mixtures with proper allowance for radiative energy flux.\textsuperscript{4} A discussion of similarity parameters in radiation gasdynamics\textsuperscript{5,6} is given, with proper allowance being made for relaxation processes and radiative energy transfer.\textsuperscript{†} The radiative equilibrium of a radiating gas between parallel plates is considered as an example problem. The chapter is concluded with a brief discussion of nongray radiative transfer.

2-1 The formulation of radiative transfer problems\textsuperscript{1–3}

Consider the geometric arrangement sketched in Fig. 2-1.1. We define the spectral steradiancy or specific intensity $I_\nu$ on the area $dA$, at the point $P$, as the instantaneous magnitude of the radiant energy $dE_\nu$, incident in unit time, in unit solid angle, in the frequency interval

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\† The material appearing in Sections 2-1 to 2-3 is largely abstracted from Penner and Patch.\textsuperscript{4} The contents of Section 2-4 are taken from Penner \textit{et al.}\textsuperscript{5} and other sources referred to in the text.
between $\nu$ and $\nu + dv$, on unit area of the projected surface normal to the direction of observation. Thus

$$I_{\nu} = \lim_{dA,dQ,dv \to 0} \left. \frac{dE_{\nu}}{\cos \theta \ dA \ d\Omega \ dv \ dt} \right|_{dI_1,dA,d\Omega,dv \to 0}.$$  \hspace{1cm} (2-1.1)$$

For isotropic radiation, $I_\nu$ is independent of the direction of the vector $\mathbf{L}$. The radiation field is said to be homogeneous if $I_\nu$ has the same value everywhere.

2-1A ABSORPTION OF RADIATION. Appealing either to experimental observations or to molecular principles concerning absorption of radiation, we may assume that the rate of change of $I_\nu$ with distance (in a uniform gas) in absorption is proportional to $I_\nu$ for a beam of photons propagating in a specified direction. Thus

$$\frac{dI_{\nu}}{ds} = -(P_\nu \rho) I_{\nu} = -\kappa_{\nu} \rho I_{\nu} = -k_{\nu} I_{\nu},$$  \hspace{1cm} (2-1.2)$$

where the distance $s$ (in cm) is measured in the direction of propagation, $P_\nu$ (in cm$^{-1}$-atm$^{-1}$) is the spectral absorption coefficient per unit pressure per unit length, $\rho$ (in atm) is the pressure, $\kappa_\nu$ (in cm$^2$/g) is the spectral mass absorption coefficient, $\rho$ (in g/cm$^3$) is the gas density, and $\kappa_{\nu} \rho = k_{\nu}$ (in cm$^{-1}$) is the linear absorption coefficient.
2-1B ABSORPTION AND EMISSION OF RADIATION. If we allow for emission of radiation, as well as for absorption, then we must replace Eq. (2-1.2) by the following expression:

$$\frac{dI_v}{ds} = -\kappa_v \rho I_v + j_v,$$

(2-1.3)

where $j_v$ is called the emission coefficient. For an isothermal system in thermodynamic equilibrium, $dI_v/\!\!ds = 0$ and, therefore,

$$j_v = \kappa_v I_v = \kappa_v B_v,$$

(2-1.4)

where $B_v$ is the equilibrium value of $I_v$ at the temperature $T$ of the system, i.e., $I_v = B_v$ corresponds to the specific intensity derived from the Planck blackbody radiation law, viz.,

$$B_v = \frac{R^0}{\pi} = \frac{2h \nu^3}{c^2} \frac{1}{[\exp(h\nu/kT)] - 1}$$

and

$$B = \int_0^\infty B_v \, dv = \frac{\sigma T^4}{\pi}.$$

(2-1.5)

where $h$ is Planck's constant, $c$ is the velocity of light, $k$ denotes the Boltzmann constant, and $\sigma$ is the Stefan-Boltzmann constant. Thus, in the case where "local" thermodynamic equilibrium is maintained but temperature variations* are present, the basic transfer equation takes the form

$$- \frac{1}{\rho \kappa_v} \frac{dI_v}{ds} = I_v - B_v$$

(2-1.6)

if absorption and emission of radiation occur without scattering.

2-1C ABSORPTION AND ISOTROPIC SCATTERING WITHOUT EMISSION OF RADIATION. Consider now the extreme case where thermal emission of radiation is totally absent and the emission coefficient is associated exclusively with isotropic scattering. The emission coefficient may be shown† to be given by the relation

$$j_v = \kappa_v \tilde{\omega}_v \int B \frac{d\Omega}{4\pi}$$

(for scattering without thermal emission),

* The concept of "local" thermodynamic equilibrium implies the existence of a meaningful local temperature.

† Kourganoff, p. 9–13.
where the integration is to be performed over the complete solid angle and the factor \( \bar{\omega}_\nu \) represents the albedo for single scattering. The basic transport equation is now of the form

\[
- \frac{1}{\rho \kappa_\nu} \frac{dI_\nu}{ds} = I_\nu - \bar{\omega}_\nu \int I_\nu \frac{d\Omega}{4\pi}.
\]  

(2-1.7)

2-1D Absorption and emission of radiation and isotropic scattering. In the general case where emission, isotropic scattering, and absorption of radiation occur, let \( \bar{\omega}_\nu \) denote the fraction of \( I_\nu \) that is (singly) scattered. Then

\[
j_\nu = \bar{\omega}_\nu \kappa_\nu \int I_\nu \frac{d\Omega}{4\pi} + (1 - \bar{\omega}_\nu) \kappa_\nu B_\nu \quad \text{(both scattering and thermal emission occur)}
\]

and the general transport equation now assumes the form

\[
- \frac{1}{\rho \kappa_\nu} \frac{dI_\nu}{ds} = I_\nu - \bar{\omega}_\nu \int I_\nu \frac{d\Omega}{4\pi} - (1 - \bar{\omega}_\nu) B_\nu.
\]  

(2-1.8)

2-1E Radiative energy transfer from gases confined between two infinite parallel plates.\(^7\) The irradiancy or incident flux on a surface is defined as

\[
dF_\nu = \frac{dE_\nu}{dA \, dv \, dt} = I_\nu \cos \theta \, d\Omega
\]  

(2-1.9)

where the angle \( \theta \) is shown in Fig. 2-1.2. Setting

\[
d\Omega = \sin \theta \, d\theta \, d\phi
\]

![Fig. 2-1.2. Schematic diagram relevant to a description of the radiative transfer problem between two parallel plates, when multiple reflections of rays from the bounding walls are considered (from Goulard\(^7\)).](image)
and integrating Eq. (2-1.9) over all directions, we find that the spectral radiant flux between the bounding plates at \( y = 0 \) and at \( y = y \) becomes

\[ F_y = F_y^+ + F_y^- , \]

(2-1.10)

where

\[ F_y^+ = \int_0^{2\pi} d\phi \int_0^{\pi/2} I_y^+ (\theta, \varphi) \cos \theta \sin \theta d\theta \]

(2-1.11)

and

\[ F_y^- = \int_0^{2\pi} d\phi \int_0^{\pi/2} I_y^- (\theta, \varphi) \cos \theta \sin \theta d\theta \]

(2-1.12)

denote, respectively, the flux in the downward \( (0 < \theta < \pi/2) \) and upward \( (\pi/2 < \theta < \pi) \) directions. If \( I_y \) is independent of \( \varphi \), the integration over \( \varphi \) may be performed immediately in Eqs. (2-1.11) and (2-1.12).

The preceding expressions constitute the starting relations for radiative transfer analysis. We refer to the excellent texts of Chandrasekhar,\(^1\) Kourganoff,\(^2\) and Sobolev\(^3\) for elaboration of methods and results in radiative transfer analysis. However, in order to provide an elementary illustration of conventional procedures, we shall indicate the form of \( F_y^+ \) and \( F_y^- \) in a one-dimensional problem without scattering.\(^7\)

For a one-dimensional problem, at local equilibrium, without scattering, Eq. (2-1.6) applies. Let

\[ d\tau_v = - \rho \kappa_v \cos \theta ds , \]

(2-1.13)

where \( \theta \) is the angle between the vector \( s \) and the outward normal to the plane of constant optical depth \( \tau_v \) (see Fig. 2-1.2). Using Eq. (2-1.13), Eq. (2-1.6) becomes

\[ \cos \theta \frac{dI_v}{d\tau_v} = I_v(\tau_v) - B_v(\tau_v) . \]

(2-1.14)

The solution of this differential equation is

\[ I_v(\tau_v) = C(\tau_v) \left( \exp \frac{\tau_v}{\cos \theta} \right) , \]

(2-1.15a)

\[ C(\tau_v) = C(\tau_v^*) - \int_{\tau_v^*}^{\tau_v} B_v(\eta) \left( \exp - \frac{\eta}{\cos \theta} \right) d\eta , \]

(2-1.15b)

where \( \tau_v^* \) and \( C(\tau_v^*) \) must be determined after imposition of suitable boundary conditions.
Neglecting reflected radiation, the intensity of radiation toward the upper wall \((\pi/2 < \theta < \pi)\) is determined by the boundary conditions at the lower wall \(C(0) = I_v^{-}(0)\) at \(\tau_v = 0\), where \(I_v^{-}(0)\) is the specific intensity in the upward direction from the lower wall (see Fig. 2-1.2). Thus

\[
I_v^{-}(\tau_v) = \left[ \int_0^{\tau_v} \frac{B_s(\eta)}{-\cos \theta} \left( \exp - \frac{\eta}{\cos \theta} \right) d\eta + I_v^{-}(0) \right] \left( \exp \frac{\tau_v}{\cos \theta} \right). \quad (2-1.16)
\]

Similarly, the intensity of radiation toward the lower wall \((0 < \theta < \pi/2)\) is determined by the boundary condition at the upper wall

\[
I_v^{+}(\tau_{v_2}) = C(\tau_{v_2})[\exp(\tau_{v_2}/\cos \theta)],
\]

where \(I_v^{+}(\tau_{v_2})\) is the specific intensity in the downward direction from the upper wall. Thus

\[
I_v^{+}(\tau_v) = I_v^{+}(\tau_{v_2}) \left( \exp - \frac{\tau_{v_2} - \tau_v}{\cos \theta} \right) + \int_{\tau_{v_2}}^{\tau_v} \frac{B_s(\eta)}{-\cos \theta} \left( \exp - \frac{\eta - \tau_v}{\cos \theta} \right) d\eta. \quad (2-1.17)
\]

Next we use Eqs. (2-1.16) and (2-1.17) in order to compute the corresponding spectral radiant fluxes. Thus we find [cf. Eqs. (2-1.11) and (2-1.12)] that

\[
F_v^- = -2\pi \int_0^{1} \int_{\tau_v}^{\tau_{v_2}} B_s(\eta) \{\exp[(\eta - \tau_v)/\mu]\} \, d\eta \, d\mu - 2\pi I_v^{-}(0) \int_0^{1} \mu[\exp(- \tau_v/\mu)] \, d\mu \quad (2-1.18)
\]

and

\[
F_v^+ = 2\pi \int_0^{1} \int_{\tau_v}^{\tau_{v_2}} B_s(\eta) \{\exp \left[ -(\eta - \tau_v)/\mu \right]\} \, d\eta \, d\mu
\]

\[
+ 2\pi I_v^{+}(\tau_{v_2}) \int_0^{1} \mu\{\exp \left[ -(\tau_{v_2} - \tau_v)/\mu \right]\} \, d\mu, \quad (2-1.19)
\]

where \(I_v^{-}(0)\) and \(I_v^{+}(\tau_{v_2})\) are assumed to be independent of \(\theta\) and of \(\varphi\). Kourganoff\(^2\) has tabulated the functions

\[
E_n(t) = \int_0^{1} \mu^{n-2}[\exp(- t/\mu)] \, d\mu. \quad (2-1.20)
\]

In terms of the function \(E_n(t)\), we find that

\[
F_v^{+}(\tau_v) = -2\pi \int_{\tau_v}^{\tau_{v_2}} B_s(\eta) E_n(\tau_v - \eta) \, d\eta - 2\pi I_v^{+}(0) E_3(\tau_v) \quad (2-1.21)
\]
and

\[ F_v^+(\tau_v) = 2\pi \int_{\tau_v}^{\infty} B_v(\eta) E_2(\eta - \tau_v) \, d\eta + 2\pi I_v^+(\tau_v) E_3(\tau_v - \tau_v). \quad (2-1.22) \]

The net spectral flux in the downward direction is equal to the sum of \( F_v^-(\tau_v) \) and \( F_v^+(\tau_v) \). In the absence of walls, \( I_v^-(0) = I_v^+(\tau_v) = 0 \) and the flux equations reduce to classical forms given by Milne.\textsuperscript{1,2} Further analysis of the present problem can be carried out simply only for the case of spectrally resolved radiation or for gray radiation.\textsuperscript{7}

**2-1F Radiative-energy transfer to the nose cone of a reentry vehicle in the transparent-gas approximation.** The radiative-energy transfer problem to the nose cone of a reentry vehicle may be approximated, in first order, by making proper use of the equations derived in Section (2-1E), but specialized to the case of a uniformly distributed gas for transfer in one direction only (say, the positive direction) with \( I_v^+(\tau_v) = 0 \), i.e., without emission from the assumed infinitely thin shock front. Thus the flux density at the nose cone surface (\( \tau_v = 0 \)) is seen to be

\[ F_v^+(0) = 2\pi \int_{0}^{\tau_v} B_v(\eta) E_2(\eta) \, d\eta \]

from Eq. (2-1.22). But, for uniform gases, \( B_v(\eta) = B_v \) is a constant and, therefore,

\[
\begin{align*}
F_v^+(0) &= 2\pi B_v \int_{0}^{\tau_v} d\eta \int_{0}^{1} \left[ \exp\left( - \eta / \mu \right) \right] d\mu \\
&= 2\pi B_v \int_{0}^{1} \left[ 1 - \exp\left( - \tau_v / \mu \right) \right] \mu \, d\mu \\
&= \pi B_v \left[ 1 - 2E_3(\tau_v) \right] \\
&= B_v \int_{0}^{2\pi} \left[ 1 - \exp\left( - \kappa_\rho L \sec \theta \right) \right] d\Omega \cos \theta, \quad (2-1.23)
\end{align*}
\]

where \( d\Omega \) denotes an element of the solid angle. The last form of the integrand in Eq. (2-1.23) will be recognized immediately as the spectral emissivity for uniformly distributed gases with linear absorption coefficient \( \kappa_\rho \), contained by walls a distance \( L \) apart, and referring to propagation in the direction \( \theta \) relative to the normal distance \( L \).

For a transparent gas layer, we may expand

\[ \exp\left( - \frac{\tau_v}{\mu} \right) = 1 - \frac{\tau_v}{\mu} + \ldots \]
and retain, by definition, only the first two terms. In this case,

\[ F_{\nu}^+(0) \simeq 2\pi B_{\nu} \int_0^1 \tau_{v_2} d\mu = 2\pi B_{\nu}\kappa_0 \rho L = (\kappa_0 2L) R_0^0 \]  

(2-1.24)

and it is apparent that we have obtained the well-known result that the effective geometric length that is to be used in radiative transfer calculations for uniformly distributed, transparent gases, bounded by infinite parallel planes a distance \( L \) apart, is \( 2L \).

If the spectral absorptivity of the wall is \( \alpha_{\nu} \), then the total radiant flux \( W \) absorbed by the wall may be obtained from Eq. (2-1.24) by integrating \( \alpha_{\nu} F_{\nu}^+(0) \) over all frequencies, viz.,

\[ W = (2L) \rho \int_0^\infty \alpha_{\nu} \kappa_0 R_0^0 d\nu \]

\[ = (2L) \rho \int_0^\infty \alpha_{\nu} P_{\nu}[R_0^0(T)] d\omega \]

\[ = (2L) \rho \int_0^\infty \alpha_{\omega} P_{\omega}[^{\omega}[\omega T^4 F_{\omega}(\omega)] d\omega \]  

(2-1.25)

where \( \omega = 1/\lambda = \nu/c \) is the wavenumber, \( R_0^0(T) \) is the Planck function at the temperature \( T \) at \( \omega \), \( T \) is the temperature of the uniformly distributed gases in front of the nose cone, and \( F_{\omega}(\omega) d\omega \) is the fractional energy of a blackbody radiator contained in the wavenumber range between \( \omega \) and \( \omega + d\omega \) if \( \sigma \) represents the Stefan-Boltzmann constant. The effective engineering emissivity of the shocked gases may be defined as

\[ \epsilon_{\text{eff}} = \frac{F}{\sigma T^4} = (2L) \rho \int_0^\infty P_{\omega} F_{\omega}(\omega) d\omega, \]  

(2-1.26)

where we use unit absorptivity \( \alpha_{\omega} \) at all wavenumbers when the absorbed integrated flux \( W \) is replaced by the incident integrated flux \( F \).

It is of interest* to compare the incident radiant flux, viz.,

\[ F = (2L) \rho \sigma T^4 \int_0^\infty \kappa_{\omega} F_{\omega}(\omega) d\omega, \]

with the convective energy transport to a reentry body. The shock standoff distance \( L \) turns out to be a linear function of the nose radius \( R \)

* In the following calculations, we assume that the radiant-energy exchange has no effect on the flow-field calculations. This assumption is not justified for very large reentry velocities.
and is also proportional to the ratio of free-stream to shocked-gas density. The spectral absorption coefficient $\kappa_\omega$ depends only weakly on density but varies strongly with the gas stagnation temperature. A fit to the available emissivity data for heated air shows\textsuperscript{8-10} that, roughly,

$$F \propto \rho^{3/2} R U^{10},$$

where $\rho$ is the ambient air density and $U$ denotes the vehicle velocity. On the other hand, the convective heat transfer to a reentry body is given\textsuperscript{10} by the approximate relation

$$F_c \propto \left(\frac{\rho}{R}\right)^{1/2} U^3.$$  

Hence the ratio of radiative to convective heat transfer becomes

$$\frac{F}{F_c} \propto \rho R^{3/2} U^7,$$

i.e., radiative energy transfer becomes relatively much more important as the reentry velocity increases and is also favored by high air density and a large nose radius.

### 2-2 Planck and Rosseland mean free paths\textsuperscript{1-4,11-14}

Since $I_\nu$ is a function of both position and frequency, we may rewrite Eq. (2-1.6) in the form

$$\frac{dI_\nu}{ds} = k_\nu (B_\nu - I_\nu)$$

or

$$\cos \theta \frac{dI_\nu}{dx} = k_\nu (B_\nu - I_\nu), \quad (2-2.1)$$

where $k_\nu = \rho \kappa_\nu$ is the spectral absorption coefficient per unit length, $\theta$ is the angle between the normal to the slab of thickness $dx$ and the direction of propagation in which an incremental length has been denoted by $ds$. In radiative transfer analysis, the Planck and Rosseland mean absorption coefficients are often employed since they represent useful approximations for optically thin (i.e., transparent) gases and for relatively opaque gases, respectively.

#### 2-2A The Planck mean absorption coefficient\textsuperscript{1-4,11}  
The Planck mean absorption coefficient is useful for optically thin gases for which

$$k_\nu L \ll 1, \quad (2-2.2)$$
where $L$ denotes the distance characteristic of the problem. In this case, for no external radiation sources, $I_v \ll B_v$, and absorption may be neglected in the equation of transfer, i.e.,

$$dI_v \approx k_v B_v \frac{dx}{\cos \theta}.$$  \hfill (2-2.3)

Consider an infinite slab of thickness $L$. The flux at $L$ is calculated by multiplying Eq. (2-2.3) by $\cos \theta$ and integrating over $x$, solid angle, and frequency [cf. Eq. (2-1.9)]. Thus

$$F(L) = \int_0^\infty \int_0^{2\pi} \int_0^L k_v B_v \frac{dx}{\cos \theta} \, d\Omega \, dv \bigg|_{0}^{2\pi}$$

$$= 2\pi \int_0^L \left(\int_0^\infty k_v B_v \, dv\right) \, dx$$

$$= 2 \int_0^L \bar{k}_p \sigma T^4 \, dx,$$ \hfill (2-2.4)

where

$$\bar{k}_p \equiv \frac{1}{l_p} = \frac{\pi}{\sigma T^4} \int_0^\infty k_v B_v \, dv$$ \hfill (2-2.4a)

defines the Planck mean absorption coefficient $\bar{k}_p$. For a constant temperature slab, the flux reduces to

$$F(L) = \varepsilon_{\text{eff}} \sigma T^4 = 2\bar{k}_p L \sigma T^4.$$ \hfill (2-2.5)

Considering now a point in the interior of the slab, we must include the flux from $4\pi$ sterad; therefore,

$$\frac{dF}{dx} = 4\bar{k}_p \sigma T^4,$$ \hfill (2-2.6)

which, in the general three-dimensional case, becomes

$$\nabla \cdot \mathbf{F} = 4\bar{k}_p \sigma T^4.$$ \hfill (2-2.7)

Equation (2-2.7) is valid at each point in an optically thin gas if there are no external radiation sources and if local thermodynamic equilibrium may be assumed.

### 2-2B The Rosseland Mean Absorption Coefficient

The Rosseland mean absorption coefficient is useful for "optically thick" gases for which

$$k_v L \gg 1$$ \hfill (2-2.8)
at all frequencies contributing to the flux. The characteristic length $L$ is given by the distance over which an appreciable change of temperature occurs. It is seen from the equation of transfer [see Eq. (2-2.1)] that $I_\nu \sim B_\nu$ in an optically thick medium. Accordingly, we write the transfer equation in the form (with $d\tau_\nu \equiv k_\nu \, ds$)

$$I_\nu = B_\nu - \frac{1}{k_\nu} \frac{dI_\nu}{ds} = B_\nu - \frac{dI_\nu}{d\tau_\nu}$$

(2-2.9)

and progressively substitute for $I_\nu$ in terms of $B_\nu$ and $dI_\nu/d\tau_\nu$, i.e.,

$$I_\nu = B_\nu - \frac{dB_\nu}{d\tau_\nu} + \frac{d^2B_\nu}{d\tau_\nu^2} + \cdots + (-1)^n \frac{d^nB_\nu}{d\tau_\nu^n} + \cdots$$

(2-2.10)

Since $B_\nu = B_\nu(T)$, the first two terms* on the right-hand side of Eq. (2-2.10) are sufficient to approximate $I_\nu$ provided

$$\frac{1}{k_\nu} \frac{\left| \nabla T \right|}{T} \ll 1.$$ 

This condition is equivalent to that given by Eq. (2-2.8) since $T/\left| \nabla T \right|$ is a characteristic length $L$ for the temperature variation.

The temperature (and, therefore, $B_\nu$) is a function of the Cartesian coordinates $x_i$. Thus the derivative of $B_\nu$ in an arbitrary direction $s$ may be expressed in terms of the partial derivatives in the direction $x_i$ and the direction cosines, $\cos \theta_i$. We may compute the spectral flux $F_{\nu \theta}$ in the direction $x_i$ by using the first two terms on the right-hand side of Eq. (2-2.10), viz.,

$$F_{\nu \theta} = B_\nu \int_0^{4\pi} \cos \theta_1 \, d\Omega - \frac{1}{k_\nu} \left( \frac{\partial B_\nu}{\partial x_1} \int_0^{4\pi} \cos^2 \theta_1 \, d\Omega + \frac{\partial B_\nu}{\partial x_2} \int_0^{4\pi} \cos \theta_2 \cos \theta_1 \, d\Omega \right)$$

$$+ \frac{\partial B_\nu}{\partial x_3} \int_0^{4\pi} \cos \theta_3 \cos \theta_1 \, d\Omega$$

(2-2.11)

in view of the spatial symmetry properties of the cosine function. Consideration of the other directions gives the result

$$F_\nu = -\frac{4\pi}{3} \frac{1}{k_\nu} \nabla B_\nu.$$ 

(2-2.12)

* Higher-order terms have been examined by Thomas. 14a
Integration over frequency yields

\[ F = - \frac{4\pi}{3} \frac{1}{k_R} \nabla B = - \frac{16\pi T^3}{3k_R} \nabla T, \quad (2-2.13) \]

where

\[ \frac{1}{k_R} = \frac{1}{|\nabla B|} \int_0^\infty \frac{1}{k_\nu} |\nabla B_\nu| \, d\nu = \left( \frac{\pi}{4\pi T^3} \right) \int_0^\infty \frac{1}{k_\nu} \frac{dB_\nu}{dT} \, d\nu \quad (2-2.14) \]

defines the Rosseland mean absorption coefficient \( k_R \), in the direction in which the rate of change of \( B_\nu \) with temperature is \( dB_\nu/dT \) and the local value of the absorption coefficient is \( k_\nu \).

**2-2C **EDDINGTON'S DERIVATION OF THE ROSSELAND MEAN ABSORPTION COEFFICIENT.\(^{1-4,11-15}\) It is instructive to derive Eq. (2-2.14) for the Rosseland mean absorption coefficient by an alternative method, namely, that originally used by Eddington.

At the steady state, in each volume element, the emitted and absorbed radiant energies must be exactly equal. But the absorbed radiant energy per unit surface area of an infinitesimal volume element, allowing for radiant energy incident from all directions and averaging over the complete solid angle, is proportional to

\[ J_\nu = \int I_\nu \frac{d\Omega}{4\pi}; \quad (2-2.15) \]

on the other hand, the emitted radiant energy per unit surface area, for this same volume element, is proportional to

\[ \int B_\nu \frac{d\Omega}{4\pi} = B_\nu \]

since \( B_\nu \) is isotropic. Hence

\[ B_\nu = \int I_\nu \frac{d\Omega}{4\pi} = J_\nu \quad (2-2.16) \]

since the same proportionality constant occurs in the expressions for the emitted and absorbed radiant energies. The quantity \( J_\nu \) is one of a set of mean values introduced by Eddington. The preceding relation is often referred to as a statement of the continuity equation for radiative transfer in a stationary system.

Again following Eddington, we also define the following mean values:

\[ H_\nu = \int I_\nu \cos \theta \frac{d\Omega}{4\pi} \quad (2-2.17) \]
and

\[ K_\nu = \int I_\nu \cos^2 \theta \frac{d\Omega}{4\pi}. \] (2-2.18)

From the basic transfer relation given in Eq. (2-2.1) and the continuity equation for radiative transfer, we obtain the relations

\[ -\frac{1}{k_\nu} \frac{dH_\nu}{dx} = 0 \]

and, since the mean value of \( \cos \theta \) is zero,

\[ -\frac{1}{k_\nu} \frac{dK_\nu}{dx} = H_\nu. \] (2-2.19)

Furthermore, since \( \cos^2 \theta = \frac{1}{3} \), we find, for optically rather opaque systems with relatively small temperature gradients for which \( I_\nu \) is not a strongly varying function of direction,

\[ K_\nu \simeq \frac{1}{3} J_\nu = \frac{1}{3} B_\nu. \] (2-2.20)

An analogous argument to that given above, but integrating over all frequencies and using the mean absorption coefficient \( \bar{k} \), leads to the expression

\[ -\frac{1}{\bar{k}} \frac{dK}{dx} = H \] (2-2.21)

where

\[ H = \int_0^\infty dv \int I_\nu \cos \theta \frac{d\Omega}{4\pi} \]

and

\[ K = \int_0^\infty dv \int I_\nu \cos^2 \theta \frac{d\Omega}{4\pi}. \]

In order to obtain a consistent set of definitions, we have thus imposed the requirements

\[ H = \int_0^\infty H_\nu \, dv \] (2-2.22)

and

\[ K = \int_0^\infty K_\nu \, dv. \] (2-2.23)

* It should be noted that it is not necessary to assume that the mean absorption coefficient is the same for absorption and emission. Rather, we require that the quantity \( (1/\bar{k}) \int_0^\infty k_\nu B_\nu \, dv \) is isotropic.
It follows now from Eqs. (2-2.21) to (2-2.23) that

\[- \frac{1}{k} \frac{d}{dx} \int_{0}^{\infty} K_v \, dv = \int_{0}^{\infty} H_v \, dv\]

and, after replacing \( H_v \) by \(-\frac{1}{k_v} \frac{dK_v}{dx}\) according to Eq. (2-2.19),

\[\frac{1}{k} \frac{dK}{dx} = \int_{0}^{\infty} \frac{1}{k_v} \frac{dK_v}{dx} \, dv. \quad (2-2.24)\]

Using now the approximations

\[K \approx \frac{1}{3} B \quad \text{and} \quad K_v \approx \frac{1}{3} B_v,\]

where

\[B = \int_{0}^{\infty} B_v \, dv,\]

and solving for

\[k = k_R,\]

we find that the reciprocal of the Rosseland mean free path \(l_R\) and the Rosseland mean absorption coefficient \(k_R\) are given by the relation

\[\frac{1}{l_R} = \frac{\int_{0}^{\infty} \frac{dB_v}{dx} \, dv}{\int_{0}^{\infty} \frac{1}{k_v} \frac{dB_v}{dx} \, dv} = \frac{\int_{0}^{\infty} \frac{dB_v}{dT} \, dv}{\int_{0}^{\infty} \frac{1}{k_v} \frac{dB_v}{dT} \, dv}, \quad (2-2.25)\]

where we have multiplied the numerator and denominator in the last relation by the rate of change of slab thickness with temperature \((dx/dT)\). The preceding considerations constitute an explicit demonstration of the physical meaning of the Rosseland mean absorption coefficient and of the fact that it is a useful, approximate, "local" value of the absorption coefficient in radiative transfer calculations for relatively opaque gases.

2.2D APPROXIMATE (CONTINUUM RADIATION) MEAN-FREE PATHS FOR POLYELECTRONIC ATOMS AT ELEVATED TEMPERATURES. A useful approximate procedure for calculating the continuum opacity resulting from bound-free and free-free transitions has been given by Raizer\textsuperscript{16,17} and has been corrected and refined by Pappert and Penner\textsuperscript{18} and by Penner and Thomas.\textsuperscript{19} The physical ideas involved in these studies have already been used in Section 1-3C for the theoretical calculation of spectral values for the continuum absorption coefficient. Armstrong's\textsuperscript{20} values for the Rosseland and Planck means for nitrogen at 5 and 20 eV and for
the number density range $\sim 5 \times 10^{20}$ to $\sim 5 \times 10^{15}$ cm$^{-3}$ will be compared with the values computed with Raizer's formulae (denoted by R) and a modified version of Raizer's formulae (denoted by MR). Pappert and Penner$^{18}$ have also compared the (R) and (MR) formulae for the Rosseland mean with calculations by Bernstein and Dyson$^{21}$ for fluorine in the number density range $\sim 10^{21}$ to $\sim 10^{18}$ cm$^{-3}$ at 10 eV. In these cases, the (MR) formulae are the better approximations; agreement with the numerical data is good to within about a factor of 2 over most of the frequency range not close to the photoelectric edges.

To illustrate that a certain measure of caution is required in applying the approximate formulae, a comparison has been made$^{18}$ between the Bernstein-Dyson data for beryllium at 10 eV and the (R) and (MR) formulae. In this case, the (MR) formula is superior to the (R) formula but it leads nevertheless to results which differ by about a factor of 10 from the Bernstein-Dyson data. The apparent reason for this discrepancy (as discussed below) is the existence of a "window." Quite generally, the more proliferous the levels, the better the approximation scheme. Thus the theoretical formulae should apply reasonably well to high-Z elements.

It should perhaps be emphasized that the approximation formulae allow only for the continuous opacity. At sufficiently low densities, the bound-bound contributions are expected to be negligibly small; however, at higher densities, the bound-bound contributions may become significant. This problem has received special attention but is beyond the scope of the present discussion.

An expression for the total linear absorption coefficient associated with the bound-free contribution from an $m$-ion (i.e., an $m$ times ionized atom) and the free-free contribution from an $(m+1)$-ion has been given in Eq. (1-3.18). Using this relation for $k(x)^{\dagger}$, we may then compute the Planck ($\bar{k}_P$) and Rosseland ($\bar{k}_R$) mean absorption coefficients. The Rosseland mean free path is defined by

$$l_R = \frac{1}{\bar{k}_R} = \int_0^\infty \frac{G(x) \, dx}{k(x)} , \quad G(x) = \frac{15}{4\pi^4} \frac{x^4 e^{-x}}{(1 - e^{-x})^2} , \quad (2-2.26)$$

* The Bernstein-Dyson data at 5 eV have not been used since omission of the weakly bound levels has presumably led to an underestimate, by about a factor of 3, of the oxygen and nitrogen opacity at this temperature.$^{21}$ However, the data at 10 eV appear to be sufficiently accurate for comparisons.

† It should be noted that we are using $k$ as the symbol for the linear absorption coefficient in this chapter. In Chapter 1, we used $\chi$ for this same quantity in order to avoid confusion with the Boltzmann constant.
and the reciprocal of the Planck mean free path is given by

\[ l_p^{-1} = \bar{k}_P = \int_0^\infty k(x) G_1(x) \, dx, \quad G_1(x) = \frac{15}{\pi^4} \frac{x^3}{(e^x - 1)}. \quad (2-2.27) \]

The Planck mean absorption coefficient may be directly evaluated from Eq. (2-2.27) with the result [see Eq. (1-3.18) for \( k(x) \)]

\[ l_p^{-1} = \bar{k}_P = \frac{45 \xi}{\pi^4 \bar{\theta}^2} \sum_m N_m \frac{g_e g_{m+1}}{g_m} (m + 1)^2 x_{1m} \exp(-x_{1m}). \quad (2-2.28) \]

An explicit evaluation of \( l_R \) is not possible without further simplification. However, it is generally reasonable to assume that the dominant contribution to \( \bar{k}_R \) comes from \( x < x_{1m} \); it should be noted that this is definitely not the case for the Planck mean which, by virtue of the definition given in Eq. (2-2.27), is quite sensitive to the high-energy photons—in fact, two-thirds of the contribution to \( \bar{k}_P \) in Eq. (2-2.28) comes from \( x > x_{1m} \). Use of the first relation in Eq. (1-3.18) for all values of \( x \) leads to the expression

\[ l_R = \frac{0.87 \theta^2}{\xi} \left[ \sum_m N_m \frac{g_e g_{m+1}}{g_m} (m + 1)^2 \exp(-x_{1m}) \right]^{-1}. \quad (2-2.29) \]

To obtain an explicit relation for \( l_p \) and \( l_R \), we now use again the Raizer approximations discussed in Eq. (1-3.21) and the sums specified in Eqs. (1-3.22) to (1-3.23). Only two terms \([m = \bar{m} - \frac{3}{2} \text{ and } m = \bar{m} - \frac{1}{2}]\) contribute to these sums and we thus find the following result for the sum occurring in the Planck mean:

\[ \sum_m N_m \frac{g_e g_{m+1}}{g_m} (m + 1)^2 x_{1m} \exp(-x_{1m}) \]

\[ \approx \frac{N^2 \bar{m}}{2 A' \bar{\theta}^{5/2}} [(\bar{m} - \frac{3}{2})^2 x_{1\bar{m} - \frac{3}{2}} + (\bar{m} + \frac{1}{2})^2 x_{1\bar{m} - \frac{1}{2}}] \quad (2-2.30) \]

if we again set the statistical weight ratio \((g_e g_{m+1}/g_m)\) equal to unity. Similarly, the sum occurring in the expression for the Rosseland mean is

\[ \sum_m N_m \frac{g_e g_{m+1}}{g_m} (m + 1)^2 \exp(-x_{1m}) \approx \frac{N^2 \bar{m}}{A' \bar{\theta}^{5/2}} (\bar{m}^2 + \frac{1}{4}). \quad (2-2.31) \]
Fig. 2-2.1. The Rosseland mean free path (in cm), as a function of number density (in cm\(^{-3}\)), for nitrogen at 5 eV. (The symbols A, R, and MR identify results obtained by Armstrong and derived from Raizer and modified-Raizer approximations, respectively).

Fig. 2-2.2. The Rosseland mean free path (in cm), as a function of number density (in cm\(^{-3}\)), for nitrogen at 20 eV.
Fig. 2-2.3. The Planck mean free path (in cm), as a function of number density (in cm$^{-3}$), for nitrogen at 5 eV.

Fig. 2-2.4. The Planck mean free path (in cm), as a function of number density (in cm$^{-3}$), for nitrogen at 20 eV.
Insertion of these expressions into Eqs. (2-2.28) and (2-2.29) leads to the following approximations for the Rosseland and Planck means:

\[
l_R = \frac{3.6 \times 10^{3.6 \theta^{7/2}}}{N^2 \bar{m} (\bar{m}^2 + \frac{1}{4})}, \quad (2-2.32)
\]

\[
l_P = \frac{1.8 \times 10^{3.6 \theta^{7/2}}}{N^2 \bar{m} [(\bar{m} - \frac{1}{2})^2 x_{\bar{m}-1} + (\bar{m} + \frac{1}{2})^2 x_{\bar{m}-1}]}. \quad (2-2.33)
\]

Equations (2-2.32) and (2-2.33) are the modified Raizer formulae. They should be contrasted with the following expressions for the Rosseland and Planck means, respectively, obtained from Raizer's original method:

\[
l_R \approx \frac{3.6 \times 10^{3.6 \theta^{7/2}}}{N^2 \bar{m} (\bar{m} + 1)^2}. \quad (2-2.34)
\]

**Fig. 2-2.5.** The Rosseland mean free path (in cm), as a function of number density (in cm\(^{-3}\)), for fluorine at 10 eV. The designation B-D refers to numerical results derived by Bernstein and Dyson.\(^{21}\)
and

\[ l_p \sim \frac{0.9 \times 10^{37} \theta^{1/2}}{N^2 \tilde{m}(\tilde{m} + 1)^2 I_{\tilde{m} - 1}}. \] (2-2.35)

Equations (2-2.34) and (2-2.35) differ from Raizer's original results in being roughly smaller by a factor of 2; this difference arises because Raizer did not use a consistent approximation procedure in treating the statistical weight ratio \( g e_{m+1}/g_m \), as has been discussed previously in Section 1-3C. The mean ionization level per atom \( \tilde{m} \) is also estimated as in Section 1-3C.

Results derived from Eqs. (2-2.32) to (2-2.35) are compared with Armstrong's results for the Rosseland and Planck means for nitrogen at 5 and 20 eV and for the number density range \( \sim 5 \times 10^{20} \text{ cm}^{-3} \) to \( 5 \times 10^{15} \text{ cm}^{-3} \). The results derived from Eqs. (2-2.32) and (2-2.35) have been checked with the Dyson-Bernstein data for fluorine at 10 eV in the density range \( \sim 10^{21} \text{ cm}^{-3} \) to \( \sim 10^{18} \text{ cm}^{-3} \). The relevant plots are

![Figure 2-2.6](image-url)

**Fig. 2-2.6.** The Rosseland mean free path (in cm), as a function of number density (in cm\(^{-3}\)), for beryllium at 10 eV. The designation B-D refers to numerical results derived by Bernstein and Dyson.\(^{21}\)
shown in Figs. 2-2.1 through 2-2.5. It is seen that the modified Raizer formulae are the better approximation formulae for the Rosseland mean and give agreement to about a factor of 2 with the numerical results. However, more important than the difference between the (R) and (MR) approximations is the fact that either set of equations provides reasonable estimates for the Rosseland and Planck means in high-temperature gases.

In conclusion, we wish to insert a word of caution. In Fig. 2-2.6 we have shown a comparison between the Bernstein-Dyson results for beryllium at 10 eV and the corresponding results obtained from Eqs. (2-2.32) and (2-2.34). The reason for the large discrepancy in this case is a "window" for values of \( x = \frac{h\nu}{\theta} > 2 \) (see Fig. 2-2.7). One of the important approximations involved in establishing the equations for the Rosseland mean is contained in Eq. (1-3.17). This relation should be used only if at least one of the ions which contribute significantly to the opacity has bound levels in the interval \( 2 \leq x \) (in eV) \( \leq 8 \); otherwise (cf. Fig. 2-2.6) the approximation formulae will seriously underestimate the Rosseland mean.

![Fig. 2-2.7. The product of absorption coefficient \( \kappa(x) \) and density \( \rho \) as a function of \( x = \frac{h\nu}{kT} \) (where \( k \) is the Boltzmann constant) for beryllium at 10 eV, for a mass density \( = 5.86 \times 10^4 \text{ g/cm}^3 \) (from Bernstein and Dyson)].
2-3 The conservation equations with radiant-energy transport and the coupling of radiative and convective energy transfer

The conservation equations for reacting, multicomponent gas mixtures without radiant energy transport may be derived either by using kinetic theory\textsuperscript{22,23} or by using continuum theory.\textsuperscript{24,25} In the following discussion, we indicate the required modifications when proper allowance is made for radiant-energy transport.

We assume that the effects of gravity, scattering of thermal radiation, the second coefficient of viscosity, and electromagnetic forces may be neglected. We postulate that the gas and the thermal radiation may be treated as two coexistent, interacting continua, the former composed of particles of matter and the latter of photons.

2-3A Species conservation equations. Using Gibbs' vector notation, the continuity equations for reacting, multicomponent gas mixtures become\textsuperscript{22-27}

\[
w_K = \frac{D}{Dt} (\rho Y_K) + \rho Y_K \nabla \cdot \mathbf{v} + \nabla \cdot (\rho Y_K \mathbf{v}_K)
\]

(2-3.1)

for each species \( K \), where \( w_K \) is the net production of mass of species \( K \) per unit volume per unit time, \( \rho \) is the total density, \( Y_K \) is the mass fraction of species \( K \), \( \mathbf{v} \) is the mass-weighted average velocity of the mixture, \( \mathbf{v}_K \) is the diffusional velocity of species \( K \), and \( D/Dt \) is the total time derivative following the mass-weighted average motion. The conservation of species equations are not altered by thermal radiation, provided we define \( w_K \) to include the production of \( K \) by photochemical reactions as well as by conventional chemical reactions.

2-3B Momentum equation. Modification of the momentum equation to include radiation is readily accomplished when it is noted that the interaction of radiation with a gas produces a body force on the gas. The momentum equation, including body forces, has the form\textsuperscript{22-27}

\[
\rho \frac{D\mathbf{v}}{Dt} = - \nabla \cdot\mathbf{p} + \mathbf{f} \equiv - \nabla p + \nabla \cdot \mathbf{\tau} + \mathbf{f},
\]

(2-3.2)

* The following discussion, derived from Penner and Patch,\textsuperscript{4} is a summary of heuristic arguments that may be used to obtain the corrections to the conservation equations produced by radiant-energy exchange. A complete derivation of the equations of radiation gas dynamics was carried out by Thomas.\textsuperscript{21b} Other derivations, with different relativistic formulations and with radiation treated either as a photon gas or else described from the standpoint of electromagnetic theory, have also been carried out.\textsuperscript{21c-21h}
where \( p \) is the gas pressure tensor, \( \dot{p} \) is the gas pressure, \( \mathbf{f} \) is the body force per unit volume, and \( \mathbf{\tau} \) is the symmetrical viscous stress tensor, each component of which is given in Cartesian coordinates by

\[
\tau_{ij} = -\frac{2}{3} \mu \left( \frac{\partial \sigma_i}{\partial x_j} + \frac{\partial \sigma_j}{\partial x_i} \right) = -p_i + \dot{p} \delta_{ij}. \tag{2-3.3}
\]

Here \( \mu \) is the viscosity coefficient of the gas (including diffusion contributions), \( \delta_{ij} \) is the Kronecker delta function (\( \delta_{ij} = 0 \) for \( i \neq j \) and \( \delta_{ij} = 1 \) for \( i = j \)), \( x_i \) represents the three Cartesian coordinates, and the appearance of \( l \) twice in the first term on the right-hand side of Eq. (2-3.3) indicates a summation over the components \( l \).

A body force on the gas arises because each photon has momentum \( \hbar \nu/c \), where \( \hbar \) is Planck’s constant, \( \nu \) is the frequency, and \( c \) is the velocity of light. Consider a thin cylindrical volume of cross section \( dA \) and length \( dL \) (see Fig. 2-3.1). Since the body force is the rate of change of photon momentum produced by absorption, the body force in the \( A \)-direction, per unit frequency, is

\[
f_{\nu} = \frac{1}{c} k_{\nu} (F_{\nu} \cdot dA) dL, \tag{2-3.4}
\]

where \( k_{\nu} \) is the linear absorption coefficient including the induced emission factor, and \( F_{\nu} \) is the radiation flux density per unit frequency. For each frequency, any infinitesimal volume may be thought of as consisting of a number of such cylindrical volumes, which may be specified to have their areas parallel to \( F_{\nu} \). Consequently, the body force per unit volume per unit frequency is

\[
f_{\nu} = \frac{1}{c} k_{\nu} F_{\nu}. \tag{2-3.5}
\]
Integrating over all frequencies, we find that

\[ f = \frac{1}{c} \int_0^\infty k_\nu F_\nu \, dv. \quad (2-3.6) \]

We shall now outline an alternative derivation of Eq. (2-3.6). We first define the radiation pressure tensor \( p^R_{ij} \) as the change in momentum of photons which are specularly reflected from a surface, viz.,

\[ p^R_{ij} = \frac{1}{c} \int_0^\infty \int_0^{4\pi} I_\nu \cos \theta_i \cos \theta_j \, d\Omega \, dv, \quad (2-3.7) \]

where \( \theta_i \) and \( \theta_j \) are the angles between the direction of \( I_\nu \) and the directions \( x_i \) and \( x_j \), respectively. The body force is determined by the divergence of the radiation pressure tensor,

\[ f_i = -\frac{\partial p^R_{ij}}{\partial x_j} = -\frac{1}{c} \int_0^\infty \int_0^{4\pi} \frac{\partial I_\nu}{\partial x_j} \cos \theta_i \cos \theta_j \, d\Omega \, dv. \quad (2-3.8) \]

We now multiply the equation of transfer, Eq. (2-2.1), by \( \cos \theta_i \) and integrate over all solid angles to obtain the result

\[ \int_0^{4\pi} \frac{\partial I_\nu}{\partial x_j} \cos \theta_j \, d\Omega = k_\nu \int_0^{4\pi} B_\nu \cos \theta_i \, d\Omega - k_\nu \int_0^{4\pi} I_\nu \cos \theta_i \, d\Omega. \quad (2-3.9) \]

Since \( B_\nu \) is isotropic, the first term on the right-hand side of Eq. (2-3.9) vanishes; the second term is \( k_\nu F_{\nu i} \) with the flux \( F_{\nu i} \) defined by Eq. (2-1.9). Combining Eqs. (2-3.8) and (2-3.9), we find that

\[ f_i = \frac{1}{c} \int_0^\infty k_\nu F_{\nu i} \, dv. \quad (2-3.10) \]

Equation (2-3.10) is equivalent to Eq. (2-3.6) and indicates that radiation effects may be introduced into the momentum equation as a radiation pressure tensor \( p^R_{ij} \), which is to be added to the ordinary gas pressure tensor \( p_{ij} \).

For the case where the diffusion approximation is applicable at every frequency,

\[ F_\nu = -\frac{4\pi}{3k_\nu} \nabla B_\nu. \quad (2-3.11) \]

Substitution of Eq. (2-3.11) into Eq. (2-3.6) gives

\[ f = -\frac{4\pi}{3c} \nabla B = -\frac{16\sigma T^3}{3c} \nabla T. \quad (2-3.12) \]
When the diffusion approximation is not valid, no generally applicable simplification of Eq. (2-3.6) is possible. It will be shown in Section 2-4B that the body force due to radiation is negligibly small in most problems of interest to us.

2-3C Energy conservation equation. The energy equation is modified in several ways when radiation is included. First of all, we must make suitable allowances for both the radiation pressure tensor given in Eq. (2-3.7) and for the body force of Eq. (2-3.6). Secondly, we must add the radiation energy density $u^R$ to the internal energy of the material particles. Since all photons travel with the velocity of light $c$, the radiation energy density may be expressed in terms of the specific intensity (steradiancy) $I_\nu$ as follows:

$$u^R = \frac{1}{c} \int_0^\infty \int_0^{4\pi} I_\nu \, d\Omega \, dv. \quad (2-3.13)$$

The most important radiation contribution to the energy equation is generally that given by the radiative transfer term. Consider an arbitrary volume of gas, $V$ moving with the mass-weighted average velocity of the gas mixture, $\vec{v}$ (see Fig. 2-3.2). Let $dA$ be an element of the surface bounding this volume. The net rate of energy transfer across $dA$ is

$$\mathbf{F} \cdot dA. \quad (2-3.14)$$

Integrating Eq. (2-3.14) over the entire surface, dividing by $V$, applying the divergence theorem, and taking the limit as $V$ is made arbitrarily small, the rate of energy loss by the gas per unit volume ($\epsilon$) is seen to be

$$\epsilon = \nabla \cdot \mathbf{F}. \quad (2-3.15)$$
Using the above modifications in the known expression for energy conservation, it is readily shown that the energy equation becomes

\[ \rho \frac{D}{Dt} \left( \epsilon + \frac{u^R}{\rho} \right) = \sum_K f_K \cdot V_K - (p_{ij} + p_{ij}^R) \frac{\partial v_j}{\partial x_i} - \nabla \cdot q_D + \nabla \cdot (\lambda \nabla T) - \nabla \cdot F, \]

(2-3.16)

where \( \epsilon \) is the internal energy of the gas mixture per unit mass and includes diffusion contributions, \( f_K \) is the radiation body force per unit volume of species \( K \), \( q_D \) is the heat flux density due to diffusion, \( \lambda \) is the thermal conductivity, and \( x_i \) represents the three Cartesian coordinates. The appearance of \( i \) and \( j \) twice each in the second term on the right-hand side of Eq. (2-3.16) indicates a double summation over the repeated indices. The body force per unit volume \( f \) is, of course, independent of the coordinate system. Consequently, despite the fact that the different species identified by the subscript \( K \) have different velocities, it follows, by analogy with Eq. (2-3.6), that

\[ f_K = \frac{1}{c} \int_0^\infty k_{\nu,K} F_\nu \, d\nu, \]

(2-3.17)

where \( k_{\nu,K} \) is the linear absorption coefficient of species \( K \).

In the next section, it will be shown that the radiation contribution to Eq. (2-3.16) is usually well approximated by the radiative transfer term \( \nabla \cdot F \) alone, since the radiation energy density and radiation body force (radiation pressure) terms are often negligibly small. It may be seen from Eq. (2-2.7) that the radiative transfer term assumes the following form in the optically thin limit:

\[ \nabla \cdot F = 4kR \rho a T^4. \]

(2-3.18)

On the other hand, when the Rosseland (diffusion) approximation is applicable, Eq. (2-2.13) gives

\[ \nabla \cdot F = - \nabla \cdot \left( \frac{16\sigma T^3}{3kR} \nabla T \right). \]

(2-3.19)

The quantity \( 16\sigma T^3/3kR \) is the radiative analogue of a thermal conductivity, as may be seen by comparing Eq. (2-3.19) with the next to the last term in Eq. (2-3.16).

In the derivations in this section, local thermodynamic equilibrium has been assumed. However, it has been found that, under some reentry conditions, for example, the \( N^+_2(B^2\Sigma) \) excited state is not in local thermodynamic equilibrium because it is depopulated by radiation faster than...
collisions can repopulate it. If this effect is neglected, radiation from the
gas may be seriously overestimated. To obviate such difficulties, three
quantities are needed for each state not in local thermodynamic equi-
librium: (a) one of the Einstein coefficients or the absorption coefficient
under conditions producing local thermodynamic equilibrium, (b) either
the excitation or the deexcitation collision cross section, and (c) the net
rate of production of a given state by chemical processes. With the help
of the principle of detailed balancing, the nonequilibrium populations
may then be calculated. For nonequilibrium processes locally, Eqs.
(2-3.4) through (2-3.19) should be replaced by expressions in terms of
Einstein coefficients. For some additional remarks on nonequilibrium
radiation in heated air, we refer to Section 3-4.

2-4 Similarity parameters for radiative-energy transfer in isothermal and nonisothermal gas mixtures

The similarity groups for multicomponent, reacting gas mixtures with radiative transport will now be derived (Sections 2-4A to 2-4D). The
resulting relations are used to consider the feasibility of scaling for flow
processes with radiative-energy transport under highly simplified,
representative conditions (Sections 2-4E to 2-4F). Next the scaling
parameters are derived for radiant-energy emission from isobaric and
isothermal gases for arbitrary opacities and various spectral-line and
molecular-band models (Section 2-4G). Scaling parameters for radiant-
energy emission from isobaric but nonisothermal systems are discussed,
for arbitrary opacities and various spectral-line and molecular-band
models, under the restrictions imposed on the allowed temperature
profiles for dispersion and Doppler lines by the Eddington-Barbier
approximation (Section 2-4H). Finally, we consider the radiative scaling
properties for representative temperature profiles for both collision-
broadened and Doppler-broadened line profiles on the basis of exact
numerical calculations that have been performed for a rotational spectral
line of a molecular vibration-rotation band (Sections 2-4I and 2-4J). It
appears that simple scaling rules generally constitute fair approximations
for dispersion lines in nonisothermal systems but corresponding relations
apply to lines with Doppler contour only in the transparent gas regime.

2-4A Determination of similarity parameters. The techniques
for identifying the similarity groups for systems described by a set of
conservation equations are well known and have been developed in
detail elsewhere for reacting, multicomponent gas mixtures.25,26 Some
generalization of procedure may be made without difficulty by following
the method described below [Eqs. (2-4.4) and (2-4.5) are not given in
Penner\textsuperscript{25}].

We start with the set of conservation equations [see Eqs. (12), (13),
(16) to (18), and (23) in Penner,\textsuperscript{26} Chapter XVIII]; next we use a binary
mixture approximation for each of the diffusion velocities [using Eq. (4)
in Penner,\textsuperscript{26} p. 243, we replace the subscript 1 by \( K (K = 1, 2, \ldots, n) \) for
an \( n \)-component mixture, and we replace \( Y_2 \) by \( 1 - Y_K \) and \( W_2 \) by \( \bar{W}' = \)
average molecular weight of the fluid mixture remaining without
species \( K \)]; finally, we write the complete heat flux vector [see Eq. (23)
in Penner,\textsuperscript{26} p. 239] in the form\textsuperscript{27}

\[
\mathbf{q} = - \left[ \lambda + \lambda_{eh} \right] \nabla T.
\]

Here \( \lambda \) is the thermal conductivity associated with molecular collisions
and \( \lambda_{eh} \) identifies the thermal conductivity associated with chemical
reaction; \( \lambda_{eh} \) is given explicitly by the expression\textsuperscript{27}

\[
\lambda_{eh} = \sum_{K=1}^{n} \left( \frac{c}{\rho} \right)^2 D_{KK'} W_{K'} H_K \frac{dX_K}{dT}
\]

where \( (c) \) is the total number of moles per unit volume of mixture; \( \rho \) denotes fluid density; \( D_{KK'} \) is the multicomponent diffusion coefficient,
which is a known function of the binary diffusion coefficients and the
mixture composition;\textsuperscript{22} \( W_{K'} \) is the molecular weight of species \( K' \);
\( H_K \) is the molar enthalpy of species \( K \); \( X_{K'} \) is the mole fraction of
species \( K' \); and \( T \) is the temperature.

Using the specified starting relations and standard procedures,\textsuperscript{25} we
find the following set of similarity parameters for multicomponent,
reacting gas mixtures without radiative energy transport, if the subscript
0 identifies suitably chosen reference conditions:

\[
\gamma_0 = \text{ratio of specific heat at constant pressure} \left( c_{p,0} \right) \text{to the specific}
\text{heat at constant volume} \left( c_{v,0} \right) \text{for the fluid mixture}; \quad (2-4.1)
\]

Reynolds numbers \( \equiv \text{Re}_i = \rho_0 \nu_i L_i / \mu_0 \) (\( \rho = \text{density}, \nu = \text{flow}
velocity, \( L_i = \text{ith characteristic length}, \mu = \text{mixture viscosity}; \)
\( i = 1, 2, \ldots, s \)); \quad (2-4.2)

Schmidt numbers \( \equiv Sc_K = \mu_0 / \rho_0 D_{K,0} \) (\( D_K = \text{diffusion coefficients}
for species} \( K, K = 1, 2, \ldots, n \)); \quad (2-4.3)

\( Sc_{K'} = \mu_0 / \rho_0 k_{TK,0} D_{K,0} \) (\( k_{TK} = \text{thermal diffusion ratio for species}
\( K \), \( D_{TK} = k_{TK} D_K = \text{thermal diffusion coefficient for species} \( K \)\)); \quad (2-4.4)
Prandtl number \( \equiv Pr = \frac{\varepsilon_{p,0}\mu_0}{\lambda_0} \);  
(2-4.5)

\( Pr' = \frac{\varepsilon_{p,0}\mu_0\lambda_{ch,0}}{\lambda_0} \) (this group will actually remain invariant if the groups \( Pr, Sc_K \) and \( D_{III,t,r} \) are fixed);  
(2-4.6)

Mach number \( \equiv M = \left( \rho_0\varepsilon_0^2/\gamma_0 p_0 \right)^{1/2} \) \((p = \text{pressure})\);  
(2-4.7)

Froude numbers \( \equiv v_0^2gL_t \) \((g = \text{gravitational acceleration})\);  
(2-4.8)

Damköhler's first similarity groups \( \equiv D_{I,t,r} = L_t U_{r,0}/v_0 \) \((U_{r,0} = \text{characteristic reaction frequency for the } r\text{th chemical process; } r = 1, 2, \ldots, m)\);  
(2-4.9)

Damköhler's third similarity groups \( \equiv D_{III,t,r} = q_r' L_t/v_0\varepsilon_{p,0} T_0 \)  
(2-4.10)

\[ \varphi = \frac{1}{2} v_0^2 \left( \frac{\varepsilon_{p,0}}{\gamma_0} \right) T_0. \]  
(2-4.11)

2-4B Radiative-energy transport in the diffusion approximation. Throughout the following discussion, we neglect photochemical reactions. In the diffusion approximation, the radiative transfer term \( \nabla \cdot F \) is added to the effective conductivity in the form of a radiative conductivity \( \lambda_r \) (cf. Section 2-3), where

\[ \lambda_r = 16\sigma T^3/3k_R. \]

Hence an additional analogue of the Prandtl number will appear, viz.,

\[ Pr^* = \frac{\varepsilon_{p,0}\mu_0}{\lambda_{ra,0}}. \]  
(2-4.12)

Thus \( Pr^* \) is a measure of the relative importance of the viscous terms to the radiative transfer term \( \nabla \cdot F \) in the energy equation [see Eq. (2-3.16)]. The Rosseland mean absorption coefficient \( k_R \) is obtained from \( k_r \) by the integral over frequency given in Eq. (2-2.14). Thus the constancy of the similarity group \( Pr^* \) imposes severe restrictions on the allowed values of \( k_r \), although it is not required that the products of the linear spectral absorption coefficients and the characteristic lengths be maintained invariant, as is necessary for radiative transfer problems with arbitrary opacities (see Section 2-4D) in isothermal systems.

An alternative way of describing the relative effect of the radiative transfer term is to form the ratio of \( \nabla \cdot F \) to the enthalpy flow rate per unit volume, \( \rho_0 v_0 \varepsilon_{p,0} T_0/\varepsilon_{p,0} L_t \) [in flow problems, it is convenient to use the enthalpy flow rate rather than the internal energy flow rate, as given by the left-hand side of Eq. (2-3.16)]. We obtain the parameter

\[ \Gamma' = \frac{16}{3} \left( \frac{\sigma T_0^3}{\rho_0 v_0 \varepsilon_{p,0}} \right) \frac{1}{k_R T_0 L_t}. \]  
(2-4.13)
Other important similarity parameters in flows with radiant energy transfer are the Bouguer numbers

\[ Bu_{e,i} = k_{e,i}L_i \]  
(2-4.14)

or

\[ Bu_{P,i} = k_{P,i}L_i \]  
(2-4.14a)

or

\[ Bu_{R,i} = k_{R,i}L_i \]  
(2-4.14b)

and the Boltzmann number

\[ Bo = \left( \frac{\rho_0 v_0 \tilde{u}_{e,0}}{\sigma T_0^3} \right). \]  
(2-4.15)

From the preceding definitions, it is apparent that

\[ \Gamma'_i = (16/3)(Bo) Bu_{R,i}^{-1}. \]

We shall now investigate the contributions made to the conservation equations by the radiation body force (radiation pressure gradient) and the radiation energy density terms. In the diffusion approximation, the radiation body force given by Eq. (2-3.12) has a magnitude of order

\[ 16\sigma T_0^4/3cL, \]

which is to be compared with the gas-pressure gradient of order \( p_0/L. \) It is, therefore, expedient to form the parameter

\[ \mathcal{R} = \frac{16\sigma T_0^4}{3cp_0} = \frac{16}{3} \left( \frac{\gamma_0}{\gamma_0 - 1} \right) \left( \frac{\sigma T_0^3}{p_0 v_0 \tilde{u}_{e,0}} \right) \frac{v_0}{c} . \]

This parameter, which is a measure of the ratio of radiation pressure \( p_0^R \) to gas pressure \( p_0 \), is also a measure of the ratio of radiation energy density \( u_0^R \) to the gas internal energy density \( \rho_0 e_0 \). In most applications, the gas velocities are decidedly nonrelativistic, i.e., \( v_0/c \) is so much smaller than unity that \( \mathcal{R} \) will generally also be much smaller than unity. Accordingly, we shall usually be able to neglect the radiation body force and radiation energy density terms, even when the radiative-transfer term is very important, i.e., even when \( \Gamma'_i \) is not small. Of course, the radiation pressure and energy density may become important in astrophysical problems and in some applied problems (e.g., the perturbation of a satellite orbit by radiation pressure is important in “free space” where the gas pressure is negligibly small). However, for all of the problems considered in this monograph, the radiation body force and radiation energy density terms may be neglected.
2-4C Radiative-energy transport for transparent gases. In this section, we consider optically thin (transparent) gases with no external radiation sources. The specific intensity for an optically thin gas is smaller than for the corresponding optically thick gas considered in the preceding section; therefore, the radiation body force and radiation energy density are smaller and will be neglected for the problems considered by us in this volume.

For the optically thin limit, we use Eq. (2-3.18) to obtain the following expression for the ratio of $\nabla \cdot \mathbf{F}$ to the enthalpy flow rate per unit volume $\rho_0 v_0 c_{p,0} T_0 / L_i$:

$$
\Gamma_i = 4 \left( \frac{\sigma T_0^4}{\rho_0 v_0 c_{p,0}} \right) k_{p,0} L_i .
$$

(2-4.16)

The parameter $\Gamma_i$ plays an important role in problems on stellar turbulence,\textsuperscript{11} as well as in many of the problems in radiation gas dynamics considered in Chapter 6.

2-4D Radiative-energy transport for arbitrary opacities.\textsuperscript{*}

When neither the diffusion approximation nor the transparent gas approximation are applicable, proper allowance for radiative-energy transport becomes particularly difficult since local radiant-energy contributions are determined by integrals over the accessible field of view. Neglecting photochemical processes, it appears now that complete simulation is possible, even for isothermal systems, only if a set of similarity groups involving the parameters $B_{u,v,i}$ remains invariant for all important lengths $L_i$ at all frequencies $\nu$. For nonisothermal systems, the problem becomes still further complicated because simulation now requires invariance of similarity groups involving the spectral radiant flux per unit solid angle at $x_1$, with the solid angle measured (see Fig. 2-4.1) in the direction $(x_2 - x_1)/|x_2 - x_1|$; thus the following integral must be constant:

$$
\int_{x_1,C} [B_\nu(x_2)] [k_i(x_2)] \left\{ \exp - \int_{x_2,C} [k_i(x)] dx \right\} dx_2 ,
$$

where the symbol $C$ indicates that integration is to be performed along the path defined by a straight line drawn between the ends of (initially chosen) vectors $x_2$ and $x_1$. Here the spectral blackbody steradiancy, $B_\nu$, and the scalar linear spectral absorption coefficients are, in general, complicated functions of the spacial location.

\textsuperscript{*} This important problem is considered further in Sections 2-4F and 2-4G for isothermal emitters and in Section 2-4H for nonisothermal systems.
2-4E Scaling of flow with maintenance of constant values of $D_{1,i,x}$ and $T_i$ for transparent gases.\(^5\) An interesting special case, which is of some practical importance, involves the scaling of reactive gas flows for a given chemical system in such a way that the reference temperature, velocity, and specific heats are invariant.

Let us identify by the subscripts $H$ and $Mo$ a large-scale system and the model, respectively. It follows then from Eqs. (2-4.9) and (2-4.16) that

$$\frac{L_i;H U_{r,0};H}{v_{0;H}} = \frac{L_i;Mo U_{r,0};Mo}{v_{0;Mo}}$$

and

$$\frac{\sigma T^3_0;H K_{P,0};H L_i;H}{\rho_{0;H} v_{0;H} T_{P,0};H} = \frac{\sigma T^3_0;Mo K_{P,0};Mo L_i;Mo}{\rho_{0;Mo} v_{0;Mo} T_{P,0};Mo}.$$

Let

$$n_\eta = \frac{\eta_{0;H}}{\eta_{0;Mo}} \quad (2-4.17)$$

denote the scaling parameter for the physical variable $\eta$. Furthermore, let us consider two gas mixtures that are injected at the same reference conditions with

$$n_v = n_T = 1.$$

The specified requirements for similarity now reduce to the relations

$$n_{L_i} n_{U_r} = 1 \quad (2-4.18)$$

and

$$n_{K_p} n_{L_i} = n_\rho. \quad (2-4.19)$$

Since

$$n_{U_r} \propto n_\rho^{e_r - 1}, \quad (2-4.20)$$
if \( \mathscr{O}_r \) represents the over-all order of the \( r \)th chemical process, we may rewrite Eq. (2-4.18) in the form

\[
n_{L_i} = n_p^{1-\mathscr{O}_r}.
\]  

(2-4.21)

Therefore, the similarity groups \( D_{1,i,r} \) will remain invariant if model tests are performed in small-scale experiments at elevated density since \( \mathscr{O}_r \) will generally be larger than unity. For example, for second-order processes \( (\mathscr{O}_r = 2) \), \( n_{L_i} = 10 \) if the model tests are performed on a scaled-down device in which all lengths have been reduced by a factor of 10. At the same time, \( n_p^{1-\mathscr{O}_r} = n_p^{-1} = 10 \) if \( n_p = 10^{-1} \), i.e., model tests should be carried out at densities ten times larger than those which are of interest for the full-scale device.

Combining Eqs. (2-4.19) and (2-4.21), we find that

\[
n_{k_p} = n_p^{\mathscr{O}_r},
\]

whence it follows that simultaneous similarity with respect to the groups \( D_{1,i,r} \) and \( \Gamma_i \) can be maintained only for first-order processes since we expect, in general, that \( n_{k_p} \propto n_p \). To summarize, it is not possible to maintain simultaneously similarity in the model tests and in the large-scale experiments with respect to the important similarity groups \( D_{1,i,r} \) and \( \Gamma_i \), except in the unrealistic case that the effective, over-all reaction order is unity.

2-4F Scaling of Flow with Maintenance of Constant Values of \( D_{1,i,r} \) and \( B_u p_{i,t} \) for Transparent Gases ("Binary Scaling").\(^5\) The scaling procedures discussed in the preceding Section 2-4E were designed to assure similarity of chemical reaction profiles and of radiative-energy loss rate, per unit area, relative to the convective-energy transport rate per unit area. Thus they were designed to maintain the flow field in a reacting system invariant under the influence of radiant-energy loss.

For some applications, it may be more important to require invariance of the chemical composition profile and of the absolute value of the radiant-energy emission rate per unit area of reaction front, since the occurrence of radiant-energy loss generally constitutes only a small perturbation on the flow field. In this case, it is pertinent to demand invariance of the Bouguer numbers \( B_u p_{i,t} \), as well as of \( D_{1,i,r} \).

In order to assure invariance of \( B_u p_{i,t} \), we impose the condition

\[
k_{P:0;H}^L_{i;H} = k_{P:0;M_0}^L_{i;M_0}
\]

or

\[
n_{k_p} n_{L_i} = 1.
\]  

(2-4.22)
Equations (2-4.18) and (2-4.21) apply, as before, for $n_v = 1$. If we combine Eqs. (2-4.21) and (2-4.22), we obtain the result

$$n_{kp} = n_{p}^{e_{v}-1},$$

i.e., the condition

$$n_{kp} = n_{p}$$

is now only satisfied for effective, over-all, second-order processes. To summarize, simultaneous invariance of the similarity groups $D_{1,i,r}$ and $Bu_{p,i}$ is possible only for the important case of second-order, over-all rate processes. For this case, the groups $Re_i$ and $M$ are also maintained invariant for a given reacting mixture.

The foregoing considerations lead to the important conclusion that significant model testing of the radiative properties of reacting-gas flows is possible for those cases in which radiant-energy emission does not produce significant perturbations in the flow field (e.g., some transparent gases), provided we are satisfied in scaling the radiative properties per unit area of reacting mixture. The correlated changes in geometrical specifications ($L_i$) and operating density ($\rho$) are determined through Eq. (2-4.21) for $n_v = n_e = n_T = 1$.

2-4G SCALING PARAMETERS FOR RADIANT-ENERGY EMISSION FROM ISOBARIC AND ISOTHERMAL SYSTEMS FOR ARBITRARY OPACITIES AND VARIOUS SPECTRAL-LINE AND MOLECULAR-BAND MODELS. For isothermal systems at the temperature $T$, the emitted steradiancy may be calculated from the equation

$$I(T) = \int B_\omega [1 - \exp(-P_\omega X)] \, d\omega.$$

The results of these calculations are summarized in Table 2-4.1 for selected spectral line shapes and in Table 2-4.2 for selected vibration-rotation band models.

From the data listed in Tables 2-4.1 and 2-4.2, we may draw the following important conclusions:

1. For transparent gases, the steradiancy is directly proportional to $\rho L$, irrespective of the spectral-line contour or of the band model.
2. The important scaling parameter is $\rho L$ at all optical depths for isolated Doppler-broadened lines, and for all band models in which the spectral-line structure is effectively smeared out.
3. The steradiancy is proportional to $(\rho^2 L)^{1/2}$ at moderate to large optical depths (a) for isolated, collision-broadened lines and (b) for
<table>
<thead>
<tr>
<th>Assumed line contour</th>
<th>Function of density ((\rho)) and geometrical length ((L)) which assures constancy of the line radiancy</th>
<th>Restrictive conditions</th>
<th>Basic equations and figures$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure natural line broadening</td>
<td>(\rho L^{<em>b}) ((\rho L)^{1/2</em>})</td>
<td>((SX/2\pi b) &lt; (2/\pi)^n) ((SX/2\pi b) &gt; (2/\pi)^n)</td>
<td>Eq. (4-28) Eq. (4-29) for constant (b)</td>
</tr>
<tr>
<td>Pure collision broadening</td>
<td>(\rho L^{<em>}) ((\rho L)^{1/2</em>})</td>
<td>((SX/2\pi b) &lt; (2/\pi)^n) ((SX/2\pi b) &gt; (2/\pi)^n)</td>
<td>Eq. (4-28) Eq. (4-29 for (b \propto \rho))</td>
</tr>
<tr>
<td>Pure Doppler broadening</td>
<td>(\rho L^{<em>}) (\rho L) [\ln(P'X)]^{1/2</em>})</td>
<td>(P'X &lt; 1)</td>
<td>Eq. (4-8) Eqs. (4-10), (4-11)</td>
</tr>
<tr>
<td>Combined Doppler and collision broadening</td>
<td>(\rho L^{<em>}) ((\rho L)^{1/2</em>})</td>
<td>(P'X &lt; 1) for all (\omega) &quot;Square-root region&quot; of the curves of growth, where (P'X) is sufficiently large to make (R_L) independent of the line contour near the line center for all values of (a)</td>
<td>Eq. (4-35) Fig. 4-6</td>
</tr>
<tr>
<td>First-order Stark broadening</td>
<td>(\rho Ln_e^{*})</td>
<td>(P\omega X &lt; 1) for all (\omega)</td>
<td>Eqs. (3-48), (3-48a)</td>
</tr>
</tbody>
</table>

$^a$ Equation and figure numbers refer to S. S. Penner, "Quantitative Molecular Spectroscopy and Gas Emissivities." Addison-Wesley, Reading, Massachusetts, 1959.

$^b$ Quantities identified with an asterisk indicate that the line radiancy is directly proportional to the specified function of \(\rho\) and \(L\); without asterisk, constancy of the line radiancy is determined by the specified function of \(\rho\) and \(L\), although no simple proportionality exists.

$^c$ \(P\omega\) = spectral absorption coefficient at the wavenumber \(\omega\) (in \(cm^{-1}\)-atm\(^{-1}\)); \(X\) = optical depth (in cm-atm); \(S\) = integrated absorption of a spectral line (in \(cm^{-2}\)-atm\(^{-1}\)); \(b\) = dispersion semi-half-width of a spectral line (in \(cm^{-1}\)); \(P'\) = maximum value of the spectral absorption coefficient at the line center for a spectral line with pure Doppler broadening; \(n_e\) = number of electrons per unit volume.
### Table 2-4.2

**Radiation Scaling Rules for Selected Vibration-Rotation Band Models**

*Isothermal Emitters.*

Reproduced from Penner, Thomas, and Adomeit.*

<table>
<thead>
<tr>
<th>Assumed band model</th>
<th>Function of density ($p$) and geometrical length ($L$) which assures constancy of the band radiancy</th>
<th>Restrictive conditions</th>
<th>Basic equations and figures$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonoverlapping spectral lines</td>
<td>Applicable results for isolated spectral lines (see Table 2-4.1)</td>
<td>The assumed line contour applies for all of the spectral lines which determine the total band radiancy</td>
<td>Applicable equations, as listed in Table 2-4.1</td>
</tr>
<tr>
<td>Rectangular box model</td>
<td>$pL^{**}$</td>
<td>($\alpha X/\Delta \omega ) &lt; 1^c$</td>
<td>Eqs. (11-25), (11-26)</td>
</tr>
<tr>
<td></td>
<td>$pL$</td>
<td>None</td>
<td>Eqs. (11-25), (11-26)</td>
</tr>
<tr>
<td>Just-overlapping-line model</td>
<td>$pL^{*}$</td>
<td>$P_{\omega}X &lt; 1$ for all $\omega$</td>
<td>Eq. (11-50)</td>
</tr>
<tr>
<td></td>
<td>$pL$</td>
<td>None</td>
<td>Eq. (11-49)</td>
</tr>
<tr>
<td></td>
<td>$[\ln(C\alpha X/\Delta \omega )]^{1/2}$</td>
<td>$(\alpha X/\Delta \omega ) \rightarrow \infty$, rotational fine structure smeared out</td>
<td>Eq. (11-143) et seq.</td>
</tr>
<tr>
<td>Statistical distribution of collision-broadened lines</td>
<td>$pL^{*}$</td>
<td>$(\bar{S}X/2\pi b) &lt; 1$ and $(\bar{S}X/\delta *) &lt; 1$</td>
<td>Eq. (11-120a)</td>
</tr>
<tr>
<td></td>
<td>$pL$</td>
<td>$(\bar{S}X/2\pi b) &lt; 1$</td>
<td>Eq. (11-120a)</td>
</tr>
<tr>
<td></td>
<td>$(p^2L)^{1/2}$</td>
<td>$(\bar{S}X/2\pi b) \geq 1$</td>
<td>Eq. (11-120b)</td>
</tr>
<tr>
<td>Statistical distribution of Doppler-broadened lines</td>
<td>$pL^{*}$</td>
<td>$P_{\omega}X &lt; 1$ for all $\omega$; $(\bar{S}X/\delta *) &lt; 1$</td>
<td>Eqs. (11-118), (4-8)</td>
</tr>
<tr>
<td></td>
<td>$pL$</td>
<td>None</td>
<td>Eqs. (11-118), (4-8)</td>
</tr>
<tr>
<td>Statistical distribution of spectral lines with combined Doppler and collision broadening</td>
<td>$pL^{*}$</td>
<td>$P'X \leq 1$ and $\bar{S}X/\delta * \leq 1$</td>
<td>Eq. (11-118) and Fig. 4-6</td>
</tr>
<tr>
<td></td>
<td>$(p^2L)^{1/2}$</td>
<td></td>
<td>Eq. (11-118) and Fig. 4-6</td>
</tr>
<tr>
<td>All other band models</td>
<td>$pL^{*}$</td>
<td>$P_{\omega}X &lt; 1$ for all $\omega$</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ Equation and figure numbers refer to S. S. Penner, "Quantitative Molecular Spectroscopy and Gas Emissivities," Addison-Wesley, Reading, Massachusetts, 1959.

$^b$ Quantities identified with an asterisk indicate that the band radiancy is directly
isolated lines with combined collision and Doppler broadening falling in the "square-root region" of the curves of growth. Also, for statistical distributions of the lines described under (a) and (b), \((\rho^2L)^{1/2}\) is the important scaling parameter.

2-4H Scaling parameters for radiant-energy emission from isobaric but nonisothermal systems for arbitrary opacities and various spectral-line and molecular-band models.\(^5\) One of the classical approaches to the theoretical calculation of radiant-energy emission from nonisothermal systems is exemplified by the Lundblad series development for the solar photosphere.\(^1\textsuperscript{--3.5.11}\)

The spectral steradiancy at the frequency \(v\) and at the optical depth \(\tau_v = 0\), corresponding to the geometrical length \(s = 0\), in the direction \(\theta\) (see Fig. 2-4.2), is given by the relation

\[
I_v(0, \theta) = \int_0^\infty B_v(\tau_v) \left[ \exp\left(-\tau_v \sec \theta \right) \right] (\sec \theta) \, d\tau_v
= \int_0^\infty B_v(s) \left\{ \exp \left[ - \int_0^s k_v(s') \, ds' \right] \right\} k_v(s) \, ds,
\]

\[(2-4.23)\]

![Fig. 2-4.2. Schematic diagram showing the geometric configuration discussed in the text.](image)

proportional to the specified function of \(\rho\) and \(L\); without asterisk, constancy of the band radiancy is determined by the specified function of \(\rho\) and \(L\), although no simple proportionality exists.

\(P_\omega\) = spectral absorption coefficient at the wavenumber \(\omega\) (in cm\(^{-1}\)-atm\(^{-1}\)); \(X\) = optical depth (in cm-atm); \(C\) = Euler's constant; \(\alpha\) = integrated absorption of a vibration-rotation band (in cm\(^{-2}\)-atm\(^{-1}\)); \(\Delta \omega\) = effective width of a vibration-rotation band (in cm\(^{-1}\)); \(S\) = constant value of the integrated absorption for each of the spectral lines contributing to the statistical distribution (in cm\(^{-2}\)-atm\(^{-1}\)); \(b\) = dispersion semi-half-width of the spectral lines (in cm\(^{-1}\)); \(\delta^*\) = mean spacing of spectral lines (in cm\(^{-1}\)); \(P'\) = maximum value of \(P_\omega\) at the line center for a spectral line with pure Doppler broadening.
where \( B_v \) is the blackbody steradyn for local thermodynamic equilibrium at the optical depth \( \tau_v = \cos \theta \int_0^s k_v(s') \, ds' \) corresponding to the geometric length \( s \), along the beam of the emitting system, which has been taken to extend in depth without limit.

If \( B_v(\tau_v) \) is developed in a (Lundblad) power series in \( \tau_v \), viz.,

\[
B_v(\tau_v) = \sum_{i=0}^{\infty} a_i \tau_v^i, \quad (2-4.24)
\]

then

\[
I_v(0, \theta) = \sum_{i=0}^{\infty} a_i \cos^i \theta \int_0^{\infty} y^i e^{-y} \, dy = \sum_{i=0}^{\infty} a_i \cos^i \theta, \quad (2-4.25)
\]

where the substitution \( y = \tau_v \sec \theta \) has been used. Comparison of Eqs. (2-4.24) and (2-2.25) shows that

\[
I_v(0, \theta) = B_v(T_v = \cos \theta) \quad \text{for} \quad a_i = 0 \quad \text{if} \quad i \geq 2, \quad (2-4.26)
\]

i.e., if only the first two terms are used (Eddington-Barbier approximation) in the power series given in Eq. (2-4.24). The physical interpretation of Eq. (2-4.26) is the following: the spectral steradyn at \( \tau_v = 0 \), observed at an angle \( \theta \), for a nonisothermal system, is identically equal to the numerical value of the spectral blackbody steradyn \( B_v(T_v = \cos \theta) \) at the optical depth \( \tau_v = \cos \theta \) or at the geometrical length defined by \( \int_0^s k_v \, ds' = 1 \),* provided only two terms are used in the power-series expansion shown in Eq. (2-4.24).

It is interesting to consider the possible temperature profiles for selected spectral line shapes that are consistent with the statements

\[
B_v(\tau_v) = a_0 + a_1 \tau_v = a_0 + a_1 \cos \theta \int_0^s k_v(s') \, ds' = B_v(0) + [B_v(T_v = \cos \theta) - B_v(0)] \int_0^s k_v(s') \, ds' \quad (2-4.27)
\]

and

\[
\int_0^s k_v(s') \, ds' = 1. \quad (2-4.28)
\]

We assume a monotone variation of \( T \) and \( \tau_v \) with \( s \) (and thus also of \( \tau_v \) with \( T \)) and we impose the boundary condition \( T = T_0 \) at \( \tau_v = s = 0. \)

---

* Note that \( s \) is measured along the direction \( \theta \) shown in Fig. 2-4.2.
Differentiation of Eq. (2-4.27) with respect to $T$ yields the differential equation

$$(\cos \theta) [k_s(T)] \frac{ds}{dT} = \frac{1}{a_1} \frac{dB_v}{dT} = \frac{1}{a_1} \frac{2h^2
u^2}{c^2} \frac{1}{kT^2} \frac{\exp(h\nu/kT)}{[\exp(h\nu/kT) - 1]^2}. \quad (2-4.29)$$

It is now possible to specify $k_s(T)$ for various spectral line profiles belonging to various assumed atomic or molecular emitters. We may then integrate Eq. (2-4.29) in order to find $s$ as a function of $T$. Finally, Eq. (2-4.28) may be used to obtain the proper value of $s$, and hence of $T$, for which

$$I_s(0, \theta) = B_v \left[ T \text{ for } \int_0^s k_s(s') \, ds' = 1 \right].$$

In the derivation of Eqs. (2-4.23) to (2-4.29), it has been assumed that the quantities $a_1$ are constants. For an emitting system with structure, this statement can be true only spectrally, i.e., a different value of $a_1$ must be chosen at a different frequency for any specified temperature dependence on geometrical length. The implications of this fact may be clarified by referring to the schematic diagram shown in Fig. 2-4.3. The temperature profile must, of course, be independent of frequency in any

---

**Fig. 2-4.3.** Schematic diagram showing the relation between temperature $T$ and distance $s$ for a specified angle $\theta$. The values of $s$ and $T$ required to satisfy Eq. (2-4.28) are shown for the representative frequencies $\nu_0 + \Delta \nu_1$ in the near wing and $\nu + \Delta \nu_2$ in the far wing of a line. Reproduced from Penner, Thomas, and Adomeit.\textsuperscript{5}
physically meaningful problem. However, the physical location \( s \) and the temperature \( T \) at which Eq. (2-4.28) is satisfied are strongly dependent on frequency. In the near line wing at the frequency \( \nu_0 + \Delta \nu_1 \), the integral condition of Eq. (2-4.28) will be met for small values of \( s \) and \( T \); on the other hand, in the far wings of spectral lines where \( \nu = \nu_0 + \Delta \nu_2 \), much larger values of \( s \), and hence of \( T \), are required (compare Fig. 2-4.4). In other words, the contributions to \( I_s(0, \theta) \), calculated according to Eq. (2-4.26), arise from regions of different temperature at different frequencies in such a way that the far line wings will make relatively larger contributions since they may be “seen” at greater geometrical depths and, correspondingly, at higher temperatures.

Since \( a_1 \) may vary with frequency, it is convenient to introduce a frequency dependence for \( a_1 \) deliberately in such a way as to allow a universal representation of a reduced distance variable (which is a function of \( \nu \)) as a function of \( T \).

Detailed calculations, which are described elsewhere, lead to the conclusions (a) that the Eddington-Barbier (E-B) approximation applies exactly to collision-broadened spectral lines belonging to diatomic emitters only for a very special temperature profile and (b) that it cannot apply exactly for Doppler-broadened lines, for any physically meaningful temperature profile, that is independent of frequency.

**Fig. 2-4.4.** Representative temperature profiles described by Eq. (2-4.31) for \( m = 1 \), 2, and 4; reproduced from Penner, Thomas, and Adomeit.⁵
Representative errors introduced by using the E-B approximation for line radiation are discussed in the following Section 2-4I.

2-4I Radiative scaling properties for representative temperature profiles. The integral expression for the spectral steradiancy $I_\nu$ is the formal solution to the linear, first-order differential equation

$$\frac{dI_\nu}{ds^*} = L_0 k_\nu (B_\nu - I_\nu), \quad (2-4.30)$$

where $s^* = s/L_0$; $s$ is the distance along the line of sight; and $L_0$ is a characteristic length of the system. Equation (2-4.30) has been integrated numerically by means of a fourth-order Runge-Kutta method for representative temperature profiles. The temperature profiles are represented by the expressions

$$T = (T_{\text{max}} - T_0) \left( 1 - |s^* - 1|^m \right) + T_0, \quad m = 1, 2, 4. \quad (2-4.31)$$

The specified temperature profiles are sketched in Fig. 2-4.4.

For local thermodynamic equilibrium, we find for diatomic emitters, to the harmonic-oscillator and rigid-rotator approximations, the following relation for the spectral absorption coefficient of collision-broadened lines:

$$k_\nu = \frac{c^2}{8\pi^2 \nu_0^2} A_{u\rightarrow l} g_u \left( \frac{p}{kT_0} \right) \frac{\sigma_0}{b_0} \left( \frac{T_0}{T} \right)^{3/2} \left( 1 - \exp \left( - \frac{u_0 T_0}{T} \right) \right) \left( 1 - \exp \left( - \frac{h\nu_0}{kT} \right) \right)$$

$$\times \left[ \exp \left( - \frac{E_l}{kT} \right) \right] \left[ 1 + \frac{(\nu - \nu_0)^2}{(b_0^2 T_0/kT)} \right]^{-1}. \quad (2-4.32)$$

Here $\nu_0$ is the frequency at the center of the emitted spectral line; $A_{u\rightarrow l}$ is the Einstein coefficient for spontaneous emission for the transition producing the given spectral line; $g_u$ is the statistical weight of the upper energy level involved in the transition; $p/kT_0$ represents the number of molecules, per unit volume, at the pressure $p$, and at the reference temperature $T_0$, with $k$ denoting the Boltzmann constant; $\sigma_0 = hcB/kT_0$, where $B$ is the appropriate rotational constant for the rigid rotator; $b_0$ is the spectral line semi-half-width at the pressure $p$ and at the reference temperature $T_0$ and we have assumed that $b = b_0(T_0/T)^{1/2}$ at the constant pressure $p$; $u_0 = h\nu_0/kT_0$ where $\nu_0$ represents the normal vibration frequency of the diatomic molecule (harmonic oscillator); $E_l$ = energy of the lower state above the zero-point energy.
For Doppler-broadened lines, we find

\[
k_\nu = \frac{\epsilon^2}{8\pi v_0^2} A_{\nu-\nu_0} \left( \frac{p}{kT_0} \right) \frac{\sigma_0}{b_{D,0}} \left( \frac{\ln 2}{\pi} \right)^{1/2} \left( \frac{T_0}{T} \right)^{5/2} \left( 1 - \exp - \frac{\nu_0 T_0}{T} \right) \times \left( 1 - \exp - \frac{h\nu_0}{kT} \right) \exp \left( - \frac{E_1}{kT} \right) \exp \left[ - (\ln 2) \left( \frac{\nu - \nu_0}{b_{D,0}} \right)^2 \frac{T_0}{T} \right],
\]

(2-4.33)

where the Doppler half-width under reference conditions is given by

\[
b_{D,0} = \left( \frac{2kT_0 \ln 2}{mc^2} \right)^{1/2} \nu_0.
\]

The spectral and total line steradiancies at \( s^* = 2 \) have been computed for a typical strong line of the vibration-rotation spectrum of the hydrogen fluoride molecule. We have chosen the values \( T_0 = 300^\circ \text{K} \) and \( T_{\text{max}} = 3000^\circ \text{K} \). Representative results of the calculation are shown in Figs. 2-4.5 to 2-4.7 for the \( R3(v = 0, J = 3 \rightarrow v = 1, J = 4) \) line of HF. For this line, in case of dispersion broadening, \( b_0/c = 0.132p \text{ cm}^{-1} \) (\( p \) in atm), whereas for Doppler broadening we have used \( b_0/c = 5.85 \times 10^{-3} \text{ cm}^{-1} \). The reference optical depth is defined as
FIG. 2-4.6. The spectral steradiancies $I_{s*}^m$ for $m = 2$ for the $R3$ line of HF at $s^* = 2$ as a function of $(\nu - \nu_0)/\hbar_0$ for pure Doppler broadening and various values of the reference optical depth $\tau_0$; reproduced from Penner, Thomas, and Adomeit.

FIG. 2-4.7. The total steradiancy divided by $2\hbar_0(=I_{s*}^m/2\hbar_0)$ at $s^* = 2$ as a function of $\tau_0$ for the dispersion- and Doppler-broadened $R3$ line of HF; $T_0 = 300^\circ$K, $T_{\text{max}} = 3000^\circ$K, for various values of $m$. Reproduced from Penner, Thomas, and Adomeit. 
\[ \tau_0 = (L_0 / \pi b_0) \int k_{\nu,0} d\nu \quad (k_{\nu,0} = \text{absorption coefficient evaluated at } T_0), \]

\[ I_{\nu,0}^* \] is the spectral steradiancy at \( s^* = 2 \), and \( I_{\nu,0}^* \) represents the integrated steradiancy at \( s^* = 2 \) for the entire line.

Examination of Fig. 2-4.5 shows that the R3 line of HF retains a typical dispersion contour until \( \tau_0 \) becomes greater than about 10, when self-reversal becomes important. Hence \( \tau_0 \approx 10 \) may be said to define the upper limit of the transparent-gas regime. From Fig. 2-4.7 it is seen that the pressure and length dependence of the steradiancy for \( \tau_0 \lesssim 10 \) is

\[ I_{s^*} \propto \rho_0 L_0, \quad (2-4.34) \]

t.e., it is the same as for an isothermal transparent gas. In general, \( \tau_0 = 10 \) corresponds to a small physical length for a strong spectral line. For the R3 line of HF, the value of \( L_0 \) at \( \tau_0 = 10 \) is about 0.015 cm.

The center of the line is essentially completely self-absorbed for \( \tau_0 \approx 100 \). Figure 2-4.7 shows that, for \( \tau_0 \lesssim 100 \),

\[ I_{s^*} \propto (\rho_0^2 L_0)^{1/2}, \quad (2-4.35) \]

which is the same as for the Eddington-Barbier approximation or for the isothermal case for large optical depths. For a strong line, the transition to the regime described by Eq. (2-4.35) occurs at a small physical length (\( \approx 0.15 \) cm for the R3 line of HF).

The plot in Fig. 2-4.6 shows that self-reversal for the Doppler-broadened line becomes important for \( \tau_0 \approx 5 \). However, reference to Fig. 2-4.7 indicates that Eq. (2-4.34) remains valid up to \( \tau_0 \approx 10 \) which, at \( p = 0.1 \) atm, corresponds to \( L_0 \approx 0.007 \) cm for the R3 line of HF. For larger values of \( \tau_0 \), however, the Doppler-broadened line does not approximate the behavior of the dispersion-broadened line, i.e., \( I_{s^*} \) does not become simply proportional to a power of \( L_0 \). Rather, \( I_{s^*} \) becomes a weaker and weaker function of \( \tau_0 \) as saturation is approached.

The more complicated behavior of the Doppler-broadened line, compared to the dispersion-broadened line, is the result of the fact that the Doppler-line half-width increases with temperature, while the dispersion-line half-width decreases with temperature at constant pressure. Hence, when a Doppler-broadened system is viewed at \( s^* = 2 \), the radiation emitted from the higher temperature regions is "seen" at all optical depths since this radiation is not extensively reabsorbed.

* The condition that the steradiancy is proportional to \((\rho_0^2 L_0)^{1/2}\), for values of \( \tau_0 \) exceeding the value required to make the line center "black," may be used for the derivation of an approximate relation for the critical minimum value of \( \tau_0 \) above which Eq. (2-4.35) applies for various temperature profiles (for details, see Thomas).
We may, therefore, conclude that the numerical calculations indicate that a similarity analysis utilizing isothermal approximations for the radiative-energy transport constitutes an acceptable approximation for dispersion-broadened lines for a variety of important temperature profiles. On the other hand, excepting only the transparent-gas regime, it is not possible to represent a nonisothermal radiating system with Doppler-broadened lines by an equivalent isothermal distribution. Hence, simplified procedures for studying the interplay between flow, chemical reactions, and radiative-energy transport can generally be employed only for nonisothermal emitters with dispersion lines.

2-4J Quantitative estimates of the deviation of the Eddington-Barbier approximation from exact results. Typical results obtained from the exact calculations are compared in Fig. 2-4.8 with those derived from the E-B approximation for a dispersion line with \( \tau_0 = 50 \) and \( m = 2 \). Reference to Fig. 2-4.8 shows that the E-B approximation yields very poor results at the chosen small reference optical

![Graph](image)

**Fig. 2-4.8.** The spectral radiancy as a function of the reduced frequency displacement from the line center for the \( R3 \) line of HF; dispersion contour, \( T_0 = 300^\circ \text{K}, T_{\text{max}} = 3000^\circ \text{K}, \rho = 0.1 \text{ atm}, s^* = 2, b_0 = 0.132 \text{ pc sec}^{-1}, \tau_0 = 50, m = 2 \). Results derived from exact numerical calculations are shown, as well as data obtained by use of the Eddington-Barbier approximation. Reproduced from Thomas and Penner.
depth for the spectral distribution. The line steradiancy, divided by 2$b_0$, in the E-B approximation is $4.9 \times 10^{-14}$ watt-sec/cm$^2$ compared with the correct value of $7.6 \times 10^{-14}$ watt-sec/cm$^2$.

As the value of $\tau_0$ is increased, the E-B approximation tends to yield progressively better results near the line center. For example, for the same dispersion line at $\tau_0 = 1000$, use of the E-B approximation for the calculation of the spectral radiancy leads to results that are in error by only about 2% at the line center for $m = 1$ or 2.

The numerical work emphasizes the fact that the E-B approximation is generally useful for highly absorbing media but becomes relatively poor for less opaque gases with spectral structure. The conclusion for large $\tau_0$ may, of course, be derived from detailed considerations of the behavior of the transfer equation in the limit as the diffusion approximation applies.

2-5 Radiative-energy transfer at the steady state between two infinite, parallel, isothermal plates, separated by a gray absorbing gas

As an interesting illustration of the use of the techniques described in the present chapter, we consider a generalized version of the problem treated in Section 2-1F by investigating the heat-transfer problem between two infinite, parallel, black, isothermal plates, separated by a gray absorbing medium, which is in radiative equilibrium.

2-5A Calculations for transparent gases. The geometric arrangement for the problem under discussion is sketched in Fig. 2-5.1. We consider a gray gas for which $\tau_\nu$ is independent of frequency and for

![Fig. 2-5.1. Schematic diagram illustrating the geometric configuration considered in the heat-transfer problem between two infinite, parallel, isothermal plates, which are separated by an absorbing gas of optical depth $\tau = \tau_0$; reproduced from Olfe and Penner.](image)
which the integration over frequency has been performed. The net transfer \( F(\tau) \) in the direction of the cold wall at \( T_1 \) is obtained by adding Eq. (2-1.21) to Eq. (2-1.22). In this manner, we obtain the Milne-Eddington transfer equation, viz.,

\[
\frac{F(\tau)}{2\pi} = -\int_0^\tau B(\eta) E_2(\tau - \eta) \, d\eta + \int_\tau^\infty B(\eta) E_2(\eta - \tau) \, d\eta \\
- I^{-}(0) E_3(\tau) + I^{+}(\tau_0) E_3(\tau_0 - \tau).
\]  

(2-5.1)

In the transparent gas approximation, \( B(\eta) \) is determined by a heat balance for an infinitesimal gas slab,* i.e.,

\[
B(\eta) = \frac{1}{2} [B(T_2) + B(T_1)] = \frac{\sigma/2\pi}{(\sigma/\pi)} (T_2 + T_1);
\]  

(2-5.2)

also \( I^{-}(0) = (\sigma/\pi) T_1^4 \) and \( I^{+}(\tau_0) = (\sigma/\pi) T_0^4 \). Integration of Eq. (2-5.1), using the constant value for \( B(\eta) \) given in Eq. (2-5.2), leads to the result

\[
F(\tau) = \sigma(T_2^4 - T_1^4) [E_3(\tau) + E_3(\tau_0 - \tau)],
\]

where we have used the relation \( dE_3(t)/dt = -E_2(t) \). But, in the transparent gas approximation, both \( \tau \) and \( \tau_0 \) must be small, whence (see, for example, Kourganoff,2 p. 255)

\[
E_3(\tau) + E_3(\tau_0 - \tau) = 1 - \tau_0 + O(\tau_0^2)
\]

and

\[
F \sim \sigma(T_2^4 - T_1^4) (1 - \tau_0).
\]  

(2-5.3)

The flux is seen to be independent of the location \( \tau \), as it must be in this radiative-equilibrium limit.

2-5B CALCULATIONS IN THE DIFFUSION APPROXIMATION. In the diffusion approximation, we may use Eq. (2-5.1) in conjunction with the Eddington-Barbier approximation

\[
B(\eta) = a\eta + b = \frac{B(T_2) - B(T_1)}{\tau_0} \eta + B(T_1),
\]

(2-5.4)

where \( a \) and \( b \) are constants. Integration of Eq. (2-5.1) now leads to the result

\[
F(\tau) = 2\pi a[\frac{3}{8} - E_4(\tau) - E_4(\tau_0 - \tau)]
\]

* The result given in Eq. (2-5.2) may be verified by a straightforward perturbation calculation, which also yields higher-order terms in \( \tau_0 \).
or, for \( \tau \gg 0, (\tau_0 - \tau) \gg 0 \), to

\[
F \approx \frac{3}{4} (\sigma/\tau_0) (T_2^4 - T_1^4) .
\]  

(2-5.5)

It is interesting to observe that

\[
\lim_{\tau \to 0} [F(\tau)] = \lim_{\tau \to 0} [F(\tau)] = 2\pi a[\frac{3}{2} - \epsilon(\tau_0)] = \frac{3}{8} (\sigma/\tau_0) (T_2^4 - T_1^4)
\]

and that this erroneous result is obtained because the E-B approximation becomes invalid as the boundaries are approached.

2-5C "Radiation Slip" and Interpolation Formulae for Heat Transfer. Probstein\textsuperscript{30} and Eckert\textsuperscript{31} have introduced the notion of a "radiation slip" in order to specify the boundary conditions near a wall for confined gases of arbitrary opacity. Probstein assumed, adjacent to the solid surface, a temperature discontinuity proportional to the product of the local photon mean free path and the steradiancy gradient in the gas. By evaluating a constant in the limit of zero optical depth (no intervening gas), Probstein used the given model to obtain the following expression for the heat transfer at all optical depths \( \tau \):

\[
F = \frac{\sigma(T_2^4 - T_1^4)}{1 + (3\tau_0/4)} .
\]  

(2-5.6)

Although this equation gives the correct limit at \( \tau_0 = 0 \), it does not give the correct linear term (transparent gas contribution) as \( \tau_0 \to 0 \), since the term linear in \( \tau_0 \) must have a coefficient of \(-1\) rather than \(-\frac{3}{4}\). A somewhat better interpolation formula is\textsuperscript{29}

\[
F = \frac{\sigma(T_2^4 - T_1^4)}{1 + (3\tau_0/4) + [\tau_0/4(1 + 3.8070\tau_0)]} .
\]  

(2-5.7)

although Eq. (2-5.7) does not differ greatly from Eq. (2-5.6) and, unlike Eq. (2-5.6), it cannot be derived from a heuristic physical model.

Stewart,\textsuperscript{31a} after comparing the interpolation formulae given in Eqs. (2-5.6) and (2-5.7) with exact numerical calculations by Chandrasekhar, by King, and by Sobolev, has proposed an improved interpolation formula of the form

\[
F = \frac{\sigma(T_2^4 - T_1^4)}{1 + (3\tau_0/4) + [\tau_0/4(1 + 3.8070\tau_0)]} .
\]  

(2-5.7a)

which leads to errors smaller than 0.3\% for all values of \( \tau_0 \).
In the preceding analysis, the optical depth $\tau$ was used as the independent variable. In order to present results in terms of the geometric depth $x$, the defining relation for the optical depth, $d\tau = k \, dx$, may be integrated after expressing the gray absorption coefficient $k$ as a function of $\tau$ by solving for $T(\tau)$, i.e., $k(\tau) = k[T(\tau)]$, and

$$x = \int_0^\tau \frac{1}{k(\eta)} \, d\eta. \quad (2-5.8)$$

Radiative transfer in a slab is considered further in Section 6-2F, where we solve for the temperature distribution and flux in the non-stationary case as well as in the steady-state (radiative equilibrium) limit.

2-6 Radiative transfer for non-gray gases

Choice of the appropriate method for solving radiant-heat transfer problems in non-gray gases depends on the spectral character of the radiation, as well as on the range of optical depths considered. It has been shown in Sections 2-2A and 2-2B that, for gases which are optically thin or optically thick at all frequencies, the flux may be determined as for gray gases except that the Planck or Rosseland mean absorption coefficients, respectively, must be used. On the other hand, when the medium is optically thin at some frequencies but optically thick at other frequencies, then no general procedure is known for evaluating an effective absorption coefficient which reduces the "non-gray problem" to an equivalent "gray problem." In the following discussion, we shall distinguish between "slightly non-gray" and "strongly non-gray" gases.

2-6A Slightly non-gray gases. For a slightly non-gray gas, the absorption coefficient has a sufficiently slow variation with frequency that a perturbation solution may be constructed with the gray-gas solution used in the first approximation. This "slightly non-gray analysis" may be applied to the solar atmosphere, where the dominant contribution to the radiation is made by photodetachment from $\text{H}^-$. Chandrasekhar has treated this problem by writing the absorption coefficient in the form $k_\nu = \bar{k}(1 + \delta_\nu)$, where $\bar{k}$ is a mean absorption coefficient and $\delta_\nu$ measures the departure from the gray case. The integrated transfer equation for a one-dimensional problem becomes now

$$\mu \frac{dI}{d\tau} = B - I + \int_0^\infty \delta_\nu(B_\nu - I_\nu) \, d\nu, \quad (2-6.1)$$
where \( dr = k \, dx \). [In the astrophysical literature the definition \( dr = -h \, dx \) is generally used and, therefore, the terms appearing on the right-hand side of Eq. (2-6.1) have the opposite sign]. A first-order solution is obtained by neglecting the term in \( \delta \); thus

\[
\mu \frac{dI^{(1)}}{dr} = B^{(1)} - I^{(1)},
\]

(2-6.2)

where the superscript \((1)\) identifies quantities obtained in the first approximation. Equation (2-6.2) is the gray-gas transfer equation, which may be solved by using standard techniques, such as the method of discrete ordinates.\(^1\) The first-order solution is used to compute the term involving \( h \), and substitution of the results into Eq. (2-6.1) then yields the following equation for the second approximation:

\[
\mu \frac{dI^{(2)}}{dr} = B^{(2)} - I^{(2)} + \int_{0}^{\infty} \delta \left[ B^{(1)} - I^{(1)} \right] \, dv.
\]

(2-6.3)

This equation may be written in terms of discrete ordinates and solved by the method of variation of parameters.\(^1\) The proper choice for \( k \) is found to be given by the Chandrasekhar mean, which is defined as

\[
k^*_C \equiv \frac{1}{F} \int_{0}^{\infty} k \, f^{(1)} \, dv.
\]

(2-6.4)

It should be noted that Chandrasekhar's treatment of the slightly non-gray problem does not require that \( \delta \) be small compared with unity at every frequency, but rather that the last term on the right-hand side of Eq. (2-6.1) be small compared with \( |B - I| \) (for details, see the discussion by Münch\(^32\)).

The slightly non-gray problem has also been treated by other methods involving iterative and variational techniques. Discussion of these methods is given in Kourganoff\(^2\) and Münch\(^32\).

2-6B STRONGLY NON-GRAY GASES. Under most conditions, the radiation emitted from a gas will be strongly non-gray because it is associated with molecular bands, atomic lines, or discontinuous bound-free radiation at photoelectric edges. In these cases, the absorption coefficient may vary by orders of magnitude over a relatively small spectral range. Radiative transfer calculations for strongly non-gray gases are usually carried out for particular forms of the absorption coefficient. For example, in Section 2-4I we discussed transfer calculations for isolated dispersion lines and Doppler-broadened lines in a gas
with given temperature profiles. As is shown in Figs. 2-4.5 and 2-4.6, line reversal may occur, giving maximum radiant-heat transfer at frequencies where the value of the absorption coefficient is intermediate between the maximum and minimum values. The complex nature of the radiant-heat transfer obtained in these calculations illustrates the fact that the particular spectral dependence of the absorption coefficient is of fundamental importance and that generally no simple mean absorption coefficient will describe a strongly non-gray gas adequately.

The influence of a discontinuity in the absorption coefficient has been examined by Stone and Gaustad\textsuperscript{33} and by Carrier and Avrett.\textsuperscript{34} In both of these studies, a spectral absorption coefficient is considered which has a constant value \( k_1 \) from \( \nu = 0 \) to \( \nu_0 \), and a different constant value \( \alpha k_1 \) from \( \nu_0 \) to \( \infty \). Stone and Gaustad use a moment method which yields accurate results for \( 10^{-2} \lesssim \alpha \lesssim 10^3 \), whereas Carrier and Avrett carried out a boundary-layer type analysis, which is useful for \( \alpha \gtrsim 10^2 \).

Calculations of radiative equilibrium for finite and semiinfinite atmospheres have been performed by King,\textsuperscript{35,36} who used an absorption coefficient corresponding to the Elsasser band model (equally intense lines with dispersion profiles, constant half-width, and uniform spacing). King used the Milne-Eddington assumption

\[
k_\infty(T, \rho) = k(T, \rho) \frac{1}{\alpha(\nu)},
\]

which, on substitution into the equation of transfer, yields the relation

\[
\mu \alpha(\nu) \frac{dI_\nu(\tau, \mu)}{d\tau} = B_\nu(\tau) - I_\nu(\tau, \mu),
\]

where \( d\tau \equiv k(T, \rho) \, dx \). The condition for radiative equilibrium gives

\[
B_\Delta(\tau) = \frac{1}{2} \int_{-1}^{1} \int_{0}^{1} \frac{I_\nu(\tau, \mu)}{\alpha(\nu)} d\left(\frac{\nu}{\Delta\nu}\right) d\mu,
\]

where \( \Delta\nu \) is the width of the Elsasser band. Equations (2.6.6) and (2.6.7) may be solved by using the method of discrete ordinates for both \( \mu \)- and \( \nu \)-configuration spaces; that is, the integrals over the continuous variables are replaced by appropriate finite-sum, quadrature formulae.

A method for treating strongly non-gray gases has recently been proposed by Stewart.\textsuperscript{37} Utilizing the Milne-Eddington assumption given in Eq. (2.6.5), Stewart was able to transform the transfer and radiative equilibrium equations into equations of the same forms as those applicable to nonisothermal, gray gases. Stewart has applied his method to the
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problem of radiative equilibrium in a semiinfinite atmosphere with line radiation described by "picket-fence" and statistical models.

A different approach for molecular band models, without a high degree of spectral structure, has been described in Section 1-11, and involves an approximation equivalent to that specified in Eq. (2-6.5). This procedure, which entails rather special assumptions about the form of $k_v(T, \rho)$, leads to a problem similar to that for heat transfer in isothermal, non-gray gases.

REFERENCES

REFERENCES