Chapter 3

RADIATIVE HEAT
TRANSFER IN HEATED AIR*†

The calculation of radiative heat transfer rates can be carried out systematically only for a few limiting cases. In this chapter, we summarize existing data relevant to the optically thick, optically thin, and isothermal cases for air in local thermodynamic equilibrium (LTE), and close with some brief remarks on non-LTE radiation.

3-1 Limiting radiative transfer cases

Under the assumption of LTE, the radiative-energy transfer rate in a system depends on the spectral value of the photon mean free path $l_\nu \equiv 1/k_\nu$ and on the blackbody steradiancy $B_\nu$, suitably averaged over frequencies and positions. As was shown in detail in Section 2-2, the averaging process becomes simple for two important limiting cases.

(a) An optically thick system is one for which $l_\nu \ll L$ at all $\nu$, where $L$ is a typical system dimension. The radiative-energy flux vector at a point well inside such a system is given, in terms of local quantities, by the expression

$$F = -\frac{4\pi}{3} I_R \text{grad } B$$  \hspace{1cm} (3-1.1)

* Chapter 3 is by S. S. Penner and J. C. Stewart.
† A first draft of this chapter was prepared by S. S. Penner. The chapter was then extensively rewritten and revised by J. C. Stewart. It should be noted that the discussion in Sections 3-1 and 3-2 duplicates some of the analysis presented in Section 2-2. However, the point of view is somewhat different and, furthermore, the presentation is now completely self-contained and should allow reading of Chapter 3 without reference to the contents of Chapters 1 and 2.
where $B$ is the integrated blackbody steradiancy ($= \sigma T^4/\pi$) and $l_R$ is the Rosseland mean free path. As was shown in Eq. (2-2.14),

$$l_R = \int_0^\infty \frac{(dB_v/dT) l_v dv}{dB/dT} = \frac{1}{k_R}, \quad (3-1.2)$$

where $k_R$ is the Rosseland mean absorption coefficient. Since the conductive energy flux is also proportional to the local temperature gradient, the radiative and conductive flux terms are directly additive. The optically thick case is sometimes called the opaque, diffusion, or Rosseland limit.

(b) For an optically thin system, $l_v \ll L$ at all $\nu$. In this case, all radiation emitted by the system escapes without reabsorption, and the local radiative rate of energy loss, per unit volume and time, is $4\pi k_p B_v$, where $k_p$ is the Planck mean absorption coefficient given in Eq. (2-2.4a) viz.,

$$k_p = \int_0^\infty B_v k_v dv = \frac{1}{l_p} \quad (3-1.3)$$

where $l_p$ is the Planck mean free path and $k_v$ is the linear absorption coefficient. The optically thin case is called the transparent, emission, or Planck limit.

(c) An isothermal and isobaric system has the same $B_v$ and $k_v$ everywhere; the specific intensity $I_v(L)$ of a ray, which has traversed a length $L$ in such a system, after entering with incident intensity $I_v(0)$, is given by

$$I_v(L) = [I_v(0) \exp(-k_v L)] + [B_v [1 - \exp(-k_v L)]] \quad (3-1.4)$$

For heat-transfer computations, it is convenient to define a spectral transmission function $\tau_v(L)$ by the expression

$$\tau_v(L) = \frac{1}{\Delta \nu} \int_{\Delta \nu} \exp(-k_v L) \, dv, \quad (3-1.5)$$

where $\Delta \nu$ is sufficiently large to contain, for example, many lines of a molecular band [and thereby smoothes out the most fine-grained part of the frequency dependence of $\exp(-k_v L)$] but small enough that $B_v$ may be taken as constant within $\Delta \nu$. If $I_v(0) \neq 0$, it must likewise be nearly constant in $\Delta \nu$ for $\tau_v(L)$ to be relevant. The average value of $I_v(L)$ in $\Delta \nu$, $\langle I_v(L) \rangle$, is given by the relation

$$\langle I_v(L) \rangle = \langle I_v(0) \rangle \tau_v(L) + \langle B_v \rangle [1 - \tau_v(L)]. \quad (3-1.6)$$

The quantity $1 - \tau_v(L)$ is the spectral emissivity (or absorptivity) for radiation emitted (in $\Delta \nu$), in a direction normal to the face of a slab of
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thickness \( L \). The emissivity for a direction making an angle \( \theta \) with respect to the outward normal is \( 1 - \tau_{\nu}(L \sec \theta) \); the total radiative rate of energy loss from the slab (both faces), per unit volume and time, is then obtained by deriving a proper average over all angles [cf. Eq. (2-1.23)]. The result is

\[
2F^+(0) = 4\pi B \int_0^1 [1 - \tau(L,\mu)] \mu d\mu
\]

(3-1.7)

where \( \mu = \cos \theta \), and \( \tau \) is the Planck-weighted average transmission

\[
\tau(L) = \int_0^\infty \tau_{\nu}(L) B_v dv/B.
\]

(3-1.8)

We note that the energy-loss rate reduces properly to the Planck limit; thus, if \( k_vL < \varepsilon \) (for all \( \nu \)), then \( \tau(L) \to 1 - k_vL \). In general, however, the ratio of actual to Planck-limit rates, \( \mathcal{R} \), is less than unity, where

\[
\mathcal{R} = \frac{1}{k_vL} \int_0^1 \left[1 - \tau(L,\mu)\right] \mu d\mu \leq 1.
\]

(3-1.9)

The spectral transmission may also be used to obtain the Rosseland mean, as follows:

\[
\int_0^\infty \tau_{\nu}(L) dL = \langle 1/k_v \rangle = \langle l_{\nu} \rangle.
\]

(3-1.10)

Here \( \langle l_{\nu} \rangle \) may be substituted for \( l_{\nu} \) in the defining integral of Eq. (3-1.2) for \( l_R \).

The frequency-averaged emissivity per unit path length,

\[
e(L) \equiv [1 - \tau(L)]/L,
\]

depends on \( L \) and approaches \( k_v \) as \( L \to 0 \). The replacement of \( \epsilon(L) \) by \( k_v \) is valid only if \( k_vL \ll 1 \) (for all \( \nu \)); in the literature, the incorrect and less stringent criterion \( k_vL \ll 1 \) is sometimes quoted. The difference between these criteria is especially large when most of the emission occurs from a few narrow spectral lines.

3-2 Existing absorption-coefficient data for heated air\(^{1-9}\)

The calculation of the absorption coefficient \( k_v \) and its various averages for heated air falls naturally into two temperature ranges: below about 9000° K (at normal density) air consists mostly of diatomic and triatomic molecules, and above this temperature it consists mostly of atoms, atomic ions, and free electrons. The boundary between these "molecular" and "atomic" regimes depends somewhat on density and is not sharply defined.

Because of the multitude of absorption lines which contribute strongly
to $k$, in both the atomic and molecular regimes, it is not practical to try to exhibit the full fine-grained frequency dependence of $k_v$ (or $l_v$). Instead, averages over frequency intervals $\Delta \nu$ (see Section 3-1) of $k_v$, $l_v$, or $\exp(-k_vL)$ are employed, corresponding to "local" (in frequency space) Planck, Rosseland, and transmission averages, respectively. These are not interchangeable; thus $\langle k_v \rangle \neq 1/\langle l_v \rangle$. Since $\langle k_v \rangle$, unlike the other local averages, is essentially independent of spectral line profiles and depends only on integrated line strengths, it is the local average most often presented. The full Planck and Rosseland means ($\bar{k}_p$, $\bar{k}_R$) are still more widely used.

Numerical data for the various averages of $k_v$ have been revised markedly in recent years through the inclusion of formerly neglected absorption mechanisms, such as the vibration-rotation bands of the NO molecule and atomic line absorption. The revisions amount to several orders of magnitude in some cases; the Planck mean in Armstrong et al., 1 for example, is far below more recent data at most temperatures. Since further revision is possible, the "current" data should not be regarded as immutable. Some recent calculations for air* are summarized in Table 3-2.1.

The results of Gilmore 5 up to $T = 8000^\circ$K, Armstrong et al. 2 from $kT = 1$ to 20 eV, and of Cox and Stewart 3 above 20 eV, are shown in Figs. 3-2.1 and 3-2.2, where Rosseland and Planck means are given, respectively. Spectral data from Churchill et al. 4 and Gilmore 5 are shown in Figs. 3-2.3 through 3-2.22.

**TABLE 3-2.1**

CALCULATIONS OF RADIATIVE ABSORPTION COEFFICIENTS FOR HEATED AIR

<table>
<thead>
<tr>
<th>Type of average</th>
<th>Molecular regime</th>
<th>Atomic regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rosseland</td>
<td>—</td>
<td>Armstrong, et al.; Armstrong, et al.; 2 Cox and Stewart; Freeman 6</td>
</tr>
<tr>
<td>Planck</td>
<td>Gilmore 5</td>
<td>Armstrong et al.; 4 Freeman 6</td>
</tr>
<tr>
<td>Local Rosseland</td>
<td>Gilmore 5; Churchill et al. 7</td>
<td>Freeman 6</td>
</tr>
<tr>
<td>Local Planck</td>
<td>Gilmore 5; Churchill et al. 7</td>
<td>Churchill et al. 7 (without atomic lines)</td>
</tr>
<tr>
<td>Local transmission</td>
<td>Churchill et al.; 4 Churchill and Hagstrom, including the NO vibration-rotation bands</td>
<td>—</td>
</tr>
</tbody>
</table>

* After this chapter was completed, an additional relevant publication appeared by Buttrey and McChesney. 9a
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Fig. 3-2.1. Rosseland mean absorption coefficient of air as a function of temperature and density; plotted from the data listed in Armstrong et al. and Cox and Stewart.

Fig. 3-2.2. Planck mean absorption coefficient of air as a function of temperature and density; plotted from the data listed in Armstrong et al. and Gilmore.
**Fig. 3-2.3.** Absorption coefficients of high-temperature air as a function of wavelength at 2000°K; plotted from the data listed in Gilmore.5

**Fig. 3-2.4.** Absorption coefficients of high-temperature air as a function of wavelength at 3000°K; plotted from the data listed in Gilmore.5
Fig. 3-2.5. Absorption coefficients of high-temperature air as a function of wavelength at 4000°K; plotted from the data listed in Gilmore.\(^6\)

Fig. 3-2.6. Absorption coefficients of high-temperature air as a function of wavelength at 6000°K; plotted from the data listed in Gilmore.\(^6\)
FIG. 3-2.7. Absorption coefficients of high-temperature air as a function of wavelength at 8000°K; plotted from the data listed in Gilmore.  

FIG. 3-2.8. Emission coefficient of high-temperature air as a function of wavelength at 2000°K; plotted from the data in Gilmore.
Fig. 3-2.9. Emission coefficient of high-temperature air as a function of wavelength at 3000°K; plotted from the data in Gilmore.5

Fig. 3-2.10. Emission coefficient of high-temperature air as a function of wavelength at 4000°K; plotted from the data in Gilmore.5
Fig. 3-2.11. Emission coefficient of high-temperature air as a function of wavelength at 6000°K; plotted from the data in Gilmore.  

Fig. 3-2.12. Emission coefficient of high-temperature air as a function of wavelength at 8000°K; plotted from the data in Gilmore.
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Fig. 3-2.13. Total emission rate of air as a function of temperature (neglecting self-absorption, induced emission, and infrared radiation beyond 2.5 μ); plotted from the data listed in Gilmore.

Fig. 3-2.14. Emission rate of high-temperature air in the visible range (0.40–0.65 μ), neglecting self-absorption and induced emission; plotted from the data listed in Gilmore.
The atomic Planck mean results of Armstrong et al.\textsuperscript{2} do not join smoothly with the molecular Planck mean of Gilmore;\textsuperscript{5} the older atomic Planck mean,\textsuperscript{1} without atomic lines, does join smoothly but lies some 2.5 orders of magnitude lower than the data of Armstrong et al.\textsuperscript{2} at $kT = 1$ eV. These observations suggest that the values of $k_P$ at tempera-
tures around 5000 to 8000°K would be revised sharply upward if atomic lines were included. The fact that atomic-line radiation dominates the optically thin emission rate at $kT \sim 1$ eV and above, especially at low densities, was noted for air by Armstrong et al.\textsuperscript{1} and for other light elements by Stewart and Pyatt.\textsuperscript{9}

Fig. 3-2.17. Average transmission of optical radiation through a slab of air, heated to 8000°K, as a function of optical path length for normal density; reproduced from Churchill et al.\textsuperscript{4}

Fig. 3-2.18. Average transmission of optical radiation through a slab of air, heated to 4000°K, as a function of optical path length for one-tenth of normal density; reproduced from Churchill et al.\textsuperscript{4}
It is clear that the existing data for air absorption coefficients, though extensive, are far from complete. The range of system dimensions, $L$, for which the optically thick and optically thin criteria both fail is obviously larger than the range $l_p < L < l_R$ (and, as noted above, sometimes much larger); the ratio $I_R/I_p$ is itself large, e.g. $\sim 10^4$ at the lowest density.
shown. For values of \( L \) in this neither-thick-nor-thin range, none of the averages discussed above is valid, except that the calculated transmission data apply. Calculations such as those of Churchill et al.\(^4\) and Churchill and Hagstrom\(^8\) are therefore needed for a wide range of parameters—including the atomic regime—with realistic estimates of line profiles.
As noted above, it is probably necessary to add atomic lines in the hotter part of the molecular regime. Also, Rosseland-mean data are needed in the molecular regime; these may be constructed from spectral transmission data when the latter are sufficiently complete. Particular attention must be given to the transmission in relatively transparent spectral regions, since these regions dominate in the Rosseland averaging.

3-3 Thermal conduction

A process which competes with radiative heat transfer is gas-kinetic thermal conduction, which is a diffusive process in any gas dense enough to be treated by continuum hydrodynamics. The conductive-energy flux vector is then

\[ \mathbf{F}_{\text{cond}} = -\lambda \nabla T \]  

(3-3.1)

with \( \lambda \) = thermal conductivity. At temperatures above 2 eV, \( \lambda \) is dominated by the contribution of free electrons and may be computed by means of the following expression from Spitzer and Härm:\textsuperscript{11}

\[ \lambda = \frac{80k^3(2k/m\pi)^{1/2} T^{5/2} \delta_T}{n e^4 \bar{z} \ln[9h^4 T^4/4e^6 n_e^2 p_e(1 + \bar{z})]} \]  

(3-3.2)

where \( k \) = Boltzmann's constant; \( T \) = absolute temperature; \( \delta_T \) = a coefficient calculated numerically (by interpolation from Table III of Spitzer and Härm\textsuperscript{11}) as a function of \( \bar{z} \); \( e \) = electronic charge; \( \bar{z} \) = mean ion charge = \( \Sigma n_i z_i^2 / \Sigma n_i z_i \), \( p_e \) = equilibrium electron partial pressure; and \( n_i \) = concentration of particles with charge \( z_i \). This expression for \( \lambda \) applies in the presence of direct current in a completely ionized gas. For partially ionized gases, corrections to the Spitzer and Härm relation may be found in Shkarofsky,\textsuperscript{12} but these may be shown to be unimportant for our purposes. Below 2 eV, the values of \( \lambda \) were taken from the work of Peng and Pindroh,\textsuperscript{13} who used a shielded coulomb potential for electron-ion interactions. These estimates for \( \lambda \) are considered to be more accurate than those derived earlier by Hansen.\textsuperscript{14} Equation (3-3.2) for \( \lambda \) may be rewritten in the form

\[ \lambda = \frac{2.465 \times 10^6 \delta_T^{5/2}}{\bar{z} \log_{10} \left[ \frac{2.401 \times 10^{20} \rho^3}{\bar{z}^2 \bar{m}(1 + \bar{z}) \bar{N}} \right]} \text{ erg cm-sec}^{-2} \text{K} \]  

(3-3.3)
where $\theta$ is the temperature (eV), $\delta_T$ has been defined previously, $\bar{m} =$ mean particle charge $= \Sigma n_i z_i / \Sigma n_i$, and $N =$ particle concentration $= 5.38 \times 10^{19} \rho / \rho_0$ (where $\rho$ is the mass density and $\rho_0 = 1.293 \times 10^{-3}$ g cm$^{-3}$).

If an electric field builds up sufficiently to restrain the flow of electric current, the values of $\lambda$ from Eq. (3-3.3) should be multiplied by 0.4. In many practically important problems, Eq. (3-3.3) is known to yield values for the total thermal conductivity that are good to within about a factor of 2.

We may estimate the relative importance of conductive and radiative energy transport in heated air and gain some insight into the conditions under which radiative-energy transport must be considered in stationary systems. It should be noted that the results are directly applicable to flow problems with radiant-energy transport only in the Rosseland (diffusion) limit since, in this case, the radiative and conductive heat transfer coefficients are additive. In the transparent gas limit, the important similarity group measuring the ratio of radiative-energy loss from the system, per unit surface area, to the free-stream rate of enthalpy transport, per unit area, is the more meaningful parameter in gas-dynamic studies (cf. Section 2-4C).

In the diffusion limit, the ratio of heat transfer by thermal conduction to heat transfer by radiation is given by the expression

$$\alpha_1 = \frac{3\lambda}{16\sigma^2 T^4 l_R}$$  \hspace{1cm} (3-3.4)

on the other hand, in the emission limit, this ratio is

$$\alpha_2 = \frac{(1/L)\lambda |\nabla T| l_p}{4\sigma T^4}$$  \hspace{1cm} (3-3.5)

Here $\sigma$ is the Stefan-Boltzmann constant, $T$ equals the absolute temperature, $L$ is a characteristic length, and $\nabla T$ represents a typical temperature gradient.

Values of $\alpha_1$ obtained from Eq. (3-3.4) and Figs. 3-2.1 and 3-3.1 are all small compared to unity for the density range $(\rho / \rho_0 \leq 10)$ considered; so we do not quote them. The conclusion that, in the optically thick limit, radiation dominates conduction up to still higher densities was obtained by Cox and Stewart.\(^3\)

Values of $\alpha_2 L |\nabla T|$ have been given elsewhere\(^10\) on the basis of the Planck mean without atomic lines.\(^1\) These values must be revised down-
ward by up to 3.5 orders of magnitude to conform to the revised $l_p$ of Armstrong et al.\textsuperscript{2} For order-of-magnitude estimates, $|\nabla T| \approx T/L$, and we may define a critical length $L_0$ by the condition $\alpha_2 = 1$ at $L = L_0$. In layers thicker than $L_0$ (but still optically thin), radiation dominates conduction, and in thinner layers the reverse holds; the ratio $\alpha_2$ is $L_0^2/L^2$. The critical length is given by

$$L_0 = \left( \frac{\lambda p}{4\sigma T^3} \right)^{1/2} \tag{3-3.6}$$

and depends on temperature and density. In the temperature range 1–2 eV, for example, an order-of-magnitude fit to $L_0$ is

$$L_0 \approx \left( \frac{1 \text{ eV}}{KT} \right)^2 \left( \frac{\rho_0}{\rho} \right)^{1/2} 10^{-2} \text{ cm} \tag{3-3.7}$$

so that, for most layer thicknesses of interest in hypersonic flow problems (e.g., $L \sim 1/10$ cm, $\rho/\rho_0 \sim 1/10$) at these temperatures, radiation is dominant. At lower or much higher temperatures, $l_p$ increases and conduction becomes more important.

### 3-4 Examples of nonequilibrium radiation in heated air

The basic relations referring to individual excited energy levels, which have been developed in Chapters 1 and 2, are, of course, equally applicable to nonequilibrium systems and to equilibrium systems. The only essential requirement for the calculation of opacities in nonequilibrium systems is the requirement that the nonequilibrium system be defined. There is clearly a multiplicity of possible nonequilibrium systems ranging all the way from the completely uncorrelated arrangement, in which the state of each individual atom must be specified, to systems in intermediate states of relaxation which may be characterized by rotational and translational “temperatures” but not by vibrational and electronic “temperatures”.

In view of the limitless number of possible variants of nonequilibrium systems, it is not particularly interesting to prepare a catalog of the radiative properties for nonequilibrium systems. Examples of this type of computation have been published.\textsuperscript{15} Of more immediate interest is an analysis of experimental measurements, particularly of data referring to entry into planetary atmospheres, from which information may be obtained on the relaxation phenomena that dominate the population distributions of the emitting species and hence the observed radiant-
energy flux. Examples of studies of this type are provided by recent experimental investigations of the phenomena associated with super-satellite reentry.\textsuperscript{16,17,18a–18d}

3-5 Neglected band systems\textsuperscript{10}

All of the results listed in the preceding sections refer to a “pure air mixture” without CO\textsubscript{2}, H\textsubscript{2}O, etc. Hence the equilibrium opacity estimates are not applicable to “real” air. Gilmore has recently emphasized this fact and has called attention to other band systems that have been neglected in the published opacity literature on high-temperature air.\textsuperscript{19}

An exhaustive account of opacity calculations on high-temperature air has been published recently.\textsuperscript{20}

\textbf{References}

REFERENCES


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