Chapter 6

RADIATION GASDYNAMICS*

In Chapter 2 the conservation equations for a radiating gas were developed and similarity parameters were discussed. These conservation equations are used in this chapter to study several problems in the rapidly expanding field of radiation gasdynamics. Much of the recent interest in radiation gasdynamics has been produced by the problem of very high velocity reentry, although radiation gasdynamics also has application to astrophysical problems and to nuclear explosions. Accordingly, in Section 6-1 we investigate the conditions under which radiative transfer has important effects on the flow about a reentry body. The approximations used for the radiative transfer are discussed in Section 6-2. The specific problems of the effect of radiant-energy transfer on acoustic waves, shock wave structure, shock layers, and boundary layers are discussed in Sections 6-3 through 6-6, respectively.

6-1 Radiative transfer during reentry

At sub-satellite reentry velocities the radiant heat transfer is less than the aerodynamic heat transfer to the vehicle, and the radiant-energy loss in the shock layer is far too small to affect the flow field. However, as the reentry velocity is increased, radiation heat transfer rapidly overtakes the aerodynamic heat transfer (cf. Section 2-1F). At velocities greater than the earth parabolic velocity \( V_{\text{parabolic}} = 11.19 \text{ km/sec} = 36,650 \text{ ft/sec} \) the radiation can appreciably affect the flow field. To illustrate this fact, the parameter \( \Gamma_n \) is given in Fig. 6-1.1 (from Goulard\(^1\) as a function of altitude and velocity for a blunt body; \( \Gamma_n = 4a T q s k_s A / \rho \alpha V q s h_n \), which is the ratio of the radiant-energy flux emitted from an optically thin shock layer of thickness \( A \) to the enthalpy flux across the shock front. It is observed from the figure that for a

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A reentry vehicle of 1-ft nose radius, \( \Gamma_n \) approaches \( 10^{-2} \) for a moon satellite and unity for a Mars probe. Thus as the velocity of a blunt reentry vehicle increases above the parabolic value, the radiant energy loss from the shock layer approaches the magnitude of the flow energy.

In order to determine the importance of absorption within the blunt-body shock layer, Goulard has constructed the diagram shown in Fig. 6-1-2. In this figure the transparent gas approximation is assumed to be good until the optical depth \( \tau_d \) of the shock layer reaches the value 0.2. For an "optically thick" gas, the Rosseland diffusion approximation will be good until the optical depth \( \tau_b \) of the thermal boundary decreases below the value 5. When the Rosseland approximation is valid, instead of \( \Gamma_n \) the appropriate parameter is \( F_w/\rho \alpha V \propto h_s \), where \( F_w \) is the radiant-heat flux to the vehicle surface. Goulard shows that the ratio \( F_w/\rho \alpha V \propto h_s \) is equal to \((\Gamma_k/2\pi)^{1/2}\), where \( \Gamma_k = (16\sigma T_b^4/3\rho \alpha V \propto h_s)(1/k_s \Delta) \) (see Section 6-5B). When the optical depth is too small to validate the use of the Rosseland approximation but too large for the gas to be optically thin, the radiant-energy flux to the vehicle will be approximately
the blackbody value, so the relevant parameter is the reciprocal of the Boltzmann number \( B_0^{-1} = \sigma T^3 / \rho_\infty V_\infty h_s \). These radiative transfer regimes are illustrated on the velocity-altitude plot shown in Fig. 6-1.2.

Yoshikawa and Chapman\(^2\) have carried out calculations for a hypersonic normal shock wave to determine the regimes in which decay (energy depletion by radiative transfer) and absorption are important. They define a heat transfer coefficient \( \lambda_{w,s} = F_{w,s} / \rho_\infty V^3_\infty \), where \( F \) is the net radiant-energy flux to the surface (subscript \( w \)) or to the region upstream of the shock front (subscript \( s \)). Since \( \frac{1}{2} V^2_\infty \approx h_s \) for a strong shock, \( \lambda \) corresponds to the parameters used by Goulard, which are discussed above. In Fig. 6-1.3 \( \lambda \) is given as a function of velocity and altitude for a shock standoff distance \( L \) of 0.1 ft. The characteristic length \( L_c = 1 \text{ ft} = 10L \) shown in Fig. 6-1.3 equals the absorption length \( L_{abs} = 1/2h_s \) at low velocities and equals the decay length at
high velocities \( (L_{\text{dec}} = \rho_{\infty} V_{\infty} h_{\infty} / 4 k_{\text{e}} \sigma T_{\infty}^4) \), which is approximately the flow distance over which a gas element loses all of its energy by radiation. Thus, at low velocities the characteristic length \( L_c = 1 \text{ ft} \) in Fig. 6-1.3 divides the absorbing region \([L > (1/10) L_{\text{abs}}]\) from the nonabsorbing region \([L < (1/10) L_{\text{abs}}] \), whereas at high velocities \( L_c = 1 \text{ ft} \) divides the region of decay \([L > (1/10) L_{\text{dec}}]\) from the region of nondecay \([L < (1/10) L_{\text{dec}}]\). The curves of constant \( \lambda \) in Fig. 6-1.3 are seen to be in fair agreement with the two curves of Goulard shown in Fig. 6-1.2. Yoshikawa and Chapman summarize their results by stating that for the altitude range of severe heating for shallow entry trajectories of manned vehicles (60 to 80 km), neither decay nor absorption of radiation is important for a standoff distance of 1 ft or less and entry velocities of about 11 km/sec (corresponding to entry upon return from the moon).* At higher entry velocities of about 15 km/sec (corresponding to entry upon return from a relatively short-time trajectory from Mars), decay of radiation is important but absorption is not. For objects such as large meteorites, which enter at higher velocities and also at relatively steep angles, the most severe heating occurs at altitudes of about 20 to 40 km, where both decay and absorption are important.

* Recent calculations described in Hoshizaki and Wilson indicate that decay and absorption can be important for an Apollo-type entry.
Entry speeds well in excess of parabolic speed are required if interplanetary travel times are to be appreciably shortened (cf. Fig. 6-1.4). As discussed above, at these high entry velocities the radiant-heat transfer to blunt bodies becomes prohibitively high, so the use of relatively slender, pointed bodies is indicated. The heat transfer to conical bodies at high entry speeds has been studied by Allen et al., whose results are shown in Figs. 6-1.5 and 6-1.6. The energy fractions $\eta_l$, $\eta_t$, $\eta_e$, and $\eta_n$ for laminar convection, turbulent convection, equilibrium radiation, and nonequilibrium radiation heat transfer, respectively, are defined as the fractions of the vehicle kinetic energy which appears as heat transfer to the vehicle during an entry trajectory. Neglecting the acceleration of gravity during slowdown, the change in vehicle kinetic energy equals the integral of the drag force along the trajectory:

$$\frac{1}{2}mV_E^2 \simeq A \int C_D \frac{1}{2} \rho V^2 \, ds = C_D A \int \frac{1}{2} \rho V^3 \, dt,$$

$$\eta = \frac{H}{\frac{1}{2} m V_E^2} = \frac{S \int q_w \, dt}{C_D A \int \frac{1}{2} \rho V^3 \, dt}. \quad (6-1.1)$$

Here $C_D$ is the drag coefficient; $A$ and $S$ are the cone base and surface (excluding the base) areas, respectively; $H = S \int q_w \, dt$ is the total radiative and convective heat transfer to the conical body. Therefore, when radiative heat transfer is dominant, $\eta$ corresponds to the "average" value of the heat transfer coefficient $\lambda_w = F_w / \frac{1}{2} \rho V^3$ times $S/C_D A$ (we note that $S/C_D A \simeq (2 \sin^3 \theta_c)^{-1}$ for a cone of half-angle $\theta_c$). From Fig. 6-1.5 it is seen that for a fixed cone angle the equilibrium radiation will sharply
FIG. 6-1.5. Variation of the energy fractions $\eta_l$, $\eta_t$, $\eta_e$, and $\eta_n$ with entry speed for a 30° half-angle cone with a Teflon ablator, from Allen et al.\textsuperscript{3} (a) Laminar flow, ballistic parameter = 200. (b) Turbulent flow, ballistic parameter = 20.
become the dominant form of heat transfer for entry velocities greater than a certain value. The boundary layer will be laminar for large values of the ballistic parameter and turbulent for small values of the ballistic parameter; the ballistic parameter $B$ is defined by $B = C_d \rho_0 A/\beta m \sin \gamma$, where $\rho_0$ is the sea level density, $\beta$ the inverse of the atmospheric scale height, $m$ the entry mass, and $\gamma$ the flight path angle measured from the horizontal. If the entry velocity is large enough for a given cone, the radiation can affect the flow field (radiation decay). However, the heat transfer can be minimized by reducing the cone angle as the entry

Fig. 6-1.6. Variation of the total energy fraction $\eta$ with entry speed for cones having Teflon ablators and various half angles, from Allen et al.\textsuperscript{3} (a) Laminar flow, ballistic parameter $= 200$. (b) Turbulent flow, ballistic parameter $= 20$. 
velocity is increased (see Fig. 6-1.6). Recent measurements\(^4\) of radiation from the shock layers of 30° half-angle cones traveling at 7 km/sec in a free-flight facility verify the large reduction in radiant-heat transfer for cones compared with that for blunt bodies at comparable free-stream conditions. We expect, however, that the regime where radiation affects gasdynamics will be important for pointed bodies such as cones, as well as for blunt bodies.

6-2 Approximations for the radiative transfer term

Adding the effects of radiative transfer to gasdynamics problems results in increasing the complexity by so much that usually solutions can be obtained only if approximations are made. One approximation which is used in a number of problems is to treat the radiation as a perturbation, with the radiant-energy flux determined from the unperturbed flow field. When the radiative transfer is too large to be considered as a perturbation, approximations are made for the radiative transfer term \( \nabla \cdot \mathbf{F} \), which appears in the energy equation [see Eq. (2-3.16)]. The general solution to the transfer equation gives \( \mathbf{F} \) in terms of an integral over the unknown flow field [cf. Eqs. (2-1.21) and (2-1.22), or Eq. (6-2.7) below]. This integral is eliminated in the approximations which will be discussed.

6-2A The Planck and Rosseland Limits; The Differential Approximation. Probably the most common approximation, which also results in the greatest simplification, is the transparent or optically thin gas approximation (Planck limit). The transparent gas approximation is applicable when the photon mean free path is large compared with the dimensions of the emitting gas. The radiant-energy emission per unit volume reduces to

\[
\nabla \cdot \mathbf{F} = 4\sigma T^4 \bar{k}_p,
\]

(6-2.1)

where \( \bar{k}_p \) is the Planck mean absorption coefficient (see Section 2-2A).

At the other limit, when the photon mean free path is small enough for the gas to be "optically thick," the Rosseland or diffusion approximation holds:

\[
\mathbf{F} = -\frac{4}{3} \frac{\sigma}{\bar{k}_R} \nabla T^4
\]

or

\[
\nabla \cdot \mathbf{F} = -\frac{4}{3} \sigma \nabla \cdot \left( \frac{1}{\bar{k}_R} \nabla T^4 \right),
\]

(6-2.2)
where \( \bar{k}_R \) is the Rosseland mean absorption coefficient (see Section 2-2B). It should be noted that the diffusion approximation is not valid near surfaces, where higher-order derivatives of \( T^4 \) are important.

As observed from Eqs. (6-2.1) and (6-2.2), the radiative transfer term becomes greatly simplified in the Planck and Rosseland limits, with the spectral variation of the absorption coefficient being accounted for by the use of the appropriate average absorption coefficient. In situations where both small and large optical depths are important, no simple average absorption coefficient is adequate, and generally the problem must be solved for a particular spectral variation of the absorption coefficient (see Section 2-6). The simplest and most widely treated case is that of a gray gas, defined by an absorption coefficient which does not vary with frequency, i.e., \( k_v = k = \text{constant} \). A gray gas may often be used as a first approximation for a gas with a slowly varying spectral absorption coefficient, such as that produced by photodetachment or free-free transitions. When strongly non-gray radiation is dominant, such as that produced by molecular bands, atomic lines, or bound-free transitions with large absorption edges, then the use of a gray-gas model may give completely erroneous results. In this section we consider a differential approximation to the equation of radiative transfer which is appropriate for a gray gas. Later, in Section 6-2E, we shall consider modifications of this differential approximation for application to (1) slightly non-gray gases, and (2) gases with radiation in a gray band, which may approximate a bound-free continuum or a band of well-overlapped lines.

The equation of radiative transfer for a gray gas in local thermodynamic equilibrium is (see Section 2-1)

\[
\frac{1}{k} \frac{d I}{ds} = B - I, \tag{6-2.3}
\]

where \( I \) is the frequency-integrated specific intensity (steradiancy) propagating in the direction \( s \), and \( B = \sigma T^4/\pi \) is the blackbody steradiancy. Inclusion of the photon travel time at the velocity of light \( c \) would introduce the time derivative \( (kc)^{-1} \partial I/\partial t \) to the left-hand side of Eq. (6-2.3); however, this term will be completely negligible for the time scales of interest in the problems considered in this monograph. The net radiative flux in the \( x_t \)-direction may be obtained by taking the component of \( I \) in the \( x_t \)-direction and integrating over solid angle [cf. Eq. (2-1.9)],

\[
F_t = \int_0^{4\pi} \cos \theta, I \, d\Omega, \tag{6-2.4}
\]
where $\theta_i$ is the angle between the direction $s$ of $I$ and the $x_i$-direction (see Fig. 6-2.1 for the one-dimensional case), and $d\Omega$ is the element of solid angle, which is integrated over $4\pi$ sterad. By appropriately averaging Eq. (6-2.3) over all directions, the following differential approximation to the transfer equation may be obtained:

$$\frac{1}{k} \frac{\partial}{\partial x_i} \left( \frac{1}{k} \frac{\partial F_j}{\partial x_j} \right) = \frac{4\sigma}{k} \frac{\partial T^4}{\partial x_i} + 3F_i,$$  (6-2.5)

where the subscript $i$ denotes $i = 1, 2, \text{or} 3$, and the appearance of the subscript $j$ twice in the first term indicates summation over the components $j = 1, 2, \text{and} 3$. For one-dimensional problems, Eq. (6-2.5) reduces to the following form:

$$\frac{d^2F}{d\tau^2} = 4\sigma \frac{dT^4}{d\tau} + 3F$$  (6-2.6)

where $d\tau = k \, dx$ is the element of optical depth in the $x$-direction.

The above differential approximation is often called the "Eddington approximation" in the astrophysical literature, and in the Soviet literature it is called the "diffusion approximation," which is not to be confused with the Rosseland diffusion approximation.

Equations (6-2.5) and (6-2.6) are relatively simple differential equations for the net flux. For the one-dimensional case, integration of the transfer equation (6-2.3) and substitution into Eq. (6-2.4) yields the following exact integral equation [cf. Eqs. (2-1.21) and (2.1-22)], which is much more difficult to utilize than is the approximate differential equation (6-2.6):

$$F = 2\sigma \int_0^\tau T^4 E_0(\tau - t) \, dt - 2\sigma \int_\tau^0 T^4 E_0(t - \tau) \, dt$$

$$+ 2\pi \int_0^{\pi/2} I^{-0}(0) \left[ \exp(-\tau/cos \theta) \right] \cos \theta \sin \theta \, d\theta$$

$$+ 2\pi \int_{\pi/2}^\pi I^+(\tau_0) \left[ \exp((\tau_0 - \tau)/cos \theta) \right] \cos \theta \sin \theta \, d\theta,$$  (6-2.7)

where the tabulated exponential functions $E_n(t)$ are defined by

$$E_n(t) = \int_0^1 \mu^{n-2} \exp(-t/\mu) \, d\mu.$$  (6-2.8)

In the derivation of Eq. (6-2.7) the gas is assumed to lie between $\tau = 0$ and $\tau_0$, with the boundary walls emitting or reflecting the specific
intensities $I^-(0)$ and $I^+(\tau_0)$ at $\tau = 0$ and $\tau_0$, respectively. When the wall radiation is diffuse, i.e., independent of $\theta$, then the last two terms in Eq. (6-2.7) reduce to $2\pi I^-(0) E_3(\tau)$ and $-2\pi I^+(\tau_0) E_3(\tau_0 - \tau)$, respectively [if the walls are blackbodies at temperatures $T_1$ and $T_2$, then $I^-(0) = \sigma T_1^4/\pi$ and $I^+(\tau_0) = \sigma T_2^4/\pi$]. Thus $I^-(0)$ and $I^+(\tau_0)$ will be known, or can be related to the radiative flux incident on the surfaces (in the case of reflection).

The appropriate boundary conditions for the one-dimensional form (Eq. 6-2.6) of the differential approximation are

$$
\frac{dF}{d\tau} = 2F + 4\pi T^4 - 4F^-(0) \quad \text{at} \quad \tau = 0
$$

$$
\frac{dF}{d\tau} = -2F + 4\pi T^4 + 4F^+(\tau_0) \quad \text{at} \quad \tau = \tau_0
$$

where $F^-(0)$ and $F^+(\tau_0)$ are the radiant fluxes from the boundary surfaces. That is,

$$
F^-(0) = 2\pi \int_{\sigma/2}^{\pi} I^-(0) \cos \theta \sin \theta \, d\theta
$$

$$
F^+(\tau_0) = 2\pi \int_{\pi/2}^{\pi} I^+(\tau_0) \cos \theta \sin \theta \, d\theta
$$

which reduce to $F^-(0) = \sigma T_1^4$ and $F^+(\tau_0) = -\sigma T_2^4$ for black walls at temperatures $T_1$ and $T_2$, respectively. For the three-dimensional case, the boundary condition at the surface $x_i = 0$ is

$$
\frac{1}{k} \frac{\partial F_j}{\partial x_j} = 2F_i + 4\pi T^4 - 4F^+_i(0) \quad \text{at} \quad x_i = 0,
$$

where $F^+_i(0)$ is the flux from the surface.

In the next three sections, the one- and three-dimensional differential approximations, as well as the boundary conditions, will be derived by various methods. Each of these methods involves a first-order directional averaging of the transfer equation.

6-2B The Schuster-Schwarzchild and Eddington methods. The first approximate method of solving radiative transfer problems was developed by Schuster and by Schwarzchild. This method, commonly called the Schuster-Schwarzchild method, yields an equation identical to Eq. (6-2.6) except for a different (less accurate) constant in the last term. An analysis of the errors in the Schuster-Schwarzchild method has been given by Milne.
The Schuster-Schwarzchild method involves consideration of the total intensities from the left and right directions for a one-dimensional problem. That is, for a gray gas, the following quantities are defined:

\[
I_1(\tau) \equiv \frac{1}{2\pi} \int_0^{2\pi} I(\tau, \theta) d\Omega = \int_0^{\pi/2} I(\tau, \theta) \sin \theta d\theta
\]

\[
I_2(\tau) = \int_{\pi/2}^{\pi} I(\tau, \theta) \sin \theta d\theta
\]

(6-2.12)

where the subscripts 1 and 2 are used to denote the total intensities from the left and right directions, respectively (see Fig. 6-2.1). The basic approximation of the Schuster-Schwarzchild method is the replacement of \(|\cos \theta|\) by the average value of \(\frac{1}{2}\) whenever the product \(I(\tau, \theta) \cos \theta\) is averaged over \(2\pi\) sterad, i.e.,

\[
\int_0^{\pi/2} I(\tau, \theta) \cos \theta \sin \theta d\theta = \frac{1}{2} I_1(\tau)
\]

(6-2.13)

\[
\int_{\pi/2}^{\pi} I(\tau, \theta) \cos \theta \sin \theta d\theta = -\frac{1}{2} I_2(\tau).
\]

In view of Eqs. (6-2.13), the net flux given by Eq. (6-2.4) may be written as

\[
F = 2\pi \int_0^{\pi} I(\tau, \theta) \cos \theta \sin \theta d\theta = \pi(I_1 - I_2).
\]

(6-2.14)
The integral of the specific intensity over $4\pi$ sterad is

\[
I_0 = \int_0^{4\pi} I(\tau, \theta) \, d\Omega = 2\pi(I_1 + I_2). \quad (6-2.15)
\]

For one-dimensional problems the transfer equation (6-2.3) becomes

\[
\cos \theta \frac{\partial I(\tau, \theta)}{\partial \tau} = B - I(\tau, \theta). \quad (6-2.16)
\]

Integrating Eq. (6-2.16) over the angles $\theta = 0$ to $\pi/2$ and $\pi/2$ to $\pi$,

\[
\frac{1}{2} \frac{dI_1}{d\tau} = B - I_1,
\]

\[
-\frac{1}{2} \frac{dI_2}{d\tau} = B - I_2. \quad (6-2.17)
\]

By adding and subtracting the above equations, and using Eqs. (6-2.14) and (6-2.15), the following equations are obtained:

\[
\frac{dF}{d\tau} = 4\sigma T^4 - I_0, \quad F = -\frac{1}{4} \frac{dI_0}{d\tau}, \quad (6-2.18)
\]

where the identity $B = \sigma T^4/\pi$ has been used. Elimination of $I_0$ from the above equations gives the differential approximation (6-2.6), except with the last term $3F$ replaced by $4F$. For some problems it may be more convenient to work with $I_0$ rather than with $F$ (see Section 6-2F).

It should be noted that the first of Eqs. (6-2.18) is the exact integral of the transfer equation, whereas the second equation is an approximate equation of the diffusion type. This result has led Soviet authors to call Eqs. (6-2.18) the diffusion approximation (here the flux is proportional to the gradient of $I_0$, rather than to the gradient of $B$, as in the Rosseland diffusion approximation).

The boundary conditions (6-2.9) are obtained for the Schuster-Schwarzchild method by adding Eqs. (6-2.17), evaluating the result at $\tau = 0$ and $\tau_0$, and identifying $\pi i_1(0)$ and $\pi i_2(\tau_0)$ with $F^-(0)$ and $-F^+(\tau_0)$, respectively [cf. Eqs. (6-2.10) and (6-2.13)].

Reference to Eqs. (6-2.17) shows that the Schuster-Schwarzchild approximation may be considered as a division of the photons into two streams traveling in opposite directions. Chandrasekhar has generalized this procedure in the "method of discrete ordinates," which treats $2n$ streams (opposing streams at $n$ different angles). The transfer equation is reduced by this method to a system of $2n$ linear differential equations.
The Eddington method, which involves a more accurate directional averaging procedure than the Schuster-Schwarzchild method, yields an equation identical to the differential approximation (6-2.6). In the Eddington method it is the integral over solid angle of $I(\tau, \theta) \cos^2 \theta$, rather $I(\tau, \theta) \cos \theta$, which is approximated. Namely,

$$\int_0^\pi I(\theta) \cos^2 \theta \sin \theta d\theta \simeq (I_1 + I_2) \int_0^{\pi/2} \cos^2 \theta \sin \theta d\theta = \frac{1}{2} I_0. \quad (6-2.19)$$

Integration of the transfer equation (6-2.16) over $4\pi$ sterad gives

$$\frac{dF}{d\tau} = 4\sigma T^4 - I_0. \quad (6-2.20)$$

Now multiplication of the transfer equation by $\cos \theta$ and integration over $4\pi$ sterad, using Eq. (6-2.19), yields

$$F = -\frac{1}{3} \frac{dI_0}{d\tau}. \quad (6-2.21)$$

Elimination of $I_0$ between Eqs. (6-2.20) and (6-2.21) gives the differential approximation (6-2.6). Eddington also carried out a second approximation (Eddington, 11, p. 332).

6-2C THE SUBSTITUTE KERNEL METHOD. Equation (6-2.6) for the differential approximation may be derived by a substitute kernel method (also called the exponential approximation), which is used often in radiation-gasdynamics problems (see, e.g., Vincenti and Baldwin).

For diffuse wall radiation, the integral expression (6-2.7) for the flux involves the $E_3$ function for the wall radiation, as well as integrals over the $E_2$ function for the gas radiation. The basic approximation of the substitute kernel method is the replacement of $E_2(t)$ by the exponential $me^{-nt}$, which also implies $E_3(t) \simeq me^{-n/\|n\}$ since $dE_3(t)/dt = -E_2(t)$. Appropriate values for the constants $m$ and $n$ will be determined later. In terms of the approximate exponential kernels, the integral expression for the flux becomes

$$F \simeq 2\sigma \left[ \int_0^\tau T^4 \exp\{-n(\tau - t)\} dt - \int_0^\tau T^4 \exp\{-n(t - \tau)\} dt \right]$$

$$+ 2 \frac{m}{n} [F^-(0) \exp\{-n\tau\} + F^+(\tau_0) \exp\{-n(\tau_0 - \tau)\}]. \quad (6-2.22)$$

By differentiating Eq. (6-2.22) twice with respect to $\tau$, and using
Eq. (6-2.22) to substitute for the resulting integrals, the following result is obtained:

\[
\frac{d^2F}{dr^2} = 4m\sigma \frac{dT}{dr} + n^2F.
\]  

(6-2.23)

In order that this equation reduce to the correct optically thin and Rosseland limits, the values \(m = 1\) and \(n = \sqrt{3}\) must be chosen; the resulting equation then agrees with Eq. (6-2.6). Some authors choose values of \(m\) and \(n\) which give the Rosseland limit \((n^2 = 3m)\), and which approximate the radiative transfer at intermediate optical depths (giving an \(m\) value less than unity).

The boundary conditions are determined by evaluating the first derivative of Eq. (6-2.22) at the walls, and using Eq. (6-2.22) to substitute for the resulting integrals. For the values \(m = 1\) and \(n = \sqrt{3}\), the boundary conditions (6-2.9) are obtained, except with the terms \(\pm 2F\) replaced by \(\pm \sqrt{3}F\).

6-2D The spherical harmonics and moment methods. The spherical harmonics method was developed for astrophysical and neutron-transport problems, and has been used in radiation gas-dynamics problems. In the first approximation, the spherical harmonics method gives the three-dimensional differential approximation (6-2.5) and the boundary condition (6-2.11).

The procedure for this method starts with the expansion of the specific intensity in terms of the spherical harmonics \(Y^m_l(\Omega)\),

\[
I(r, \Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A^m_l(r) Y^m_l(\Omega),
\]

(6-2.24)

where the coefficients \(A^m_l(r)\) are to be determined. The spherical harmonics \(Y^m_l(\Omega)\) are related to the associated Legendre functions \(P^m_l(\cos \theta)\) by

\[
Y^m_l(\Omega) = \left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} e^{im\varphi} P^m_l(\cos \theta),
\]

(6-2.25)

where the angles \(\theta\) and \(\varphi\) are shown in Fig. 6-2.2.

In the transfer equation (6-2.3) the derivative \(dI/ds\) may be expressed as \(\cos \theta_j \partial I / \partial x_j\), where \(\cos \theta_j\) are the direction cosines, and the appearance of \(j\) twice indicates summation over \(j = 1, 2,\) and 3. The spherical harmonics method proceeds with the substitution of the expansion Eq. (6-2.24) into the transfer equation. The resulting equation is multiplied by the complex conjugate of a particular spherical harmonic
function $Y^m_l$, and integrated over $4\pi$ sterad. Application of the orthogonality and recurrence relations for the spherical harmonics yields an equation involving $A^m_l$ and partial derivatives of $A^{m+1}_{l+1}, A^{m+1}_{l-1}, A^{m-1}_{l+1}, A^{m-1}_{l-1}, A^m_{l+1}$, and $A^m_{l-1}$.

The first approximation consists of using only the first four terms in the expansion equation (6-2.24), i.e., only the terms $A^0_0 Y^0_0$, $A^{-1}_1 Y^{-1}_1$, $A^0_1 Y^0_1$, and $A^1_1 Y^1_1$. Substitution of this four-term expansion into the transfer equation, multiplication in turn by the complex conjugate functions $\tilde{Y}^0_0$, $\tilde{Y}^{-1}_1$, $\tilde{Y}^0_1$, and $\tilde{Y}^1_1$, and integration over $4\pi$ sterad gives the following equations:

For $l = 0$, $m = 0$:

$$\frac{1}{\sqrt{2}} \left( \frac{\partial A^1_1}{\partial x_1} + i \frac{\partial A^1_1}{\partial x_2} \right) - \frac{1}{\sqrt{2}} \left( \frac{\partial A^{-1}_1}{\partial x_1} - i \frac{\partial A^{-1}_1}{\partial x_2} \right)$$

$$- \frac{\partial A^0_0}{\partial x_3} - 3kA^0_0 + \frac{3k\sigma T^4}{\pi} = 0,$$

For $l = 1$, $m = -1$:

$$\frac{1}{\sqrt{2}} \left( \frac{\partial A^0_0}{\partial x_1} + i \frac{\partial A^0_0}{\partial x_2} \right) + kA^{-1}_1 = 0,$$

For $l = 1$, $m = 0$:

$$\frac{\partial A^0_0}{\partial x_3} + kA^0_0 = 0,$$

For $l = 1$, $m = 1$:

$$\frac{1}{\sqrt{2}} \left( \frac{\partial A^0_0}{\partial x_1} - i \frac{\partial A^0_0}{\partial x_2} \right) - kA^1_1 = 0.$$

Now substituting the four-term expansion into Eq. (6-2.4) for $F_i$ and
into the defining relation, Eq. (6-2.15), for \( I_0 \), the following relations are obtained:

\[
I_0 = 4\pi A_0^0, \quad F_1 = \frac{2\sqrt{2}\pi}{3} (A_1^{-1} - A_1^1)
\]

\[
F_2 = -\frac{2\sqrt{2}\pi}{3} (A_1^{-1} + A_1^1), \quad F_3 = \frac{4\pi}{3} A_1^0.
\]

Equation (6-2.27)

Use of the above equations to substitute for the \( A' \)'s in Eqs. (6-2.26) yields

\[
\frac{1}{k} \frac{\partial F_j}{\partial x_j} = 4\sigma T^4 - I_0, \quad F_i = -\frac{1}{3k} \frac{\partial I_0}{\partial x_i},
\]

which reduces to the three-dimensional differential approximation, Eq. (6-2.5), upon elimination of \( I_0 \).

The relations, Eqs. (6-2.27), may be used to transform the expansion for \( I \) into the following expression:

\[
I = \frac{1}{4\pi} (I_0 + 3F_1 \cos \varphi \sin \theta + 3F_2 \sin \varphi \sin \theta + 3F_3 \cos \theta). \quad (6-2.29)
\]

This equation may be used to compute the outward flux at a wall, say \( F_3^- (x_1, x_2, 0, t) \) at \( x_3 = 0 \),

\[
F_3^- (x_1, x_2, 0, t) = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} I^-(x_1, x_2, 0, t) \cos \theta \sin \theta \, d\theta \, d\varphi
\]

\[
= \frac{1}{4} I_0(x_1, x_2, 0, t) + \frac{1}{2} F_3(x_1, x_2, 0, t). \quad (6-2.30)
\]

Upon substituting for \( I_0 \) from the first of Eqs. (6-2.28), the boundary condition, Eq. (6-2.11), is obtained.

By using more terms in the expansion (6-2.24), the spherical harmonics method may be applied in a straightforward manner to obtain higher degrees of accuracy. In one-dimensional problems, however, the discontinuity of the intensity at the boundaries may be more easily accounted for by using separate expansions in the positive and negative directions, i.e., by using two half-interval expansions over \( 2\pi \) sterad each, rather than the full-interval expansion over \( 4\pi \) sterad (see Wilson and Sen\textsuperscript{15} and LeSage\textsuperscript{16}).

The moment method\textsuperscript{17,18} involves a procedure which is very similar
to that used for the spherical harmonics method. First, the following moments of the specific intensity are defined:

\[ I_0 = \int I \, d\Omega \]
\[ I_{1,i} = \int I \cos \theta_i \, d\Omega = F_i \]  
\[ I_{2,ij} = \int I \cos \theta_i \cos \theta_j \, d\Omega \]
\[ \vdots \]
\[ I_{n,ij} = \int I (\cos \theta_i)^{n-1} \cos \theta_j \, d\Omega, \]

where the \( \cos \theta_i \) are the direction cosines, and the integration is taken over \( 4\pi \) sterad. After expressing \( dI/ds \) as \( \cos \theta \cdot dI/dx \), the transfer equation (6-2.3) may be integrated to give

\[ \frac{1}{k} \frac{\partial I_{1,i}}{\partial x_j} = \frac{1}{k} \frac{\partial F_j}{\partial x_j} = 4\sigma T^4 - I_0. \]  
\[ (6-2.32) \]

Multiplying the transfer equation by \( \cos \theta_i \) and integrating, we obtain

\[ \frac{1}{k} \frac{\partial I_{2,ij}}{\partial x_j} = -I_{1,i} = -F_i \]  
\[ (6-2.33) \]

for \( i = 1, 2, \text{ and } 3 \). This procedure may be continued by multiplying the transfer equation by progressively higher powers of \( \cos \theta_i \) and integrating, yielding equations involving higher-order moments.

In the first approximation, only Eqs. (6-2.32) and (6-2.33) are considered. The number of unknowns is reduced and made equal to the number of equations by establishing a relation between the \( I_{2,ij} \) and \( I_0 \). To obtain such a relation, it is noted that if the specific intensity is expanded in terms of the first four spherical harmonics, then evaluation of the \( I_{2,ij} \) and \( I_0 \) gives

\[ I_{2,ij} = \frac{1}{3} I_0 \delta_{ij}, \]  
\[ (6-2.34) \]

where \( \delta_{ij} \) is the Kronecker delta, defined by \( \delta_{ij} = 0 \) if \( i \neq j \) and \( \delta_{ij} = 1 \) if \( i = j \). Substitution of Eq. (6-2.34) into Eq. (6-2.33) yields the differential approximation. The derivation of the boundary conditions for the moment method follows the procedure given above for the spherical harmonics method.

6-2E MODIFICATIONS OF THE DIFFERENTIAL APPROXIMATION. In this section we consider two types of modifications to the differential approximation: (1) a modification which correctly accounts for external
radiation sources, and (2) modifications which account for "slightly non-gray" radiation and "gray band" radiation.

Reference to Eq. (6-2.5) shows that the Rosseland expression (6-2.2) is recovered under optically thick conditions when the first term may be omitted from the differential approximation. For an optically thin gas, the last term in the differential approximation may be neglected, and integration of the remaining terms gives the optically thin expression (6-2.1) plus a constant of integration. This constant can approximately account for external radiation in one-dimensional problems; however, the modification discussed here will improve the accuracy near the optically thin limit (see the example in Section 6-2F). The modification will often be necessary in two- and three-dimensional problems, where the unmodified differential approximation may introduce very large errors if the external radiation varies with position (see Olfe19). The modification consists of dividing the flux at a point into two parts: the flux contributed by the external radiation and the flux contributed by the gas emission. The flux $F_{\text{ext}}$ contributed by the external radiation sources may be directly computed from the transfer equation with the emission term $B$ omitted [for the one-dimensional case of boundary walls at $\tau = 0$ and $\tau_0$, the flux $F_{\text{ext}}$ is given by the last two terms in Eq. (6-2.7)]. The flux $F_g$ contributed by the gas emission may be adequately calculated by the use of the differential approximation (6-2.5), with the boundary condition for $F_g$ being given by Eq. (6-2.11) with the external radiation term $F_{\text{ext}}(0)$ omitted. The total flux is given by $F = F_{\text{ext}} + F_g$, which is to be used in the energy equation (see Section 6-2F for the example problem of radiative equilibrium in a one-dimensional slab).

We now consider a modification of the differential approximation for a slightly non-gray gas, i.e., a gas with a slowly varying spectral absorption coefficient. When the absorption coefficient varies slowly with frequency, the gas will become optically thin or thick for nearly the same geometric length at each frequency; i.e., the gas will not be optically thin at some frequencies and strongly optically thick at other frequencies. Therefore, the differential approximation should be adequate for this slightly non-gray case, provided that we can identify the gray-gas absorption coefficient with the Planck and Rosseland mean values in such a manner that the correct Planck limit (6-2.1) and Rosseland limit (6-2.2) are recovered. This result is achieved by writing the differential approximation in the following form given by Traugott20:

$$\frac{\partial}{\partial x_i} \left( \frac{1}{k_p} \frac{\partial F_j}{\partial x_i} \right) = 4\sigma \frac{\partial T^4}{\partial x_i} + 3k_R F_i. \quad (6-2.35)$$
A similar method has been used by Sampson,\textsuperscript{21} who carried out calculations using the one-dimensional, gray-gas integral formula for the flux with an effective optical depth \( \tau_{\text{eff}} \equiv [(b + \tau_{\text{R}})/(b + \tau_{\text{P}})] \tau_{\text{P}} \), which reduces to the appropriate Planck and Rosseland limiting values \( \tau_{\text{P}} \) and \( \tau_{\text{R}} \), respectively (\( b \) is a constant of order unity, \( d\tau_{\text{p}} \equiv \bar{k}_{\text{p}} \, dx \), and \( d\tau_{\text{R}} \equiv \bar{k}_{\text{R}} \, dx \)). By comparing with numerical calculations for free-free radiation, Sampson found that his approximation gave quite good results, even under conditions for which the gas was more than "slightly non-gray"; e.g., his approximation gave results which were accurate to within a factor of 2 for all optical depths when the spectral dependence was so severe that \( \bar{k}_{\text{p}} \sim 30\bar{k}_{\text{R}} \).

The above methods utilizing the Planck and Rosseland mean absorption coefficients will not be adequate for the "strongly non-gray" radiation associated with atomic lines, molecular bands, or bound-free transitions. For strongly non-gray radiation the particular spectral dependence of the absorption coefficient must generally be taken into account. Here we shall consider a modification of the differential approximation for a "gray band" model, i.e., for an absorption coefficient which is constant between two fixed frequencies, say \( \nu_1 \) and \( \nu_2 \), with zero absorption outside of this frequency band. Ryhming\textsuperscript{22,223} has shown that the gray-band model may be applied to the bound-free (dissociation) continuum of molecules such as oxygen. Clarke\textsuperscript{28} has indicated that the bound-free (ionization) continuum for atoms may often be approximated by gray-band radiation extending from the ionization limit \( \nu_1 \) to \( \nu = \infty \). In addition, molecular bands consisting of well-overlapped lines may be approximated by the gray-band model (see Penner,\textsuperscript{24} Chapter 11).

The transfer equation for the spectral intensity \( I_\nu \) is the same as Eq. (6-2.3), except with \( I, B, \) and \( k \) replaced by \( I_\nu, B_\nu, \) and \( k_\nu \), respectively. For the gray-band model, all of the emission and absorption takes place between frequencies \( \nu_1 \) and \( \nu_2 \), where \( k_\nu \) takes on a constant value. Integration of the spectral transfer equation from \( \nu_1 \) to \( \nu_2 \) gives Eq. (6-2.3), but with \( B \) replaced by the integral of \( B_\nu \) from \( \nu_1 \) to \( \nu_2 \). The differential approximation therefore has the form of Eq. (6-2.5), but with \( 4\sigma T^4 \equiv 4\pi B \) replaced by \( b(T) \), where

\[
b(T) = 4\pi \int_{\nu_1}^{\nu_2} B_\nu \, d\nu. \tag{6-2.36}\]

We may also consider a gas with two gray bands having different absorption coefficients. By applying the differential approximation to the flux in each band, we obtain two equations. These equations may be combined to yield a single differential equation for the total flux;
however, this resulting equation is of fourth order and therefore much more difficult to utilize than the single-band differential approximation.

It should be noted that the modifications discussed in this section may be combined to treat more general problems. That is, we may substitute the function \( b(T) \) for \( 4 \sigma T^4 \) in Eq. (6-2.35), with \( \bar{k}_p \) and \( \bar{k}_R \) representing respectively the Planck and Rosseland mean values taken over the frequency band \( \nu_1 \) to \( \nu_2 \). In addition, the external radiation may be treated separately, as discussed above. Such a combination of modifications would be applicable to a gas having a slightly non-gray band, with the external radiation having arbitrary spectral and spatial distributions.

6-2F EXAMPLE PROBLEM: RADIATIVE TRANSFER IN A FINITE, PLANE-PARALLEL MEDIUM. Here we consider the one-dimensional problem of a stationary gas bounded by parallel, black walls which are separated by an optical depth \( \tau_0 \) and held at the temperatures \( T_1 \) and \( T_2 \). This situation is shown in Fig. 6-2.1, with \( F^-(0) = \sigma T_1^4 \) and \( -F^+(\tau_0) = \sigma T_2^4 \).

First, the flux and temperature distribution within the gas will be determined in the limit of radiative equilibrium by using the differential approximation. A more accurate expression for the flux will then be obtained by using the modified differential approximation, in which the external radiation is treated separately. In the last part of this section the transient radiant heating and cooling of the gas will be considered.

The steady-state or radiative equilibrium condition exists when \( \nabla \cdot \mathbf{F} = 0 \). This equation for radiative equilibrium may be derived from the transfer equation by equating the energy emitted by a gas element to the energy absorbed; the equation may also be obtained by taking the steady-state limit of the energy equation (2-3.16) for a stationary gas, with diffusion and heat conduction neglected. For our one-dimensional problem, the radiative equilibrium condition gives \( F(\tau) = F = \text{constant} \). Therefore, the first term in the differential approximation (6-2.6) vanishes, and after integration,

\[
4 \sigma T^4 = -3F \tau + C,
\]

where \( C \) is the constant of integration. For constant \( F \) the boundary conditions (6-2.9) give the following relations, after using Eq. (6-2.37) to evaluate \( 4 \sigma T^4 \) at \( \tau = 0 \) and \( \tau_0 \):

\[
2F + C - 4 \sigma T_1^4 = 0
\]

\[
-(2 + 3 \tau_0)F + C - 4 \sigma T_2^4 = 0.
\]
Addition and subtraction of Eqs. (6-2.38) yield expressions for \( F \) and \( C \), which may be substituted into Eq. (6-2.37) for the temperature distribution. The results are

\[
F = -\frac{\sigma(T_2^4 - T_1^4)}{1 + \frac{3}{4} \tau_0}
\]  

(6-2.39)

and

\[
\left( \frac{T^4 - T_1^4}{T_2^4 - T_1^4} \right) = \left( \frac{2 + 3\tau}{4 + 3\tau_0} \right).
\]  

(6-2.40)

It should be noted that the optical depth \( \tau \) is the natural independent variable for radiative transfer problems. The geometric depth \( x \) may be determined as a function of \( \tau \) by inverting the defining relation for \( \tau \) and integrating, viz.,

\[
x = \int_0^\tau \frac{d\tau'}{k(\tau')}
\]  

(6-2.41)

where the gray absorption coefficient \( k(T) \) is given as a function of optical depth by using Eq. (6-2.40) for \( T(\tau) \).

It is seen from Eq. (6-2.40) that at the wall there will exist temperature discontinuities, which become vanishingly small in the limit \( \tau_0 \to \infty \). In the limit \( \tau_0 \to 0 \), the temperature becomes constant throughout the gas. These optically thick and thin limits will now be investigated.

For the optically thick case, integration of the Rosseland approximation (6-2.2) for constant \( F \) yields Eq. (6-2.37). With the boundary conditions \( T(0) = T_1 \) and \( T(\tau_0) = T_2 \), Eq. (6-2.37) gives

\[
F = -4\sigma(T_2^4 - T_1^4)/3\tau_0
\]

and

\[
(T^4 - T_1^4)/(T_2^4 - T_1^4) = \tau/\tau_0,
\]

which become equivalent to Eqs. (6-2.39) and (6-2.40), respectively, in the limit of large \( \tau_0 \). As mentioned in Section 2-5C, the Rosseland approximation may be used with “slip” boundary conditions to give Eqs. (6-2.39) and (6-2.40) exactly.\(^{25,26}\) These slip or jump boundary conditions are:

\[
T^4(0) = T_1^4 + K(dT^4/d\tau)
\]

and

\[
T^4(\tau_0) = T_2^4 - K(dT^4/d\tau),
\]

where the constant \( K \) is set equal to \( \frac{3}{2} \) in order to give the correct flux when there is no gas between the walls \((\tau_0 = 0)\). We note that for \( F \) constant the slip boundary conditions are the same as the boundary conditions obtained for the differential approximation by using Eq. (6-2.6) to substitute for \( F \) in Eqs. (6-2.9). Thus the radiation slip method constitutes solving the Rosseland equation using the boundary conditions derived by the more accurate differential approximation; the formalism is the same as that of solving the ordinary heat conduction equation with “radiation” boundary conditions, as given in Carslaw and Jaeger.\(^{27}\) Although the radiation slip and differential approximation methods yield identical results in
this radiative equilibrium limit, the radiation slip method will be less accurate in more complicated problems, such as the transient problem discussed below.

For the optically thin limit, equating the emission $\int \sigma(T^4_k) dx$ from a slab of thickness $dx$ to the absorption $2\sigma(T^4_k + T^4_g) k dx$ gives $T^4 = \frac{1}{2}(T^4_k + T^4_g)$, which agrees with Eq. (6-2.40) in the limit $\tau_0 \to 0$. This constant gas temperature may be used in the integral expression (6-2.7) for the flux, which becomes for $\tau_0 \to 0$,

$$ F = -\sigma(T^4_k - T^4_g)(1 - \tau_0). $$

(6-2.42)

Now Eq. (6-2.39) is seen to give the correct limit at $\tau_0 = 0$ (no intervening gas), but it does not correctly give the term of order $\tau_0$. The modified differential approximation discussed in the preceding section may be used to derive an expression for the flux which does reduce to the correct optically thin result (6-2.42) through order $\tau_0$.

In the modified differential approximation method the flux $F_{\text{ext}}$ contributed by the wall radiation is first determined from the last two terms of Eq. (6-2.7),

$$ F_{\text{ext}} = 2\sigma T^4_k E_3(\tau) - 2\sigma T^4_g E_3(\tau_0 - \tau). $$

(6-2.43)

The differential approximation is used for the flux $F_g$ contributed by the gas radiation,

$$ \frac{d^2F_g}{d\tau^2} = 4\sigma \frac{dT^4}{d\tau} + 3F_g $$

(6-2.44)

with the boundary conditions

$$ \frac{dF_g}{d\tau} = 2F_g + 4\sigma T^4 \quad \text{at} \quad \tau = 0 $$

$$ \frac{dF_g}{d\tau} = -2F_g + 4\sigma T^4 \quad \text{at} \quad \tau = \tau_0. $$

(6-2.45)

Now substituting $F_g = F - F_{\text{ext}}$, where the total flux $F$ is constant, Eq. (6-2.44) is integrated to yield an expression for $4\sigma T^4$. Upon application of the boundary conditions (6-2.45) the following result is obtained:

$$ F = -\frac{\sigma(T^4_k - T^4_g)}{(1 + \frac{2}{5}\tau_0)} [1 + E_3(\tau_0) - \frac{5}{4}E_4(\tau_0)], $$

(6-2.46)

which reduces to Eq. (6-2.42) as $\tau_0 \to 0$. Although Eq. (6-2.46) is an improved expression for the flux, the previous result (6-2.39) was already accurate to better than 5%. On the other hand, for two- and
three-dimensional problems having external radiation which varies with position along the boundary, the modified procedure is often required since the pure differential approximation may lead to completely erroneous results (see Olfe19).

In the remainder of this section we shall consider the transient radiant heating and/or cooling of the gas. The gas is assumed to be at a constant temperature $T_0$ at time $t = 0$, with the boundary walls at $x = 0$ and $x_0$ being held at temperatures $T_1$ and $T_2$, respectively, for all times $t > 0$. The gas temperature and flux are to be found as functions of $x$ and $t$, from $t = 0$ to $t = \infty$, when the steady-state or radiative equilibrium solution determined above will be established. By setting $\tau = y/u_0$, the results of this one-dimensional, time-dependent problem will apply to the steady, two-dimensional problem of constant velocity ($u_0$) flow in a channel of width $x_0$, provided the transverse radiative transfer $\partial F_x/\partial x$ predominates over transfer $\partial F_y/\partial y$ in the flow direction.

Neglecting diffusion and heat conduction, the one-dimensional, time-dependent energy equation for a constant pressure process becomes

$$\rho c_p \frac{\partial T}{\partial t} = - \frac{\partial F}{\partial x}, \quad (6-2.47)$$

where $c_p$ is the specific heat at constant pressure. The density $\rho$ may be related to $T$ by an equation of state at constant pressure. In this problem it is found to be more convenient to use $I_0$ as the dependent variable, rather than $T$ or $F$. Therefore, the differential approximation given by Eqs. (6-2.20) and (6-2.21) is used to substitute for $T$ and $F$ in the above energy equation, yielding the following differential equation for $I_0$:

$$\frac{\partial^3 I_0}{\partial t \partial \tau^2} + \left( \frac{16\sigma T^3 k}{\rho c_p} \right) \frac{\partial^2 I_0}{\partial \tau^2} - 3 \frac{\partial I_0}{\partial \tau} = 0. \quad (6-2.48)$$

where $k$ has been assumed to be constant, with $\tau = kx$ and $\tau_0 = kx_0$.

Vetlutskii and Onufriev29 have considered the problem of radiative cooling ($T_1 = T_2 = 0$), with linearization being achieved by assuming $\rho c_p$ proportional to $T^3$ in addition to $k$ constant. Here we linearize in the same manner, defining $\eta = (16\sigma T^3/\rho c_p) t$. The linearized equation for $I_0$ is

$$\frac{\partial^3 I_0}{\partial \eta \partial \tau^2} + \frac{\partial^2 I_0}{\partial \tau^2} - 3 \frac{\partial I_0}{\partial \eta} = 0. \quad (6-2.49)$$

The differential approximation given by Eqs. (6-2.20) and (6-2.21) may
be used to put the boundary conditions (6-2.9) and the initial condition 
$T(\tau, 0) = T_0$ into the following forms:

$$\frac{2}{3} \frac{\partial I_0}{\partial \tau} - I_0 + 4\sigma T_1^4 = 0 \quad \text{at} \quad \tau = 0 \quad (6-2.50)$$

$$\frac{2}{3} \frac{\partial I_0}{\partial \tau} + I_0 - 4\sigma T_2^4 = 0 \quad \text{at} \quad \tau = \tau_0 \quad (6-2.51)$$

$$\frac{1}{3} \frac{\partial^2 I_0}{\partial \tau^2} - I_0 + 4\sigma T_0^4 = 0 \quad \text{at} \quad \eta = 0. \quad (6-2.52)$$

The differential equation (6-2.49), with the above conditions, may be solved in a straightforward manner by the Laplace transform method, or more simply by constructing a solution out of elementary solutions which decay exponentially with $\eta$ and vary sinusoidally with $\tau$. This second method of solution was used by Milne\textsuperscript{30} to solve the similar problem of time-dependent radiative transfer within a gas composed of two-level atoms with transitions between the levels occurring by radiative processes only.

The total solution for $I_0$ will involve an infinite sum of transient, elementary solutions plus the steady-state (radiative equilibrium) solution considered above. Assuming elementary solutions of the form $e^{\lambda_n}(A_n \sin c_n \tau + B_n \cos c_n \tau)$, substitution into Eq. (6-2.49) shows that $\lambda_n = -c_n^2/(c_n^2 + 3)$. The $4\sigma T_1^4$ and $4\sigma T_2^4$ terms in the boundary conditions (6-2.50) and (6-2.51), respectively, are cancelled by the steady-state solution. For the transient part of the solution, Eq. (6-2.50) gives $B_n = \frac{2}{3} c_n A_n$, and Eq. (6-2.51) yields the following equation for the eigenvalues $c_n$:

$$\cot c_n \tau_0 = \frac{1}{2} \left( \frac{2}{3} c_n - \frac{1}{3c_n} \right). \quad (6-2.53)$$

Using this equation, it can be shown that the elementary solutions are orthogonal over the integration interval $\tau = 0$ to $\tau_0$. The coefficients $A_n$ may now be determined by applying the initial condition (6-2.52) for the total solution, and using the orthogonality of the elementary solutions. The solution for $I_0$ is thus determined with $F$ and $T$ being calculated from Eqs. (6-2.21) and (6-2.20). The following expression is obtained for the flux:

$$F(\tau, \eta) = -\frac{\sigma(T_2^4 - T_1^4)}{(1 + \frac{2}{3} \tau_0)} + 2\sigma \sum_{n=1}^{\infty} [(T_0^4 - T_1^4) - (-1)^n (T_0^4 - T_2^4)]$$

$$\times \exp \left( -\frac{c_n^2}{c_n^2 + 3} \eta \right) \frac{\frac{2}{3} c_n \sin c_n \tau - \cos c_n \tau}{(1 + \frac{2}{3} c_n^2) [1 + (\frac{2}{3} + \frac{1}{3} c_n^2) \tau_0]}. \quad (6-2.54)$$
Let us now consider the accuracy of Eq. (6-2.54). At large time, \( \eta \gg 1 \), we recover the radiative equilibrium solution, which is accurate to better than 5\%, as mentioned above. If the more accurate expression given in Eq. (6-2.46) is used for the steady-state term in Eq. (6-2.54), then the correct optically thin limit will be obtained, which reduces to Eq. (6-2.42) as \( t \to \infty \), and to the following expression at \( t = 0 \):

\[
F(r, 0) = -\sigma T_\lambda^4[1 - 2(\tau_0 - \tau)] + \sigma T_\lambda^4(1 - 2\tau) + 2\sigma T_\lambda^4(2\tau - \tau_0).
\]

In the optically thick limit, Eq. (6-2.54) gives the following value for the flux at \( t = 0 \) and \( \tau = 0 \):

\[
F(0, 0) = -4\sigma(T_\lambda^4 - T_0^4)/(2 + \sqrt{3}) \approx -1.07\sigma(T_\lambda^4 - T_0^4),
\]

which is in error by 7\%. It should be noted that the computed temperature distribution will be exact at \( t = 0 \), since the initial condition \( T(r > 0) = T_0 \) was used in the solution of the problem.

If the Rosseland expression (6-2.2), together with the boundary conditions \( T(0) = T_\lambda \) and \( T(r_0) = T_2 \), is used to obtain a solution in the optically thick case, the computed flux becomes accurate at large times, but it takes on the value infinity at \( t = \tau = 0 \). Now if the "radiation" boundary conditions (6-2.50) and (6-2.51) are used with the Rosseland formula, the computed flux becomes more accurate, but at \( t = \tau = 0 \) it is too large by a factor of 2 (the solution in this case has the same form as the conduction solution with radiation boundary conditions given in Carslaw and Jaeger\(^{27}\)). We therefore conclude that the full differential approximation is required in order to obtain relatively accurate flux values near \( t = \tau = 0 \).

Numerical methods of carrying out calculations for this transient problem are described by Vetlutskii and Onufriev\(^{29}\) and Einstein.\(^{31}\)

### 6.3 The propagation of acoustic waves in a radiating gas

Early work on the effect of radiation on the propagation of acoustic waves was carried out by Stokes,\(^{32,33}\) more than a century ago. Stokes showed that radiation does not affect the propagation of sound waves under ordinary conditions. To compute the radiative transfer, he assumed a transparent gas surrounded by an infinite reservoir at the temperature of the undisturbed gas. Considerable advancement beyond Stokes' work has occurred only during the past decade, with investigations being carried out both in the Soviet Union\(^{34,35}\) and the United States.\(^{12,22,36-38}\)

The equations of motion for unsteady flow including the radiant-energy flux \( F \) are

\[
\frac{Dp}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0,
\]

(6-3.1)
where \( \rho \) is the density, \( u_i \) the velocity component in the \( x_i \) direction, \( p \) the pressure, \( h \) the enthalpy per unit mass, and \( D(\ )/Dt \) the substantial derivative: \( D(\ )/Dt \equiv \partial(\ )/\partial t + u_j \partial(\ )/\partial x_j \) (summation is indicated by the subscript \( j \) appearing twice in a term). In the above equations the radiation pressure and energy-density terms have been neglected, as these terms are completely negligible under most conditions for the problems considered in this chapter (cf. Section 2-4). The acoustic approximation consists of treating the variations in flow quantities as perturbations on the rest values; i.e., \( p = p_0 + p', \rho = \rho_0 + \rho' \), etc., where \( p_0 \) is the constant rest value of the pressure and \( p' \) is the perturbation. Thus for one-dimensional problems the above equations reduce to the following linearized equations \( (u_0 = 0) \):

\[
\frac{\partial p'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \tag{6-3.4}
\]

\[
\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{6-3.5}
\]

and

\[
\rho_0 \frac{\partial h'}{\partial t} - \frac{\partial p'}{\partial x} + \frac{\partial F'}{\partial x} = 0. \tag{6-3.6}
\]

Specializing to the case of a perfect gas,

\[
h' = \frac{\gamma R T'}{\gamma - 1} = \frac{\gamma}{\gamma - 1} \left( \frac{1}{\rho_0} p' - \frac{p_0}{\rho_0} \rho' \right), \tag{6-3.7}
\]

where \( R \) is the gas constant.

Most authors find it convenient to introduce the potential function \( \varphi \) defined by

\[
u = u' = \frac{\partial \varphi}{\partial x}, \quad p' = \rho_0 \frac{\partial \varphi}{\partial t}. \tag{6-3.8}
\]

With this definition for \( \varphi \), Eq. (6-3.5) is automatically satisfied, whereas Eqs. (6-3.4), (6-3.6), and (6-3.7) may be combined to yield

\[
\frac{\partial^2 \varphi}{\partial t^2} - a_0^2 \frac{\partial^2 \varphi}{\partial x^2} = \frac{(\gamma - 1)}{\rho_0} \frac{\partial F'}{\partial x}, \tag{6-3.9}
\]
where \( a_0 = (\gamma R T_0)^{1/2} \) is the isentropic speed of sound. The temperature derivative \( \partial T' / \partial t \) may be related to the potential function by means of Eqs. (6-3.4), (6-3.7), and (6-3.8):

\[
-R \frac{\partial T'}{\partial t} = \frac{\partial^2 \varphi}{\partial t^2} - \frac{a_0^2}{\gamma} \frac{\partial^2 \varphi}{\partial x^2},
\]

(6-3.10)

where \( a_0 / \gamma^{1/2} \) is the isothermal speed of sound. It is convenient to introduce the following notation:

\[
W_s \equiv \frac{\partial^2 \varphi}{\partial t^2} - a_0^2 \frac{\partial^2 \varphi}{\partial x^2},
\]

(6-3.11)

and

\[
W_T \equiv \frac{\partial^2 \varphi}{\partial t^2} - \frac{a_0^2}{\gamma} \frac{\partial^2 \varphi}{\partial x^2}.
\]

(6-3.12)

In order to obtain an equation in terms of the single dependent variable \( \varphi \), the equation of radiative transfer must be linearized and expressed in terms of \( \varphi \). To accomplish this, Eq. (6-2.7) is first linearized, then differentiated by the operator \( \partial^2 / \partial t \partial x \), with the \( \partial T' / \partial t \partial x \) terms eliminated by the use of Eq. (6-3.10), and the result is substituted into the time derivative of Eq. (6-3.9) to yield

\[
\frac{\partial W_s}{\partial t} = -8 \frac{(\gamma - 1) k_0 T_0^3}{\rho \omega R} \left[ \int_{-\infty}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2(k_0(x - x')) \, dx' \right]
- \int_{-\infty}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2\{k_0(x' - x)\} \, dx'
\]

(6-3.13)

for an infinite gas with no boundaries, and \(^\infty\)

\[
\frac{\partial W_s}{\partial t} = -8 \frac{(\gamma - 1) k_0 T_0^3}{\rho \omega R} \left[ \int_{0}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2(k_0(x - x')) \, dx' \right]
- \int_{0}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2\{k_0(x' - x)\} \, dx'
- 2r_d \int_{0}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2\{k_0(x' + x)\} \, dx'
- r_s \int_{0}^{\infty} \left( \frac{\partial W_T}{\partial x'} \right) E_2\{k_0(x' + x)\} \, dx'
\]

(6-3.14)

for a semiinfinite gas with a wall at \( x = 0 \) which has a temperature variation \( dT'_w / dt \), an emissivity \( \epsilon \), and reflectivities \( r_d \) and \( r_s \) for diffuse
and specular reflection, respectively. Application of the substitute kernel method on Eq. (6-3.13) or Eq. (6-3.14) yields the following result, which may also be obtained directly by differentiating the linearized form of Eq. (6-2.23) with the operator $\partial^2/\partial t \partial x$ and using Eqs. (6-3.9) and (6-3.10) to eliminate the $\partial F'/\partial x$ and $\partial T'/\partial t$ terms, respectively:

$$\frac{\partial^3 W_s}{\partial x^3 \partial t} + \left( 16m(\gamma - 1) \frac{k_0T_0^3}{\rho_0R} \right) \frac{\partial^2 W_T}{\partial x^2} - n^2k_0 \frac{\partial W_s}{\partial t} = 0,$$  \quad (6-3.15)

with $W_s$ and $W_T$ given by Eqs. (6-3.11) and (6-3.12), respectively; the constants $m$ and $n$ occur in the substitute kernel, $E_2(\tau) \approx m e^{-n\tau}$. Equation (6-3.15) represents a single linear differential equation for the potential function $\phi$.

In Section 6-3A the above equations are used to study monochromatic (periodic) waves, whereas in Section 6-3B the propagation of a wave produced by an impulsively moving wall is studied.

6-3A MONOCHROMATIC WAVES. Periodic solutions are obtained by substituting $\phi \propto \text{Re}[\exp(i\omega t + c\omega x/a_0)]$ into Eq. (6-3.15) and solving for the complex constant $c = - (\delta + i\lambda)$, with $-\delta$ and $-i\lambda$ being the real and imaginary parts of $c$. The wave speed is therefore $v = a_0/\lambda$, and the wave is damped accordingly as $\exp(-\delta \omega x/a_0)$. Substitution of the trial solution into Eq. (6-3.15) results in a fourth-order algebraic equation for $c$, which has the solutions

$$\frac{c_1}{c_2} = - \left[ \frac{-(1 - \beta^2 - iK\beta) + [(1 - \beta^2 - iK\beta)^2 + 4\beta^2(1 - iK\beta/\gamma)]^{1/2}}{2(1 - iK\beta/\gamma)} \right],$$  \quad (6-3.16)

where the upper sign goes with $c_1$ and the lower with $c_2$. Following the notation of Vincenti and Baldwin$^{12}$

$$\beta \equiv \frac{n k_0 a_0}{\omega}, \quad K \equiv \frac{16(\gamma - 1) \sigma T_0^3 m}{\rho_0 R a_0}. $$  \quad (6-3.17)

The quantity $\beta$ is equal to $n/2\pi$ times the ratio of the wavelength $2\pi a_0/\omega$ of the sound wave to the photon mean free path $k_0^{-1}$. The quantity $K$ is equal to $16\gamma m/n$ times the reciprocal of the Boltzmann number $Bo^{-1}$, defined in Section 6-1 as the ratio of the blackbody radiant-energy flux $\sigma T_0^4$ to the convected enthalpy flux $\rho_0 a_0^2 RT_0/(\gamma - 1)$. In Eq. (6-3.16) two roots equal to $-c_1$ and $-c_2$, respectively, have been dropped without loss of generality since they represent leftward-moving waves.
identical to the rightward-moving waves \( c_1 \) and \( c_2 \). For the case of a semiinfinite gas to the right (positive \( x \)-direction) of an oscillating wall at \( x = 0 \), as considered in Vincenti and Baldwin\(^{12} \) only the solutions \( c_1 \) and \( c_2 \) are present.

As shown in Fig. 6-3.1 for \( K = 4 \), the \( c_1 \) root corresponds to a modified-classical wave since the wave travels at the isentropic speed of sound \( \nu_a / a_0 \equiv \lambda^{-1} = 1 \), unless the wavelength is near the magnitude of the photon mean free path \( \omega / nk_o a_0 \equiv \beta^{-1} \) near unity, in which case the wave speed approaches the isothermal speed of sound. The behavior of the modified-classical wave can be explained by the fact that for \( \beta^{-1} \equiv \omega / nk_o a_0 \ll 1 \) radiation becomes absorbed by the time it travels to a small fraction of a wavelength from the point where it was emitted; thus the radiant energy is not lost from the wave and the wave accordingly travels at the isentropic speed. For \( \beta^{-1} \equiv \omega / nk_o a_0 \gg 1 \) the product \( K \beta \equiv 16 \gamma k_o \sigma T_0^4 / \rho_0 c_p T_o \omega \) becomes much less than unity, which means that radiated energy in this optically thin case is much less than the convected energy, so the wave speed again equals the isentropic value. Thus only for \( \beta^{-1} \) near unity can the radiative transfer tend to produce an isothermal situation. As one would expect, for larger values of \( K \) (greater radiant-energy transfer) the speed of the modified-classical wave more nearly approaches the isothermal speed over a larger frequency region; whereas for smaller values of \( K \) the wave speed deviates less from the isentropic value. These results are illustrated by

![Fig. 6-3.1. Wave speeds versus the nondimensional frequency of oscillation \( \omega / nk_o a_0 \equiv \beta^{-1} \), from Baldwin.\(^{26} \)](image-url)
the calculations Vincenti and Baldwin\textsuperscript{12} shown in Fig. 6-3.2. The distances $X$ on the horizontal scales in Fig. 6-3.2 have been plotted according to the formula $X = 2\beta/(1 + \beta)$ for $\epsilon < \beta < \epsilon^{-1}$ with $\epsilon \ll 1$, whereas for $\beta < \epsilon$ and $\beta > \epsilon^{-1}$ the horizontal scales have been expanded arbitrarily to provide adequate visualization of the results.

![Graph](image-url)

**Fig. 6-3.2.** Damping and wave-speed parameters versus the nondimensional period of oscillation $\beta$, from Vincenti and Baldwin.\textsuperscript{12}

The $c_2$ root given in Eq. (6-3.16) corresponds to a radiation-induced wave which has no classical counterpart. As seen from Figs. 6-3.1 and 6-3.2, the speed of this wave varies between zero and the velocity of light (taken to be infinite in this problem). The radiation-induced wave reaches the velocity of light at a certain critical frequency, above which the wave does not exist. Curves are given in Fig. 6-3.2 for the damping
parameters $\delta_1$ and $\delta_2$ of the modified-classical and radiation-induced waves, respectively.

Ryhming\textsuperscript{22,22a} has studied the problem of a gas which is dissociating, as well as radiating. He considers properties appropriate for oxygen, and shows that the ultraviolet continuum can be approximated by a constant absorption coefficient acting over a known frequency region (see Section 6-2E). This gray-band radiation defines a value of $K$.

The results of calculations for the modified-classical wave at small values of $K$ are shown in Fig. 6-3.3. The small values of $(\lambda_1 - 1)$

![Fig. 6-3.3. Damping and wave-speed parameters versus angular frequency of the wave, computed for dissociating oxygen, from Ryhming.\textsuperscript{22a}](image)

indicate small departures from the isentropic speed of sound. The damping $\delta_1$ produced by the dissociation process is seen to peak at ultrasonic frequencies and decrease at the higher temperatures. The damping produced by radiation, on the other hand, peaks at low audible frequencies and increases with increasing temperature; i.e., the radiation damping is proportional to $K$, as also illustrated in the bottom right-hand diagram of Fig. 6-3.2. Prokof'ev\textsuperscript{34} has carried out calculations which show that in moist air at standard conditions the damping produced by radiation will be important at low audible frequencies, and at subaudible frequencies radiation damping will dominate over damping produced by viscosity and heat conduction.
In their analysis, Vincenti and Baldwin\textsuperscript{12} consider the generation of acoustic waves by an oscillating wall which has a periodic temperature variation of the same frequency as the periodic wall motion. In addition to solving for the two wave speeds and damping parameters, the relative importance of the two waves can be determined as a function of the boundary conditions. To determine this effect of the boundary conditions, the potential \( \varphi \) is expressed as a linear combination of the solutions for the two waves,

\[
\varphi(x, t) = \frac{a_0^2}{\gamma \omega} \text{Re} \left\{ \left[ C_1 \exp \left( \frac{c_1 \omega x}{a_0} \right) + C_2 \exp \left( \frac{c_2 \omega x}{a_0} \right) \right] e^{i \omega t} \right\}, \tag{6-3.18}
\]

where the constants \( C_1 \) and \( C_2 \) determine the strengths of the modified-classical and radiation-induced waves, respectively. Substitution of this expression for \( \varphi \) into Eq. (6-3.14) [for a black wall and \( E_2(\tau) \sim m e^{-\eta} \)] yields the expressions (6-3.16) for \( c_1 \) and \( c_2 \), where it is recognized that the leftward-moving waves \( -c_1 \) and \( -c_2 \) will not exist in this case of a semiinfinite gas to the right of an oscillating wall. In addition the following relation between \( C_1 \), \( C_2 \), and the wall temperature variation \( T_w(t) \) must hold for Eq. (6-3.14) to be satisfied:

\[
\text{Re} \left\{ -i \left[ \left( 1 + \frac{c_1^2}{\gamma} \right) \left( \frac{\beta}{\beta + c_1} \right) C_1 + \left( 1 + \frac{c_2^2}{\gamma} \right) \left( \frac{\beta}{\beta + c_2} \right) C_2 \right] e^{i \omega t} \right\} = \frac{T_w'(t)}{T_0}. \tag{6-3.19}
\]

The gas velocity \( \partial \varphi(0, t)/\partial x \) at the wall must follow the wall motion; therefore

\[
\text{Re}[\left( c_1 C_1 + c_2 C_2 \right) e^{i \omega t}] = \frac{\gamma u_w(t)}{a_0}. \tag{6-3.20}
\]

Since the wall temperature variation \( T_w'(t) \) and wall velocity \( u_w(t) \) are sinusoidal functions of time with angular frequency \( \omega \), Eqs. (6-3.19) and (6-3.20) can be solved to determine the values of \( C_1 \) and \( C_2 \) in terms of given amplitudes for the wall temperature and velocity variations. For nonlimiting values of \( K \), it is found that both kinds of waves will be produced by either pure wall motion (\( T_w' = 0 \)) or by pure wall temperature variation (\( u_w = 0 \)). For pure wall motion the modified-classical wave predominates at all conditions, but the radiation-induced wave is present to some extent. On the other hand, for pure wall temperature variation, the radiation-induced wave predominates at high values of \( K \), with the wave strengths becoming comparable at lower \( K \) values.
6. Radiation Gasdynamics

The two-dimensional, steady flow of a radiating gas over a wavy wall has been studied by Cheng\textsuperscript{14} (also see Olfe\textsuperscript{19}). In contrast to the situation for a nonradiating gas, the two-dimensional, steady equation for a radiating gas has a form which is different from that for one-dimensional unsteady flow given by Eq. (6-3.15).

6-3B The Acoustic Disturbance Produced by Impulsive Wall Motion. In this section we consider the effect of radiative transfer on the propagation of an acoustic disturbance produced by impulsive wall motion, i.e., by a wall which is instantaneously set in motion at a constant velocity. Our discussion will follow Baldwin\textsuperscript{36} who expresses the disturbance as a superposition of monochromatic waves of all frequencies, and uses Fourier transform theory to determine the relative contributions from the various frequencies. This problem has also been investigated by Lick\textsuperscript{37} who used the Laplace transform method to carry out solutions for large $K$, and by Moore\textsuperscript{38} who obtained solutions for the specific heat ratio $\gamma$ near unity.

The potential function is expressed as a superposition of the monochromatic potentials (6-3.18) by

$$\varphi(x, t) = \frac{a_0^2}{\gamma} \Re \left\{ \int_0^\infty \frac{1}{\omega} \left[ C_1 \exp \left( \frac{c_1 u w}{a_0} \right) + C_2 \exp \left( \frac{c_2 u w}{a_0} \right) \right] e^{i\omega t} d\omega \right\}, \quad (6-3.21)$$

where $c_1$ and $c_2$ are determined from Eq. (6-3.16). The frequency-dependent coefficients $C_1$ and $C_2$ are determined from the boundary conditions,

$$\Re \left\{ -i \int_0^\infty \left[ \left( 1 + \frac{c_1^2}{\gamma} \right) \beta \frac{C_1}{c_1} + \left( 1 + \frac{c_2^2}{\gamma} \right) \beta \frac{C_2}{c_2} \right] e^{i\omega t} d\omega \right\} = \frac{T_w(t)}{T_0} \quad (6-3.22)$$

and

$$\Re \left[ \int_0^\infty \left( c_1 C_1 + c_2 C_2 \right) e^{i\omega t} d\omega \right] = \frac{\gamma}{a_0} u_w(t). \quad (6-3.23)$$

The wall is assumed to move impulsively with the wall temperature held fixed; i.e., $u_w(t) = 0$ for $t < 0$, $u_w(t) = U$ for $t \geq 0$, and $T_w(t) = 0$ for all $t$. Baldwin also considers the case of a fixed wall with an impulsive temperature variation, for which the velocity disturbance is found to be small compared to that for the case of impulsive wall motion involving comparable gas temperature variations. For a real function $u_w(t)$ which is zero for $t < 0$, the Fourier transform $\tilde{u}_w(\omega)$ is given by

$$\tilde{u}_w(\omega) = \frac{1}{\pi} \int_0^\infty u_w(t) e^{-i\omega t} dt, \quad (6-3.24)$$
and the inverse relation may be written in the form

$$u_w(t) = \text{Re} \left[ \int_0^{\infty} \tilde{u}_w(\omega) e^{i\omega t} d\omega \right]. \quad (6-3.25)$$

Substitution of Eq. (6-3.25) into Eq. (6-3.23) gives the result

$$c_1C_1 + c_2C_2 = \frac{\gamma}{a_0} \tilde{u}_w(\omega), \quad (6-3.26)$$

and expressing $T'_w(\omega)$ is terms of its Fourier transform $T'_w(\omega)$ reduces Eq. (6-3.22) to

$$-i \left[ \left( 1 + \frac{c^2}{\gamma} \right) \left( \frac{\beta}{\beta + c_1} \right) C_1 + \left( 1 + \frac{c^2}{\gamma} \right) \left( 1 + \frac{\beta}{\beta + c_2} \right) C_2 \right] = \frac{T'_w(\omega)}{T_o}. \quad (6-3.27)$$

For the case of impulsive wall motion with fixed wall temperature, $\tilde{u}_w(\omega) = -iU/\pi\omega$ and $T'_w(\omega) = 0$. Equations (6-3.26) and (6-3.27) are easily solved for $C_1(\omega)$ and $C_2(\omega)$, which are substituted into Eq. (6-3.21) to give $\varphi(x, t)$. The flow variables can then be determined from $\varphi(x, t)$ by partial differentiation; see Eqs. (6-3.8) and (6-3.10). The complex integral (6-3.21) for $\varphi(x, t)$ cannot be evaluated exactly for all values of the parameters. Accordingly, Baldwin first considers limiting cases, then he determines the following approximate, closed-form solution for the gas velocity (which agrees well with numerical calculations):

$$\frac{u(\xi, \tau)}{U} \approx \frac{1}{2} \left[ 1 - \exp \left( - \frac{X}{b^2} \right) \right] \left[ \text{erf} \left( \frac{\tau - \xi}{2 \sqrt{X}} \right) + \text{erf} \left( \frac{\tau + \xi}{2 \sqrt{X}} \right) \right]$$

$$+ \frac{1}{2} \left[ \exp \left( - \frac{X}{b^2} \right) - \exp \left( - X \right) \right]$$

$$\times \left\{ \text{erf} \left[ b(\tau - \xi) - X \right] + \text{erf} \left[ b(\tau + \xi) + X \right] \right\}$$

$$+ \frac{1}{2} \exp(- X) \left[ 1 + \frac{\tau - \xi}{\tau - \xi} \right]. \quad (6-3.28)$$

The above quantities are defined by

$$\xi \equiv \left( \frac{2}{\gamma + 1} \right)^{1/2} n k_0 x, \quad \tau \equiv \left( \frac{2}{\gamma + 1} \right)^{1/2} n k_0 a_0 t,$$

$$b \equiv \frac{K'}{2} + \left[ \left( \frac{K'}{2} \right)^2 + 1 \right]^{1/2}, \quad X \equiv [(\gamma)^{1/2} - 1] K' \xi,$$
Equation (6-3.28) was used by Baldwin to calculate the gas velocity as a function of the dimensionless time $\tau$ and the dimensionless distance $\xi$ from the wall. Figure 6-3.4 shows curves of $u/U$ versus $\tau$ at various distances $\xi$, computed for $K' = 3$ and $\gamma = 1.4$; these values of $K'$ and $\gamma$ correspond to the value $K \approx 4$ used in Fig. 6-3.1. In order to provide good visualization of the various regions, the coordinate axes shown in Fig. 6-3.4 have been broken at several points, with different linear scales being given between the breaks. It is seen that the disturbance starts out as a discontinuity propagating at the isentropic speed of sound, given by the curve $\tau = \xi$. The jump in gas velocity at the discontinuity dies out exponentially with distance from the wall; i.e., radiative transfer rapidly smooths the wave front into a continuous variation in gas velocity. The center of the disturbance shifts toward the path of an isothermal signal, $\tau = \gamma^{1/2} \xi$, at intermediate distances, and then shifts back to the isentropic path at large distances from the wall. The center of the disturbance is taken to be the point where the gas velocity reaches half of its final value, and is indicated by a heavy vertical line in Fig. 6-3.4.

The above results can be easily explained on a physical basis. At small distances $\xi$ from the wall the wave front is so sharp that its width is small compared to a radiation mean free path, which is equal to approximately unity in units of $\xi$. Thus radiative transfer is negligible within this thin wave, and the disturbance travels at the isentropic speed of sound. At intermediate $\xi$ the width of the wave becomes of the order of a photon mean free path, so radiative transfer occurs within the wave, tending to make the wave isothermal. Thus the speed of the disturbance approaches the isothermal sound speed. At large $\xi$ the wave becomes so dispersed that reabsorption occurs near the region of emission, therefore little radiant energy is lost from the wave front, and the disturbance again travels at the isentropic sound speed. These results are consistent with the description of monochromatic waves given in the preceding section: at small $\xi$ the disturbance is primarily made up of high-frequency waves, traveling at the isentropic sound speed; at intermediate $\xi$ the most important contributions come from waves of intermediate frequencies, which travel at speeds near the isothermal sound speed; at large $\xi$ the low frequencies are predominant, so the disturbance travels at the isentropic sound speed.

At low values of $K'$ the radiative transfer is not great enough to change the disturbance speed appreciably from the isentropic value,
Fig. 6-3.4. The propagation and dispersion of a disturbance produced by impulsive wall motion; plotted for the intermediate radiation value $K' = 3$, from Baldwin.\textsuperscript{36}
FIG. 6-3.5. The propagation and dispersion of a disturbance produced by impulsive wall motion; plotted for weak radiation, $K' = 0.001$, from Baldwin.²⁶
Fig. 6-3.6. The propagation and dispersion of a disturbance produced by impulsive wall motion; plotted for strong radiation, $K' = 1000$, from Baldwin.\textsuperscript{36}
and the wave front does not become dispersed until $\xi$ becomes large, of the order of $1/K'$. These features are illustrated in Fig. 6-3.5, where the axes denote $K'\tau$ and $K'\xi$.

Figure 6-3.6 shows that for large $K'$ the disturbance will travel at the isothermal sound speed, except at very small and very large values of $\xi$. The dispersion of the wave front is illustrated for three ranges of distance from the wall.

6-4 The structure of shock waves in a radiating gas

There has been a considerable amount of work in recent years on the structure of radiating, one-dimensional shock waves. After a brief discussion of the problem, we present in Section 6-4A the simple, approximate treatment given by Raizer. Then, in Section 6-4B, the numerical solution of Heaslet and Baldwin for a gray gas is described. Finally, in Section 6-4C we briefly outline calculations for non-gray gases with nonequilibrium ionization.

Including the radiant-energy flux term $dF/dx$, the steady, one-dimensional equations of motion for a viscous, heat-conducting fluid are

$$\frac{d}{dx}(\rho u) = 0, \quad (6-4.1)$$

$$\rho u \frac{du}{dx} + \frac{dp}{dx} = \frac{d}{dx} \left( \mu \frac{du}{dx} \right), \quad (6-4.2)$$

and

$$\rho u \frac{dh}{dx} - u \frac{dp}{dx} + \frac{dF}{dx} = \mu \left( \frac{du}{dx} \right)^2 + \frac{d}{dx} \left( \lambda \frac{dT}{dx} \right). \quad (6-4.3)$$

In these equations $\mu$ represents the coefficient of viscosity and $\lambda$ is the thermal conductivity. Ionization effects are neglected here. Integration of the above equations yields

$$\rho u = \rho_1 u_1, \quad (6-4.4)$$

$$(\rho_1 u_1) u + p - \mu \frac{du}{dx} = \rho_1 u_1 c_1, \quad (6-4.5)$$

and

$$\rho_1 u_1 (h + \frac{1}{2}u^2) + F - \mu u \frac{du}{dx} - \lambda \frac{dT}{dx} = \rho_1 u_1 c_2, \quad (6-4.6)$$

where the constants of integration $c_1$ and $c_2$ are determined by the shock velocity and the state of the gas ahead of the shock wave ($\rho_1 \equiv \rho_\infty$ and $u_1 \equiv V_\infty$).
Let us first consider the results obtained from a formal substitution of the Rosseland diffusion approximation (6-2.2) into Eq. (6-4.6). The radiation and heat conduction terms accordingly have the same form, so the structure becomes the same as that for a radiationless shock wave which has an effective conductivity,

$$\lambda_{\text{eff}} = \lambda + \frac{16}{3} \phi T^3$$  \hspace{1cm} (6-4.7)

This increase in effective conductivity produced by the radiation term results in a spreading of the shock front on the upstream side. Since the photon mean free path for any real gas is much greater than the thickness of a viscous, heat-conducting (radiationless) shock front, this diffusion approximation analysis is applicable only for weak shock waves with radiant-energy transfer great enough to smooth out the shock front completely. Under these special conditions the ordinary heat conduction and viscosity may be neglected, and, therefore, the problem reduces to that of a conducting shock with the conductivity given by the radiation term in Eq. (6-4.7).

For most conditions radiation will not completely smooth out the shock front, so that the above diffusion analysis will not be applicable. Under these conditions of incomplete radiation smoothing there will be imbedded within the over-all profile a viscous, heat-conducting shock wave, which is sometimes referred to as a "shock within a shock." Here the over-all shock wave profile extends over the radiatively heated region ahead, as well as the region of radiative cooling behind the imbedded shock wave. The ordinary Rankine-Hugoniot relations for normal shock waves apply across the over-all profile, since, by definition, radiative transfer occurs only within this profile. Since the photon mean free path is much greater than the width of the imbedded shock (which is of the order of several molecular mean free paths), radiative heat transfer between the imbedded shock and the surrounding gas will be very small. Accordingly, the Rankine-Hugoniot relations apply across the imbedded shock to a high degree of accuracy. The radiation correction to the imbedded shock structure (and Rankine-Hugoniot relations) will be of the order of the ratio of molecular to photon mean free paths, with the absorption of radiation from the surrounding optically thick gas being as important as emission in this optically thin imbedded shock. The Rankine-Hugoniot relations can be modified to include the effects of changes in radiation pressure, radiation energy density, radiation flux, and magnetic fields; however, only the ordinary Rankine-Hugoniot relations are required for the problems considered in this monograph.
Since the photon mean free path is much larger than the width of the imbedded shock, the imbedded shock may be replaced by a discontinuity. The resulting over-all profile is often called a "radiation-resistant" shock wave, as all the structure will be produced by the radiation. The radiation-resistant shock wave was first studied in the Soviet Union by Prokof'ev, who obtained incorrect results because he neglected the possibility of a discontinuity; i.e., he assumed that the discontinuity (imbedded shock front) is always smoothed out by the radiation. Zel'dovich showed that a discontinuity usually occurs, and outlined the correct manner of calculating the shock structure. Approximate calculations were then carried out by Raizer. In the West, Clarke independently obtained the same results as Prokof'ev. Heaslet and Baldwin corrected Clarke's work by including the discontinuity when it occurs, and they computed numerically the resulting radiation-resistant shock structure. In Sections 6-4B and 6-4C we mention recent studies, which include non-gray radiation, real-gas effects, and nonequilibrium ionization.

6-4A AN APPROXIMATE, ANALYTICAL TREATMENT FOR STRONG SHOCK WAVES. In this section we follow the treatment of Raizer for determining the temperature and radiant-energy flux distributions through a strong, radiation-resistant shock wave. Consideration of a strong shock means that the internal energy of the gas ahead of the shock may be neglected, and consideration of a radiation-resistant shock means that viscosity and heat conduction may be neglected. Accordingly, Eqs. (6-4.4) through (6-4.6) reduce to

\[\rho u = \rho_1 u_1, \quad (6-4.8)\]

\[p + \rho u^2 = \rho_1 u_1^2, \quad (6-4.9)\]

and

\[\rho u(h + \frac{1}{2}u^2) + F = \frac{1}{2}\rho_1 u_1^3. \quad (6-4.10)\]

For a perfect gas the enthalpy is given by

\[h = \gamma RT/(\gamma - 1), \quad (6-4.11)\]

and the perfect-gas equation of state provides the following relation between \(p\), \(\rho\), and \(T\):

\[p = \rho RT. \quad (6-4.12)\]

The above equations can be combined to yield the following expressions for the temperature and the flux at a point in terms
of the compression $\eta \equiv \rho_1/\rho$ at the point and the compression $\eta_2 \equiv \rho_1/\rho_2 = (\gamma - 1)/(\gamma + 1)$ across the over-all shock wave:

$$T = T_2\eta/(1 - \eta)/\eta_2(1 - \eta_2), \quad (6-4.13)$$

and

$$-F = \rho_1 u_1 RT_2(1 - \eta) (\eta - \eta_2)/2\eta_2^2(1 - \eta_2), \quad (6-4.14)$$

where $T_2$ is the equilibrium temperature behind the shock, which is the same as the temperature behind a radiationless shock of the same over-all strength. The above quadratic equations each give two roots for $\eta$, which correspond to values in the regions ahead of and behind the discontinuity, respectively. In the region ahead of the discontinuity, elimination of $\eta$ between Eqs. (6-4.13) and (6-4.14) yields the following relation between $F$ and $T$:

$$-F \approx \rho_1 u_1 RT/(\gamma - 1) = \rho_1 u_1 e, \quad (6-4.15)$$

which is good through order $\eta_2$; e.g., for $\gamma = 1.25$ the error made in omitting terms of order $\eta_2^2$ is less than 2%. Equation (6-4.15) shows that the radiant energy absorbed ahead of the discontinuity goes only into raising the internal energy $e$ of the gas; i.e., the work of compression and the change of kinetic energy cancel each other through order $\eta_2$.

Raizer first considers the case of weak radiation, for which the gas ahead of the discontinuity does not become heated by so much that reradiation needs to be included. The gas will accordingly be heated to only about two photon mean free paths ahead of the discontinuity, and radiative cooling will occur for about two photon mean free paths behind the discontinuity, as illustrated in Fig. 6-4.1. Since the radiation flux directed upstream at the discontinuity is approximately $\sigma T_2^4$, the

![Fig. 6-4.1. Temperature profile for a strong shock wave with weak radiation.](image)
absorption of this flux by a gray gas leads to the following variation of flux ahead of the discontinuity [cf. the last term in Eq. (6-2.43)]:

\[-F \approx 2\sigma T_2^4 E_2(|\tau|), \tag{6-4.16}\]

where the exponential function \( E_2(|\tau|) \) is given by Eq. (6-2.8), with \( \tau \) being the gray-gas optical depth measured from the discontinuity (\( \tau \) is taken as negative upstream of the discontinuity, as shown in Fig. 6-4.1). Substitution of Eq. (6-4.16) into Eq. (6-4.15) gives the following temperature distribution ahead of the discontinuity:

\[ T = T_-2E_2(|\tau|), \tag{6-4.17}\]

where the temperature \( T_- \) of the gas just ahead of the discontinuity is given by

\[ \frac{T_-}{T_2} = \frac{(\gamma - 1) \sigma T_2^4}{\rho_1 u_1 R T_2} = \gamma \left( \frac{\sigma T_2^4}{\rho_1 u_1 h_2} \right). \tag{6-4.18}\]

The ratio \( T_-/T_2 \) is thus equal to \( \gamma \) times the reciprocal of the Boltzmann number \( \text{Bo}^{-1} \equiv \sigma T_2^4 / \rho_1 u_1 h_2 \).

For the above weak radiation treatment to be valid, it is required that the emission \( 4\sigma T_-^4 \) by the gas just ahead of the discontinuity be appreciably less than the local intensity \( I_0 \); i.e., the local radiation energy density \( I_0/c \) is assumed to be contributed by the radiation from the shocked gas, with the local equilibrium radiation energy density \( 4\sigma T_-^4/c \) being small. Let us now define a temperature \( T_k \) as that value of \( T_- \) at which the radiation energy density approaches the local equilibrium value. Since \( I_0(\tau = 0) = -2F(\tau = 0) \) when \( F \) is contributed by the shocked gas, an approximate value of \( T_k \) may be computed from the following relation:

\[ 4\sigma T_k^4 = -2F(\tau = 0) = 2\rho_1 u_1 R T_k/(\gamma - 1) \]

or

\[ T_k = \left[ \frac{\rho_1 u_1 R}{2(\gamma - 1) \sigma} \right]^{1/3}, \tag{6-4.19}\]

which corresponds to a Boltzmann number \( \text{Bo} \equiv \rho_1 u_1 h_k / \sigma T_k^4 \) equal to \( 2\gamma \). For strong shocks propagating into air at standard conditions, \( T_k \) is of the order of 300,000°K; however, the value of \( T_k \) decreases at lower densities due to the \( \rho_1^{1/3} \) dependence given in Eq. (6-4.19). Raizer finds that the weak radiation treatment gives good results up to values of \( T_- \) which are very close to \( T_k \). It should be noted that when \( T_- \) approaches \( T_k \), then \( T_- \) also becomes close to the value \( T_2 \); e.g., Eqs. (6-4.18) and (6-4.19) give \( T_-/T_2 = (1/2)^{1/4} (T_-/T_k)^{3/4} \).
Considering now the region behind the discontinuity, elimination of $\eta$ between Eqs. (6-4.13) and (6-4.14) gives the following relation, which is accurate through order $\eta^2$:

$$-F = \rho_1 u_1 R(T - T_2)(\gamma + 1)/(3 - \gamma)(\gamma - 1). \quad (6-4.20)$$

The flux distribution behind the discontinuity is approximated by the flux in a radiating, isothermal gas, which is given by Eq. (6-4.16) if $\tau$ now denotes the (positive) optical depth behind the discontinuity [derived from Eq. (6-2.7) by setting $T = T_2$, $I^-(0) = 0$, $I^+(\tau_0) = 0$, and $\tau_0 = \infty$]. Accordingly, the following temperature distribution is determined from Eqs. (6-4.16) and (6-4.20):

$$T - T_2 = (T_+ - T_2)2E_3(\tau), \quad (6-4.21)$$

where

$$\frac{T_+ - T_2}{T_2} = \gamma \left( \frac{3 - \gamma}{\gamma + 1} \right) \left( \frac{\sigma T_2^4}{\rho_1 u_1 h_2} \right) = \left( \frac{3 - \gamma}{\gamma + 1} \right) \frac{T_+}{T_2}. \quad (6-4.22)$$

From the above equation we see that the temperature difference $T_+ - T_2$ behind the discontinuity is nearly equal to the temperature in front for $\gamma$ near unity.

Raizer also gives an approximate treatment for the strong radiation case ($T_- \approx T_2 > T_k$), which is illustrated in Fig. 6-4.2. For this strong

![Fig. 6-4.2. Temperature profile for a strong shock wave with strong radiation.](image)
radiation case the gas upstream of the discontinuity is divided into two regions: \(-\infty < \tau \leq \tau_k\) in which \(T \leq T_k\), and \(\tau_k < \tau \leq 0\) in which \(T > T_k\). In the region \(-\infty < \tau \leq \tau_k\) only absorption is important, so the solution takes on the form given above; i.e.,

\[
T = T_k \frac{2E_d(1-\tau_k)}{\tau - \tau_k} \quad \text{for} \quad -\infty < \tau \leq \tau_k.
\]

In the region \(\tau_k < \tau \leq 0\) reradiation is as important as absorption; i.e., the radiation energy density is nearly the equilibrium value at the local gas temperature. Therefore, it is appropriate to use the Rosseland diffusion approximation (6-2.2) together with Eq. (6-4.15) to determine the temperature in this region

\[
\tau = \frac{T_k}{1 + \frac{1}{2} (\tau - \tau_k)} \quad \text{for} \quad T_k < \tau < 0. \tag{6-4.25}
\]

The optical depth \(\tau_k\) is determined from Eq. (6-4.25) by the condition that \(T = T_2 \sim T_2\) at \(\tau = 0\); therefore

\[
\tau_k = -\frac{3}{2} \left[ \left( \frac{T_2}{T_k} \right)^3 - 1 \right]. \tag{6-4.26}
\]

As discussed above, when the gas just ahead of the discontinuity heats up to the temperature \(T_k\), then \(T_2\) is close to the value \(T_k\). As \(T_2\) is increased above \(T_k\), the temperature ahead of the discontinuity follows the temperature \(T_2\) very closely; see Zel'dovich and Raizer.

As the imbedded shock thickness is much less than a photon mean free path, the flux is continuous across the imbedded shock (represented as a discontinuity here). Therefore, the temperature \(T_+\) just behind the discontinuity can be determined by equating expressions (6-4.15) and (6-4.20),

\[
T_+ = 4T_2/(\gamma + 1). \tag{6-4.27}
\]

If the exact relations (6-4.13) and (6-4.14) are used instead of the approximate relations, the value obtained is \(T_+ = (3 - \gamma) T_2\), which differs from the above value by an amount of order \(\eta_2^2\), or by less than 2\% for \(\gamma = 1.25\). In this strong radiation case the temperature equilibrates from \(T_+\) to \(T_2\) in a fraction of a photon mean free path behind the discontinuity. Since this temperature peak is very thin optically, it will not contribute much to the local radiation energy.
density, and we may therefore approximate \( I_0 \approx 4\sigma T_k^4 \). Thus the integral of the transfer equation given in Eq. (6-2.20) may be written in the following form after using Eq. (6-4.20) and (6-4.19):

\[
\frac{dT}{d\tau} = -2 \left( \frac{3 - \gamma}{\gamma + 1} \right) \left( \frac{T_4 - T_2^4}{T_k^4} \right).
\] (6-4.28)

The above equation may be directly integrated, or more simply, dropping the initially small constant term on the right-hand side and integrating gives a decrease to temperature \( T_2 \) in the approximate distance \( \Delta \tau \), where

\[
\Delta \tau \approx \frac{(\gamma + 1)}{6(3 - \gamma)} \left( \frac{T_k}{T_2} \right)^3.
\] (6-4.29)

This expression shows that the thickness of the temperature peak decreases rapidly with increasing value of the ratio \( T_2/T_k \).

In order to discuss Raizer's treatment further, let us consider the differential approximation (6-2.6) in combination with the fluid dynamic relations (6-4.15) and (6-4.20):

\[
\frac{d^2T}{d\tau^2} + 8 \left( \frac{T}{T_k} \right)^3 \frac{dT}{d\tau} - 3T = 0 \quad \text{for} \quad \tau < 0 \quad (6-4.30)
\]

and

\[
\frac{d^2(T - T_2)}{d\tau^2} + 8 \left( \frac{3 - \gamma}{\gamma + 1} \right) \left( \frac{T}{T_k} \right)^3 \frac{d(T - T_2)}{d\tau} - 3(T - T_2) = 0 \quad \text{for} \quad \tau > 0.
\] (6-4.31)

For the weak radiation case, \((T/T_k)^3 \ll 1\) and the middle (emission) terms in Eqs. (6-4.30) and (6-4.31) may be neglected. The remaining terms, together with the boundary conditions, give solutions proportional to \( e^{\sqrt{3}r} \) for \( \tau < 0 \) and \( e^{-\sqrt{3}r} \) for \( \tau > 0 \). These solutions correspond to the exponential approximations for the \( E_3 \) functions occurring in the previous solutions, Eqs. (6-4.17) and (6-4.21). For strong radiation, the gas ahead of the discontinuity (\( \tau < 0 \)) becomes optically thick so that the first term in Eq. (6-4.30) may be neglected, yielding the Rosseland formula for \( T > T_k \) (for \( T < T_k \) the middle term may be neglected, so that the first and last terms again give the weak radiation solution at large distances ahead of the discontinuity). Now for the strong radiation case behind the discontinuity, where \( T > T_k \) everywhere, the gas in the radiative cooling region becomes transparent so that the last term in Eq. (6-4.31) may be neglected. Integration of the first two terms in Eq. (6-4.31) gives Eq. (6-4.28),
which represents an optically thin gas with blackbody external radiation at temperature $T_2$. Since Eqs. (6-4.30) and (6-4.31) are nonlinear, simple analytical solutions covering the entire radiation range (all $T_k$ values) cannot be obtained.

Kogure and Osaki\(^{49}\) have obtained an approximate solution by first assuming a crude form for the source function $B(\tau) = \sigma T^\gamma(\tau)/\pi$, which is substituted into the differential approximation [Eqs. (6-2.20) and (6-2.21)] to obtain expressions for $I_0$ and $F$. The expression for $F$ is then substituted into the fluid dynamic relations (6-4.15) and (6-4.20) to obtain the temperature distribution $T(\tau)$. Kogure and Osaki consider a finite total optical depth ahead of the discontinuity. They accordingly assume a quasi-steady situation since the changing total optical depth ahead of the advancing discontinuity means that an increasing amount of radiation is lost upstream to infinity, resulting in a decreasing shock wave velocity. Using the results of this quasi-steady analysis, Kogure\(^{50}\) carried out numerical calculations for the unsteady shock problem (also, see Olfe\(^{51}\) for a calculation of unsteady shock wave propagation in the limit of a radiation loss which is small enough to be treated as a perturbation).

Using the differential approximation for radiation from a gray band, we obtain differential equations similar to Eqs. (6-4.30) and (6-4.31), but with $T^3 dT/dr$ in the middle terms replaced by $(16\sigma)^{-1} db(T)/dr$ where $b(T)$ is defined by Eq. (6-2.36). In a recent paper, Olfe and Cavalleri\(^{51a}\) have compared Raizer-type solutions with solutions obtained using the full differential approximation for a single gray band. In addition, a gas radiating in two bands is analyzed. For a single gray band, the particular case of $b(T)$ proportional to $T$ results in linear equations, with the solutions being exponentials. Although the assumption $b(T)$ proportional to $T$ corresponds to the rather unrealistic situation of a band at low frequencies ($h\nu/kT \ll 1$), calculations for this linear case are instructive. For this linear case Fig. 6-4.3 shows that Raizer's approximate solution gives good results for the strong and weak radiation cases (large and small $\Gamma$ values) and adequate results for the case of an intermediate amount of radiation. The strength of the radiation is measured by $\Gamma \equiv b(T_2)/\rho_1u_1e(T_2)$; and the calculations are carried out for $\gamma = 1$, although formulas are given by Olfe and Cavalleri\(^{51a}\) for $\gamma$ unspecified. For a gray gas $\Gamma = 2(T_2/T_k)^2$, and numerical integration of the nonlinear differential equations yields the temperature profiles shown in Figs. 6-4.4 and 6-4.5. Again the Raizer solution is seen to give quite accurate profiles. Results are also given for the case of a gray band at large frequencies ($h\nu/kT \gg 1$). The analysis produces a suitable procedure for matching the weak and strong radiation solutions in the intermediate
radiation regime. [The equations given below Eqs. (21) and (22), and the middle equation of (22) in Olfe and Cavalleri\textsuperscript{51a} are incorrect as printed; however the correct forms will be evident to the reader, and the temperature profiles were calculated with the correct equations.]

The two-band analysis by Olfe and Cavalleri\textsuperscript{51a} involves gray bands with a constant ratio $\alpha$ between the absorption coefficients: $k_2 / k_1 \equiv \alpha \geq 1$, where the subscripts denote values for the two bands, with band 2 chosen as the more strongly absorbing band. The differential approximation is applied separately to each band, and the resulting equations are combined to give a differential equation for the net flux $F = F_1 + F_2$. 

Fig. 6-4.3. Temperature profiles for $b(T) \propto T$ and $\gamma = 1$. Solid curves: full differential approximation; dashed curves: Raizer's approximate method; from Olfe and Cavalleri.\textsuperscript{51a}

Fig. 6-4.4. Upstream temperature profiles for $b(T) \propto T^4$ and $\gamma = 1$. Solid curves: full differential approximation; dashed curves: Raizer's approximate method; from Olfe and Cavalleri.\textsuperscript{51a}
Use of the fluid dynamic relations (6-4.15) and (6-4.20) then yields differential equations for $T$ which are of fourth order, as compared with the second order equations for the single band case. The case of $b_1(T)$ and $b_2(T)$ both proportional to $T$ yields linear differential equations which have solutions consisting of sum of two exponentials. The coefficients of these exponentials are determined as functions of $\alpha$ and the band radiation parameters $\Gamma_1$ and $\Gamma_2$. In the limit of weak radiation, $\Gamma_1 \ll 1$ and $\Gamma_2 \ll 1$, the effect of radiation on the temperature profile is equal to the sum of the radiation contributions determined by using the single-band analysis for each band. For the both bands radiating strongly, $\Gamma_1 \gg 1$ and $\Gamma_2 \gg 1$, the form of the temperature profile will depend on the parameter $\Gamma_1 \Gamma_2 / \alpha$. For $\Gamma_1 \Gamma_2 / \alpha \gg 1$ the temperature profile becomes identical to a single band profile with the absorption coefficient given by the Rosseland mean value in the optically thick upstream profile, and by the Planck mean value in the optically thin downstream profile. On the other hand, for $\alpha$ sufficiently large that $\Gamma_1 \Gamma_2 / \alpha \ll 1$ the profile divides into a double profile representing the profile of band 2 imbedded within the profile of band 1. For values of $\Gamma_1$, $\Gamma_2$, and $\alpha$ yielding $\Gamma_1 \Gamma_2 / \alpha = 0.1$, Fig. 6-4.6 shows that the temperature profile is quite close to the dashed profile which was obtained by superposing the single-band profiles. The increase in the upstream temperature above the equilibrium downstream value $T_2$ results from the fact that for $\Gamma_1 \Gamma_2 / \alpha \ll 1$ the radiation directed upstream in band 2 is characteristic of the higher temperatures in the cooling profile of band 1. In Olfe and Cavalleri\textsuperscript{51a} an approximate

![Figure 6-4.5](image-url)  
**Fig. 6-4.5.** Downstream temperature profiles for $b(T) \propto T^\gamma$ and $\gamma = 1$. Solid curves: full differential approximation; dashed curves: Raizer's approximate method (the dashed curves for $\Gamma = 10$ and 20 fall on the solid curves). Note the scale change at $\tau = 0.05$; from Olfe and Cavalleri\textsuperscript{51a}. 

method of solution similar to Raizer's single-band method is achieved by considering the two most important terms in the differential equation for each optical depth regime along the temperature profile. Analysis of the nonlinear cases shows that the temperature profile is determined by the sum of single band effects in the weak radiation limit; in the strong radiation case the parameter \( \Gamma_1 \Gamma_2 / \alpha \) again determines whether the profile is given by the single-band profile with the appropriate mean absorption coefficients, or by a double structure representing the profile for one band imbedded within the profile of the other band.

6-4B Numerical solutions for a gray gas. Most of the discussion in this section follows the paper by Heaslet and Baldwin. The gas is assumed to be gray, and the exponential approximation \( me^{-\kappa \tau} \) is used for \( E_2(\tau) \), as discussed in Section 6-2C. In terms of the variables \( \xi = n \tau \) and \( \nu(\xi) \equiv u(x)/c_1 \), Eqs. (6-4.4) through (6-4.6) yield the following two equations for the temperature \( T \) and gas velocity \( \nu \) of a radiation-resisted (\( \mu = \lambda = 0 \)) shock wave in a perfect gas:

\[
T = \left( c_T^2 / R \right) (\nu - \nu^2), \tag{6-4.32}
\]

and

\[
\nu^2 - \left( \frac{2\gamma}{\gamma + 1} \right) \nu + 2 \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{c_2}{c_T^2} = 4 \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{\mu c_e^k}{np_1^2 \xi^4} \int_{-\infty}^{\infty} [\nu(\xi') - \nu(\xi')]^4 \text{sgn}(\xi - \xi') e^{-|\xi - \xi'|} d\xi', \tag{6-4.33}
\]
where \( \text{sgn}(\xi - \xi') \) denotes +1 for \( \xi' < \xi \) and -1 for \( \xi' > \xi \). A radiation parameter \( K \) is defined by

\[
K = 32 \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{m_c \epsilon}{n \rho \omega u_1 R^4}
\]

\[
= 32 \left( \frac{m}{n} \right) \frac{(\gamma + 1)^3}{\gamma^2[\gamma - 1]} \left[ 2 \gamma M^2 - (\gamma - 1)^2 \right] \left( \frac{\sigma T^4}{\rho u_1 h_2} \right),
\]

where \( M \) is the shock wave Mach number relative to the undisturbed gas upstream, and \( \sigma T^4/\rho u_1 h_2 \) is equal to the reciprocal of the Boltzmann number. Equation (6-4.33) may now be written in the form

\[
-(v_1 - v)(v - v_2) = \frac{K}{8} \int_{-\infty}^{\infty} (v - v_2)^4 \text{sgn}(\xi - \xi') e^{-|\xi - \xi'|^2} \, d\xi',
\]

where \( v_1 = u_1/c_1 \) and \( v_2 = u_2/c_1 \) are the values of \( v \) ahead of (\( \xi = -\infty \)) and behind (\( \xi = +\infty \)) the over-all shock wave, respectively. In terms of \( c_2/c_1 \),

\[
v_{1,2} = \left( \frac{\gamma}{\gamma + 1} \right) \pm \left[ \left( \frac{\gamma}{\gamma + 1} \right)^2 - 2 \left( \frac{\gamma - 1}{\gamma + 1} \right) \frac{c_2}{c_1} \right]^{1/2},
\]

with the subscripts 1 and 2 being associated with the + and - signs, respectively.

Since the right-hand side of Eq. (6-4.35) is proportional to the radiation flux, which is continuous across the shock profile, \((v_1 - v)(v - v_2)\) is a continuous function of \( \xi \). Heaslet and Baldwin find it convenient to introduce the continuous variable \( \theta(\xi) \),

\[
\theta(\xi) = \frac{1}{2} (v_1 - v_2)^2 - (v_1 - v)(v - v_2)
\]

\[
= \left[ \left( \frac{\gamma}{\gamma + 1} \right) - v \right]^2.
\]

The zero value of \( \xi \) is set at the point where \( u \) is equal to the local isentropic speed of sound; therefore \( \xi \) is zero at a discontinuity, if one exists. Inverting Eq. (6-4.37),

\[
v(\xi) = \left( \frac{\gamma}{\gamma + 1} \right) - (\text{sgn} \, \xi) [\theta(\xi)]^{1/2}.
\]

Equation (6-4.35) can be put into the form,

\[
\theta(\xi) = \theta_\infty + \frac{1}{2} \int_{-\infty}^{\infty} H(\theta(\xi')) \text{sgn}(\xi - \xi') e^{-|\xi - \xi'|^2} \, d\xi',
\]
where
\[ H(\theta) = \frac{K}{4} \left[ \left(\frac{\gamma}{\gamma + 1}\right)^{\frac{\gamma - 1}{\gamma + 1}} \left(\text{sgn} \, \xi \right)^{\frac{\gamma}{\gamma + 1}} \right]^4, \] (6-4.40)

and
\[ \theta_\infty = \frac{1}{4} (v_1 - v_2)^2 = \left(\frac{\gamma}{\gamma + 1}\right)^2 \left[ \frac{M_i^2 - 1}{\gamma M_1^2 + 1} \right]^2. \] (6-4.41)

Numerical calculations are most easily carried out by replacing the integral equation (6-4.39) by an equivalent differential equation. This replacement is achieved by first differentiating Eq. (6-4.39),
\[ \frac{d\theta}{d\xi} = H - \frac{1}{2} \int_{-\infty}^{\infty} H(\theta(\xi')) e^{-|\xi - \xi'|} d\xi', \] (6-4.42)

and
\[ \frac{d^2\theta}{d\xi^2} = \frac{dH}{d\theta} \frac{d\theta}{d\xi} + \frac{1}{2} \int_{-\infty}^{\infty} H(\theta(\xi')) \text{sgn}(\xi - \xi') e^{-|\xi - \xi'|} d\xi'. \] (6-4.43)

Now using Eq. (6-4.39) to substitute for the integral appearing in Eq. (6-4.43),
\[ \frac{d^2\theta}{d\xi^2} - (\theta - \theta_\infty) = \frac{dH}{d\theta} \frac{d\theta}{d\xi}. \] (6-4.44)

The reduction to the above differential equation constitutes the substitute kernel method described in Section 6-2C; i.e., Eq. (6-4.44) can be obtained directly from Eq. (6-2.23) by changing the variables \( F, T^4, \) and \( \tau \) to \( \theta, H, \) and \( \xi, \) respectively.

Now let the integral term in Eq. (6-4.42) define a function \( G(\theta), \) which is continuous in both \( \theta \) and \( \xi; \) i.e.,
\[ G(\theta) = H(\theta) - \frac{d\theta}{d\xi}. \] (6-4.45)

Using the above equation to eliminate the \( d^2\theta/d\xi^2 \) and \( d\theta/d\xi \) terms in Eq. (6-4.44), the following result is obtained:
\[ [H(\theta) - G(\theta)] \frac{dG(\theta)}{d\theta} = (\theta_\infty - \theta). \] (6-4.46)

The boundary conditions on \( G(\theta) \) are determined by the fact that \( d\theta/d\xi \) goes to zero for \( \xi \to \pm \infty; \) therefore Eq. (6-4.45) yields
\[ \xi = -\infty: \quad \theta = \theta_\infty, \quad G = H_\alpha(\theta_\infty) \] (6-4.47)
\[ \xi = +\infty: \quad \theta = \theta_\infty, \quad G = H_\beta(\theta_\infty). \] (6-4.47)
where the subscripts $a$ and $b$ denote the two branches of $H(\theta)$. Equation (6-4.40) shows that the two branches of $H(\theta)$ meet at $\theta = 0$, as illustrated in Fig. 6-4.7. The parameter $\tilde{K}'$ given in the figure is defined by

$$\tilde{K}' = \frac{2 \sqrt{2} (\gamma + 1)^2}{\gamma^2 (\gamma - 1)} \theta^{1/2}.$$  

(6-4.48)

To obtain the two branches of $G(\theta)$, corresponding to the branches of $H(\theta)$, Eq. (6-4.46) is integrated from the two points representing $\xi = \pm \infty$. These two points are saddle points through which only the integral curves $G_a(\theta)$ and $G_b(\theta)$ shown in Fig. 6-4.7 provide a realistic solution, which may contain an imbedded compression shock. Since $G$ and $\theta$ are continuous functions of $\xi$, the transition from $G_a(\theta)$ to $G_b(\theta)$ occurs at the point $\xi = 0$ where $G_a(\theta)$ equals $G_b(\theta)$. If this transition point occurs at a nonzero value of $\theta$, then $H$ and $v$ will change discontinuously from curve $a$ to curve $b$; cf. Eqs. (6-4.38) and (6-4.40), and Fig. 6-4.7. Therefore, only under conditions for which the $G_a(\theta)$ and $G_b(\theta)$ curves meet at $\theta = 0$ (right-hand plots in Fig. 6-4.7) will
the over-all shock profile be continuous. Once the $G(\theta)$ curves are computed, $d\theta/d\xi$ is obtained from Eq. (6-4.45), which gives $\theta(\xi)$ upon integration. The velocity $v(\xi)$ may then be determined from Eq. (6-4.38), and the temperature from Eq. (6-4.32).

The calculations of Heaslet and Baldwin for the profiles of the dimensionless velocity $v = u/c_1$, temperature $\tilde{T}$, and flux $\tilde{F}$ are shown in Fig. 6-4.8, where

$$\tilde{T} = \frac{RT}{c_1^2} = \left(\frac{\gamma M_1^2}{\gamma M_1^2 + 1}\right)^2 \frac{RT}{u_1^2},$$

and

$$\tilde{F} = \frac{\theta_\infty - \theta}{\theta_\infty} = -2(\gamma^2 - 1) \left(\frac{M_1^2}{M_1^2 - 1}\right)^2 \frac{F}{\rho_1 u_1^2}.$$

The computed temperature profiles for a strong shock show the same features as Figs. 6-4.1 and 6-4.2 in the weak and strong radiation limits, respectively. Weak radiation is seen to affect the flow variables out to a distance of about two photon mean free paths on either side of the discontinuity at $\xi = 0$, with the net flux being symmetrical about $\xi = 0$. 
As the radiation becomes stronger, the net flux and resulting changes in flow variables increase to farther distances upstream. For strong shock waves the net flux is nonzero only to very small distances behind the discontinuity. On the other hand, weak shock waves will be smoothed out by strong radiation, with the net flux and changes in flow variables extending many photon mean free paths downstream from $\xi = 0$.

It is only for rather weak shocks that a continuous profile can exist. Mitchner and Vinokur\textsuperscript{52} have shown that the profile must be discontinuous if

$$M_1^2 > \frac{(2\gamma - 1)}{\gamma(2 - \gamma)}, \quad (6-4.51)$$

which becomes $M_1 > 1.21, 1.46,$ and $2.05$ for $\gamma = 1.2, 1.4,$ and $1.67$, respectively. Even at values of $M_1$ below the above limit the profile can be discontinuous if the upstream temperature is sufficiently low.

Mitchner and Vinokur have also shown that a transverse magnetic field will inhibit the smoothing tendency of the radiation. Ryhming\textsuperscript{53} and Emanuel\textsuperscript{54} have studied radiation-resisted shock structure including dissociation and vibrational nonequilibrium effects, respectively. Traugott\textsuperscript{55} has analyzed the equations of shock structure in a radiating gas including the viscous and heat-conducting terms.

By direct numerical calculation, Pearson\textsuperscript{56} has shown that the approximation of $E_2(\tau)$ by $m e^{-n_\tau}$, as used by Heaslet and Baldwin, results in velocity profile errors of less than about 1%. In addition to the numerical calculations described above, Heaslet and Baldwin obtained closed-form solutions by carrying out expansions of Eq. (6-4.46) for small and large values of $\mathcal{K}$. For small $\mathcal{K}$, their results are similar to those derived in the preceding section following the more physical development of Raizer. This weak radiation limit has also been studied by Ryhming.\textsuperscript{57}

\textbf{6-4C Shock waves with ionization and non-gray radiation.} Consider a monatomic gas which may become singly ionized to produce the atom, ion, and electron number densities $n_a$, $n_i$, and $n_e = n_i$, respectively. The degree of ionization $\alpha \equiv n_i/(n_i + n_a)$ will enter into the calculation of the thermodynamic functions. Accordingly, the state equations (6-4.11) and (6-4.12) are to be replaced by the following:

$$h = \frac{5}{2} (1 + \alpha) RT + e_{el} + \alpha RT,$$  \hspace{1cm} (6-4.52)

and

$$p = (1 + \alpha) \rho RT,$$  \hspace{1cm} (6-4.53)
where \( RT_i = h\nu_i/m_a \) is the ionization energy [\( h \) is Planck's constant, which is not to be confused with the enthalpy \( h \) appearing in Eq. (6-4.52); \( m_a \) is the mass of the atom and \( \nu_i \) is the frequency of the photoionization absorption edge; i.e., \( h\nu_i \) is the minimum photon energy which will produce ionization of the atom]. The electronic energy \( e_{e1} \) of the atoms and ions often may be neglected.

If the density is sufficiently high that collisions establish local thermodynamic equilibrium (LTE), then the degree of ionization is the equilibrium value \( \alpha_E \) given by the Saha equation,

\[
\left( \frac{\alpha_E^2}{1 - \alpha_E^2} \right) \rho = \frac{2(2\pi m_e)^{3/2}}{h^3} \left( \frac{Q_1}{Q_0} \right) (kT)^{5/2} \exp \left( -\frac{T_i}{T} \right). \tag{6-4.54}
\]

Here \( m_e \) is the mass of the electron, \( k \) the Boltzmann constant, and \( Q_0 \) and \( Q_1 \) the partition functions for the atoms and ions, respectively.

A number of investigations of radiative cooling have been carried out in this LTE limit. The radiative cooling of optically thin argon was calculated by Petschek et al.,\(^58\) who used Unsöld's approximate formula for the continuum radiation. The calculations of Petschek et al. compare favorably with their shock tube measurements, in which the shocked argon reached temperatures up to 18,000\(^\circ\)K. Pomerantz\(^59\) has extended the analysis for argon to higher densities by calculating the absorption using a gray-band model, and by accounting for the lowering of the ionization potential in a plasma. Using the continuum emission for a transparent gas, Whitney and Skalafuris\(^60\) calculated the radiative cooling for hydrogen at the low densities of interest in astrophysical problems. For the conditions occurring in the experiments of Petschek et al.,\(^58\) Sevastyanenko and Yakubov\(^61\) have computed radiative cooling profiles including line radiation as well as continuum radiation. Yakubov\(^62\) also studied line radiation and absorption effects in hydrogen for the conditions considered by Whitney and Skalafuris.

Clarke and Ferrari\(^23\) have calculated radiative cooling under conditions for which the degree of ionization \( \alpha \) departs from the local equilibrium value \( \alpha_E \). That is, the collision rates considered are not high enough to ensure equilibrium ionization as the gas cools by means of free-bound (recombination) radiation [however, the collisions are assumed to establish translational equilibrium among the species, i.e., \( T(\text{atoms}) = T(\text{ions}) = T(\text{electrons}) \)]. Nonequilibrium ionization often occurs in the precursor region ahead of the discontinuity. The precursor region has been studied by Clarke and Ferrari,\(^23\) Wetzel,\(^63\) Whitney and Skalafuris,\(^60\) Largar'kov and Yakubov,\(^64\) and Yakubov.\(^62\) In order to study nonequilibrium ionization, the ionization rate equation must be considered.
The net ionization rate is equal to the difference between the radiative ionization (photoionization) and recombination rates plus the difference between the collisional ionization and recombination rates. The ionization rate equation may be written in the following form:

\[
\rho \frac{D\alpha}{Dt} = -m_a \nabla \cdot \int_{\nu_1}^{\infty} \frac{F_v}{h\nu} d\nu + \frac{\rho \alpha^2}{\tau_c} \left[ \frac{(1 - \alpha) \alpha_E^2}{(1 - \alpha_E)} - \alpha^2 \right]. \tag{6-4.55}
\]

For steady, one-dimensional problems the substantial derivative \( D\alpha/Dt \) reduces to \( u \, d\alpha/dx \) since continuity gives \( \rho u = \text{constant} \). The first term on the right-hand side of Eq. (6-4.55) gives the difference between the radiative ionization and recombination rates, expressed in terms of the radiative flux \( F_v \) in the ionization continuum \( \nu = \nu_1 \) to \( \infty \). The collisional ionization and recombination rates are expressed in terms of the characteristic time \( \tau_c \) and the equilibrium degree of ionization \( \alpha_E \) (the ionization and recombination rates are equal at \( \alpha = \alpha_E \)). The ionization rate is proportional to \( \alpha(1 - \alpha) \) as it involves collisions between electrons and atoms, and the recombination rate is proportional to \( \alpha^3 \) since it involves electron-ion-electron triple collisions \( [n_i = n_e \propto \alpha \text{ and } n_a \propto (1 - \alpha)] \).

The equation of radiative transfer must be used to obtain a relation for the flux \( F_v \) to be substituted into the ionization rate and energy equations. The transfer equation may be written in the form:

\[
\frac{\partial I_v}{\partial s} = k_v \left[ \alpha^2 \left( \frac{1 - \alpha}{\alpha_E^2} \right) B_v - (1 - \alpha) I_v \right], \tag{6-4.56}
\]

where \( k_v \) is the absorption coefficient per unit mass of neutral atoms.

The emission is proportional to \( \alpha^2 \) since it consists of ion-electron recombinations, whereas absorption is proportional to \( (1 - \alpha) \) since it involves photoionization of the atoms. The flux \( F_v \) is obtained by multiplication of \( I_v \) by the direction cosine and integration over \( 4\pi \) sterad [cf. Eq. (6-2.4)].

Equations (6-4.52) through (6-4.56) may be combined with the conservation equations (6-4.4) through (6-4.6) to solve the radiation-resisted shock structure with nonequilibrium ionization. In the precursor region, the collisional ionization and recombination will usually be unimportant if the degree of ionization is not too great. Neglecting collisional ionization and recombination, integration of Eq. (6-4.55) for steady flow with \( \alpha \ll 1 \) gives:

\[
\rho_i u_i RT_1 (\alpha - \alpha_1) = -h\nu_1 \int_{\nu_1}^{\infty} \frac{F_v}{h\nu} d\nu = -\int_{\nu_1}^{\infty} F_v d\nu \left[ 1 + O \left( \frac{T}{T_1} \right) \right], \tag{6-4.57}
\]
where the relation $RT_1 = hv/m_\alpha$ has been used. For the conditions of shock tube experiments, $T/T_1 \ll 1$ and the ionization $\alpha_1$ infinitely far upstream is negligible. If radiative recombination (emission) is negligible in the precursor region, then $F_v$ may be calculated easily from the transfer equation by considering only absorption.

At the other extreme, Goldsworthy suggests that very strong radiation of the right spectral quality might establish a "radiative ionization front" upstream of the discontinuity. The gas becomes fully ionized and therefore nearly transparent to the radiation as it passes through such a front. Goldsworthy has theoretically studied the properties of ionization fronts, and their existence has been indicated by the electromagnetic shock tube experiments of Medford et al.

In the analyses of Whitney and Skalafuris and of Clarke and Ferrari, the precursor region is joined to the radiative cooling region by applying the conservation equations across the discontinuity, with no change in the degree of ionization. As mentioned above, Whitney and Skalafuris consider a transparent radiative cooling region which is in local thermodynamic equilibrium. Therefore, directly behind the discontinuity there exists a short region of internal relaxation to the equilibrium degree of ionization $\alpha_E$. Skalafuris has studied this internal relaxation region by allowing for the different translational temperatures of the electrons, ions, and atoms. The inelastic (ionizing) collisions cool the electrons, which then cool the ions through elastic collisions, with

\begin{align*}
T_1 &= 300^\circ K \\
p_1 &= 10^{-9} \text{ atm} \\
T_1/T_4 &= 10 \\
M_1 &= 28.9
\end{align*}

\begin{align*}
T &= 72,300^\circ K \\
P_1 &= 3.94 \\
T_4 &= 18,200^\circ K
\end{align*}

\begin{align*}
\alpha_\text{max} &= 0.770 \\
\alpha_4 &= 0.720 \\
\alpha_1 &= 0.0977
\end{align*}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig6-4-9}
\caption{Shock structure for argon, which has the value $T_1 = 182,000^\circ K$; from Clarke and Ferrari.}
\end{figure}
the strong ion-atom interaction resulting in nearly equal atom and ion temperatures.

In their analysis, Clarke and Ferrari consider a radiative cooling region characterized by translational equilibrium, nonequilibrium ionization, and nontransparent radiation given by the recombination continuum. An iterative procedure is used to combine the precursor and radiative cooling calculations. Clarke and Ferrari solve for $T$ and $\alpha$ from the conservation equations and Eqs. (6-4.52) through (6-4.55), using the flux $F$ and the electron source function $\int_{v_1}^{\infty} (F_v/hv) \, dv$ computed from the previous iterative step. Their zeroth approximation is the radiationless shock, which has constant properties ahead of and behind the discontinuity. Calculated profiles of the shock structure in argon and helium are shown in Figs. 6-4.9 and 6-4.10, respectively. The element

![Graph showing shock structure for helium.](image)

**Fig. 6-4.10.** Shock structure for helium, which has the value $T_1 = 285,000^\circ\text{K}$; from Clarke and Ferrari.23

of optical depth is defined by $d\tau \equiv (1 - \alpha) k \, dx$, where $k$ is the (atom) absorption coefficient at the peak of the absorption edge at $v_1$. The normalized flux quantities are defined by

$$\tilde{F} = -(\rho u RT_1)^{-1} \int_{v_1}^{\infty} F_v \, dv = -(\rho u RT_1)^{-1} F$$

$$\tilde{F}_1 \equiv -(\rho u RT_1)^{-1} h v_1 \int_{v_1}^{\infty} \frac{F_v}{h v} \, dv. \quad (6-4.58)$$
Reference to Eq. (6-4.57) shows that $\alpha = \bar{F}_x$ for negligible collisional ionization and recombination, since $\alpha_1 = 0$ for the conditions considered.

In Fig. 6-4.9 collision effects in the precursor region are small, giving a difference between $\alpha$ and $\bar{F}_x$ of only a few percent at the discontinuity. On the other hand, collision processes are dominant behind the discontinuity, producing an ionization which rapidly approaches the equilibrium value. Radiative cooling changes the variables out to about $\tau = 2$; however, cooling effects are small because the radiative flux is small compared with the total flow energy for the conditions considered ($\bar{F} = 0.111$ at the discontinuity). For the profiles shown in Fig. 6-4.10, it is seen that collisions are unimportant in the precursor region ($\alpha = \bar{F}_x$), and the collisional relaxation length behind the discontinuity is not small compared with the photon mean free path. For this case, radiative cooling is very small ($\bar{F} < 0.01$).

6-5 Radiating inviscid shock layers

The three-dimensional aspects of the shock layer problem introduce significant differences from the purely one-dimensional shock structure problem considered in the preceding section. First, the absorption of radiation ahead of the transparent shock front (discontinuity) will generally be unimportant since the photon mean free path in the unshocked gas is usually much greater than the body dimension; i.e., very little of the energy radiated upstream through the shock front will be absorbed by the portion of gas which flows into the shock layer. Accordingly, absorption of radiation ahead of the shock front is neglected in shock-layer studies.

Three-dimensional aspects of the problem are also of prime importance behind the shock front, affecting the distribution of radiation, as well as the flow field. For example, the simplest approximation to the shock layer of a blunt body, as considered in Section 6-5A, is one-dimensional flow through a normal shock wave, with no absorption ahead of the shock front, and emission of radiation occurring only over a distance $L$ behind the shock front. Thus in this first approximation the flow model is kept completely one-dimensional, but the length of the radiating region represents the finite shock layer thickness, which is produced by the three-dimensional aspects of the real flow problem.

The shock layer flow in the stagnation region of a blunt body is determined in Section 6-5B by an approximate analytical method, which reduces the energy equation to a one-dimensional form. Numerical blunt body solutions are reviewed in Section 6-5C. The shock layers of
wedges and conical bodies are analyzed by a perturbation treatment and by a thin shock layer treatment in Sections 6-5D and 6-5E, respectively. Similarity solutions for slender bodies are considered in Section 6-5F.

6-5A A SIMPLE, ONE-DIMENSIONAL MODEL FOR BLUNT BODIES. Here we consider the one-dimensional model for shock layer flow given by Yoshikawa and Chapman. In their analysis, absorption ahead of the shock front is neglected, and the gas is assumed to radiate to a distance \( L \) behind the shock; i.e., a porous, cold, blackbody wall represents the blunt body at a distance \( L \) behind the shock wave. A perfect, gray gas is considered, with thermal conduction assumed to be negligible compared with radiant-energy transfer. As discussed in Section 6-1, radiant-energy transfer affects the flow field only for bodies traveling at hypersonic speeds; thus we consider a strong shock, behind which

\[ h \gg \frac{1}{2} u^2. \]  

For this inviscid strong shock case, the continuity (6-4.8) and momentum (6-4.9) equations yield the following inequality:

\[ \frac{1}{\rho} \left| \frac{dp}{dx} \right| \simeq \frac{1}{V_\infty} \left| \frac{du}{dx} \right| \ll \frac{1}{L}, \]  

where \( V_\infty \equiv u_1 \) is the (hypersonic) shock velocity relative to the undisturbed gas. Thus behind the shock we have essentially constant pressure flow. In view of the above inequalities, Yoshikawa and Chapman approximate the energy equation (6-4.10) by

\[ \rho_\infty V_\infty \frac{dh}{d\tau} = \frac{dF}{d\tau} 
\]

\[ = \frac{d}{d\tau} \left[ 2\sigma \int_{0}^{\tau} T^4(t) E_2(t - \tau) \, dt - 2\sigma \int_{\tau}^{\infty} T^4(t) E_2(t - \tau) \, dt \right] \]

\[ = 4\sigma T^4(\tau) - 2\sigma \int_{0}^{\tau} T^4(t) E_4(|t - \tau|) \, dt, \]  

where \( \tau_w \equiv \int_{0}^{L} k(x) \, dx \) is the optical depth from the shock front at \( x = 0 \) to the porous wall at \( x = L \). In the above equation radiation from the wall surface is neglected because the wall is assumed to be relatively cool.

After we divide by \( 2\sigma T^4(\tau) \) and integrate, Eq. (6-5.3) takes on the following integral form:

\[ f_1(p_s, h_s; h) = f_2(\tau_w, \tau), \]  

\( (6-5.4) \)
where

\[ f_1(p_s, h_5; h) = \rho_\infty V_\infty \int_h^{h_s} \frac{dh}{2\sigma T^4} \]  

(6-5.5)

and

\[ f_2(\tau_w, \tau) = \int_0^{\tau_w} \left[ 2 \cdot \frac{1}{T^4(\tau')} \int_0^{\tau_w} T^4(t) E_1(|t - \tau'|) dt \right] d\tau'. \]  

(6-5.6)

The subscript \( s \) denotes values just behind the shock front.

Yoshikawa and Chapman utilize a method of successive approximations to solve Eq. (6-5.4). First, \( f_1 \) can be determined as a function of \( T \) and \( p \) by the use of tables of equilibrium air properties; or conversely, \( T \) may be expressed as a function of \( f_1 \) and \( p \). The following truncated series is used:

\[
\frac{T^4}{T^4_s} = a_0 + a_1 f_1 + a_2 f_1^2 + a_3 f_1^3
\]

(6-5.7)

with the \( a_n \) being known functions of the shock layer pressure.

The first approximation to \( f_2 \) is obtained by taking \( T \) to be constant. Then direct integration of Eq. (6-5.6) gives

\[
f_2(\tau_w, \tau) = \frac{1}{2} - E_3(\tau_w - \tau) + E_3(\tau_w - \tau).
\]  

(6-5.8)

In order to simplify calculations \( E_3(\tau) \) is approximated by the exponential \( \frac{1}{2} e^{-2\tau} \), so the first approximation to \( f_2 \) becomes

\[
f_2(\tau_w, \tau) \approx \frac{1}{2} \left[ \frac{1 - 2E_3(\tau_w)}{1 - \exp(-2\tau_w)} \right] ^x \{1 - \exp(-2\tau_w) - \exp(-2\tau) + \exp(-2(\tau_w - \tau))\}.
\]  

(6-5.9)

which reduces to Eq. (6-5.8) at \( \tau = \tau_w \). The energy equation (6-5.4) gives \( f_1 = f_2 \); thus replacing \( f_1 \) in Eq. (6-5.7) by the right-hand side of Eq. (6-5.9) gives the first approximation to the temperature distribution \( T^4(\tau) \).

This first approximation to the temperature distribution is then used to determine the value of \( f_2 \) at the wall and the derivatives of \( f_2 \) at the shock front and at the wall; i.e., the following integrals are evaluated:

\[
f_{2w} = f_2(\tau_w, \tau_w) = \int_0^{\tau_w} \left[ 2 \cdot \frac{1}{T^4(\tau')} \int_0^{\tau_w} T^4(t) E_1(|t - \tau'|) dt \right] d\tau'.
\]  

(6-5.10)
\[ f'_{2s} = f'(\tau_w, 0) = 2 - \frac{1}{T'_{s}} \int_0^{\tau_w} T^4(t) E_1(t) \, dt, \]  
(6-5.11)

and

\[ f'_{2w} = f'(\tau_w, \tau_w) = 2 - \frac{1}{T^4(\tau_w)} \int_0^{\tau_w} T^4(t) E_1(\tau_w - t) \, dt, \]  
(6-5.12)

where \( T(\tau_w) \) is temperature of the gas adjacent to the wall.

The following function is used for the second approximation to \( f_2 \):

\[ f_2(\tau_w, \tau) = b[1 - \exp(-\eta_2 \tau_w) - \exp(-\eta_1 \tau) + \exp(-\eta_2 (\tau_w - \tau))], \]  
(6-5.13)

which takes on the values

\[ f_{2w} = b[2 - \exp(-\eta_2 \tau_w) - \exp(-\eta_1 \tau_w)], \]  
(6-5.14)

\[ f'_{2s} = b[\eta_1 + \eta_2 \exp(-\eta_2 \tau_w)], \]  
(6-5.15)

and

\[ f'_{2w} = b[\eta_1 \exp(-\eta_1 \tau_w) + \eta_2]. \]  
(6-5.16)

The values of the constants \( b, \eta_1, \) and \( \eta_2 \) for the second approximation to \( f_2 \) may be determined by matching the values of \( f_{2w}, f'_{2s}, \) and \( f'_{2w} \) given by Eqs. (6-5.14) through (6-5.16) to the values computed using the first approximation to the temperature distribution in Eqs. (6-5.10) through (6-5.12). For the determination of \( b, \eta_1, \) and \( \eta_2, \) Yoshikawa and Chapman find it more expedient, however, to use a mean slope \( f'_{2w} \) instead of \( f'_{2w} \) in order to compensate for small but sudden changes of temperature near the wall.\(^2\) The value of \( f'_2 \) at \( \tau = 0.95 \tau_w \) is found to provide a good mean slope to use for \( f'_{2w} \). The second approximation to \( f_2 \) is substituted for \( f_2 \) in Eq. (6-5.7) in order to yield the second approximation to the temperature distribution \( T^4(\tau) \). This second approximation to \( T^4(\tau) \) could be used in Eqs. (6-5.10) through (6-5.12) to give further approximations for \( f_{2w}, f'_{2s}, \) and \( f'_{2w} \) (or \( f''_{2w} \)); however, it was found that the values computed from the first approximation to \( T^4(\tau) \) give sufficient convergence so that no further iterations upon these values is necessary.\(^2\)

The upstream flux \( F_s \) at the shock front and the downstream flux \( F_w \) at the wall are computed from the temperature distribution by means of the formulae,

\[ F_s \equiv -F(\tau_w, 0) = 2\sigma \int_0^{\tau_w} T^4(t) E_2(t) \, dt \]  
(6-5.17)

and

\[ F_w \equiv F(\tau_w, \tau_w) = 2\sigma \int_0^{\tau_w} T^4(t) E_2(\tau_w - t) \, dt. \]  
(6-5.18)
Yoshikawa and Chapman have carried out calculations using tabulated values for the Planck mean absorption coefficient and thermodynamic properties of equilibrium air. Plots of the radiation fluxes $F_s/2\sigma T_s^4$ and $F_w/2\sigma T_s^4$ versus the shock layer optical depth $\tau_w$ are given in Fig. 6-5.1.

![Figure 6-5.1](image)

Fig. 6-5.1. First and second approximations for the radiation fluxes $F_s$ and $F_w$ as functions of the optical thickness $\tau_w$, from Yoshikawa and Chapman.

It is shown that accurate flux values may be computed from the first approximation to the temperature distribution; this result indicates the accuracy of the $f_{2w}$, $f'_{2s}$, and $f'_{2w}$ values computed from the first approximation to $T^4(\tau)$, as discussed above. The calculated values of the nondimensional radiation fluxes $\lambda_s \equiv F_s/\frac{1}{2} \rho \infty V_\infty^3$ and $\lambda_w \equiv F_w/\frac{1}{2} \rho \infty V_\infty^3$ are presented on altitude-velocity charts such as the one shown in Fig. 6-1.3.

Temperature profiles behind the shock front are shown in Fig. 6-5.2, with check points illustrating the accuracy of the second approximation. It is seen that optically thin shock layers have a linear decrease in temperature behind the shock front, whereas optically thick layers have temperature variations only within about two photon mean free paths of the shock front or the wall.

6-5B Approximate, analytical solutions for the stagnation region of a blunt body. In the shock layer of a blunt body, the peak temperatures and therefore the greatest radiant-energy losses occur near the stagnation point. Accordingly, in this section we shall consider the
stagnation flow of a radiating gas as described by the approximate model developed by Goulard. As in the preceding section, hypersonic flow conditions are of interest, with the shock layer dynamic pressure $\rho V^2$ and kinetic energy $\frac{1}{2}V^2$.
being small compared with the static pressure $p$ and static enthalpy $h$, respectively. Thus the pressure can be treated as constant across the shock layer; cf. Eq. (6-5.2). In addition, the shock layer thickness $\delta$ will be small compared with the nose radius $R$ of the blunt body, as illustrated in Fig. 6-5.3.

For nonradiating hypersonic flow, the shock layer near the stagnation point is nearly of constant density, with the vertical ($V$) and horizontal ($U$) components of velocity approximating the incompressible, potential flow values,

$$V = - (\rho_\infty V_\infty/\Delta \rho_s) Z \quad \text{and} \quad U = \frac{1}{2} (\rho_\infty V_\infty/\Delta \rho_s) r.$$  \hspace{1cm} (6-5.19)

Here $Z$ is distance from the surface, $\Delta$ the shock layer thickness, and $\rho_s$ the (constant) density for the nonradiating case. When radiation losses become important, the cooling of the gas as it approaches the stagnation point produces an increased density, resulting in a shock detachment distance $\delta$ which is smaller than the value $\Delta$ for the nonradiating layer. Although the temperature and density will vary across the shock layer, the pressure will remain essentially constant, as indicated above. Therefore, the velocity field should not differ appreciably from that for the nonradiating case, provided radiation losses are not too great. Accordingly, Goulard assumes that the vertical
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Component of velocity $v$ will be approximately independent of $r$ for radiating stagnation flow. To the same approximation the enthalpy will depend only on $z$, since for $v = v(z)$ only, the time over which a particle has been radiating behind the shock depends only on $z$. Only the stagnation region is considered here, so the shock front is essentially normal to all the streamlines. In addition, since the shock layer thickness $\delta$ is assumed to be much smaller than the body nose radius $R$, the radiative transfer will be described approximately by the one-dimensional equation (6-5.3), with the optical depth $\tau \equiv \int_0^\delta k(z') \, dz'$ now measured from the surface.

The approximations described above reduce the energy equation to the following one-dimensional form:

$$\frac{\rho v}{k} \frac{\partial h}{\partial z} = \frac{\partial F}{\partial r} = 4\sigma T^4(\tau) - 2\sigma \int_0^\delta T^4(t) E_4(|t - \tau|) \, dt, \quad (6-5.20)$$

where $\tau_\delta$ is the optical depth from the surface at $z = 0$ to the shock front at $z = \delta$. In order to integrate the above equation, the conservation equations for mass and momentum are used to relate the variables in the radiating flow to the variables in the corresponding nonradiating flow. Conservation of mass gives (see Fig. 6-5.3),

$$\rho \omega V_\infty \pi r_c^2 = \rho v \pi r^2 = 2\pi r \int_0^z \rho u \, dz'; \quad (6-5.21)$$

whereas for the nonradiating case, the potential flow result is

$$\rho \omega V_\infty \pi r_c^2 = \rho_s V \pi r^2 = 2\pi r \rho_s U Z. \quad (6-5.22)$$

From the above equations, the following relations are obtained between the radiating and the nonradiating stagnation flows:

$$\rho v(z) = \rho_s V(Z), \quad (6-5.23)$$

$$\int_0^z \rho u \, dz' = \rho_s U Z \quad \text{or} \quad \rho u \, dz = \rho_s U \, dZ. \quad (6-5.24)$$

To obtain another relation, Goulard notes that the pressure gradient in the $r$-direction (as given by the Newtonian value, say) will be the same in the radiating and nonradiating cases. Since $\partial u/\partial z \ll \partial u/\partial r$ and $\partial U/\partial Z \ll \partial U/\partial r$ [cf. Eqs. (6-5.19)], the momentum equations in the $r$-direction for the two flows give

$$-\frac{\partial p}{\partial r} = \rho u \frac{\partial u}{\partial r} = \rho_s U \frac{\partial U}{\partial r}. \quad (6-5.25)$$
As \( \rho \) is independent of \( r \) to the first approximation, the above equation integrates to

\[
\rho^{1/2} u = \rho_s^{1/2} U. \tag{6.5.26}
\]

Eliminating \( u \) and \( U \) between Eq. (6-5.26) and the differential form of Eq. (6-5.24) yields the relation

\[
dz = (\rho_s/\rho)^{1/2} dZ. \tag{6.5.27}
\]

Integration across the shock layer gives the reduced thickness \( \delta \) of the radiating shock layer,

\[
\delta = \int_0^\delta (\rho_s/\rho)^{1/2} dZ. \tag{6.5.28}
\]

The left-hand side of Eq. (6-5.20) may now be expressed in terms of \( \rho, T, \) and \( Z \) by means of the relation \( \Delta h = c_p \Delta T \) and Eqs. (6-5.19), (6-5.23), and (6-5.27),

\[
\frac{\rho_c V_c}{\Delta} \left( \frac{\rho}{\rho_s} \right)^{1/2} \frac{c_p}{k} \frac{\partial T}{\partial Z} = 4\sigma T^4 - 2\sigma \int_0^\delta T^4(t) E_1(|t - \tau|) \, dt. \tag{6.5.29}
\]

The specific heat \( c_p \) and the absorption coefficient \( k \) are assumed to have the following power law variations:

\[
\frac{c_p}{(h_s/T_s)} = \left( \frac{T}{T_s} \right)^{\alpha} \tag{6.5.30}
\]

and

\[
\frac{k}{k_s} = \frac{\kappa \rho}{\kappa \rho_s} = \left( \frac{\rho}{\rho_s} \right)^{n+1} \left( \frac{T}{T_s} \right)^{\beta}. \tag{6.5.31}
\]

where \( \kappa \) is the mass absorption coefficient, and the subscript \( s \) denotes the values just behind the shock front. Since the pressure is essentially constant across the shock layer,

\[
\frac{\rho}{\rho_s} \approx \frac{T_s}{T}. \tag{6.5.32}
\]

Substitution of Eqs. (6-5.30) through (6-5.32) into Eq. (6-5.29) yields

\[
T^{(\alpha-\beta+\eta-4+1)}Z \frac{\partial T}{\partial Z} = \Gamma_n \left[ 1 - \frac{1}{2T^4} \int_0^\delta T^4(t) E_1(|t - \tau|) \, dt \right], \tag{6.5.33}
\]
where $\mathcal{T} = T/T_s$ and $\mathcal{Z} = Z/A$. As given in Section 6-1,

$$\Gamma_n = \frac{4\sigma T^4 \rho_0 \mathcal{L}}{\rho \infty V \omega \mathcal{A}} = 4\mathcal{A} \cdot Bo^{-1}$$  \hspace{1cm} (6-5.34)

is the ratio of the radiant energy flux emitted from an optically thin shock layer with no radiative cooling to the enthalpy flux flowing into the shock layer.

For an optically thin shock layer the integral on the right-hand side of Eq. (6-5.33) becomes negligible so direct integration along the stagnation streamline gives the following temperature profile:

$$T = [1 + (\alpha - \beta + n - 3 + \frac{1}{2}) \Gamma_n \ln \mathcal{Z}]^{1/(\alpha - \beta + n - 3 + \frac{1}{2})}.$$  \hspace{1cm} (6-5.35)

This temperature profile may be used to obtain $(\rho_0/\rho)^{1/2} = \mathcal{T}^{1/2}$ in terms of $\mathcal{Z}$ for substitution into Eqs. (6-5.27) and (6-5.28), which upon integration yield $z$ as a function of $\mathcal{Z}$, and $\delta$ in terms of $\Delta$, respectively. The temperature profile (6-5.35) may also be used to compute the reduction in stagnation point radiant-heat transfer $F_w$ produced by the radiative cooling. If $F_{w,0}$ is the radiant-heat transfer for no cooling, then

$$F_w = F_{w,0} = \frac{1}{2} \int_0^z 2\sigma T^4 k \Delta z = \int_0^{\mathcal{T} \mathcal{D}^{(n - 3 + \frac{1}{2})}} d\mathcal{Z}.$$  \hspace{1cm} (6-5.36)

If the change in shock detachment distance were not accounted for, use of the nonradiating velocity field (as in Goulard) would give the above results, except that the $\frac{1}{2}$ would be absent from the $(\alpha - \beta + n - 3 + \frac{1}{2})$ and $(\beta - n + 3 + \frac{1}{2})$ terms in Eqs. (6-5.35) and (6-5.36), respectively, and $\mathcal{Z}$ would represent the actual coordinate $z$.

The relations (6-5.35) and (6-5.36) constitute an approximate solution to the problem of radiating stagnation flow in the limit of an optically thin gas. Since the velocity field for the radiating flow was assumed to differ by little from that for the corresponding nonradiating flow, we would expect this approximate solution to be accurate only under conditions of relatively small radiative cooling; i.e., the above analysis is essentially a perturbation calculation in terms of the (small) parameter $\Gamma_n$. It is shown below, however, that Eq. (6-5.36) gives good results even when $\Gamma_n$ is not very small. The validity of this treatment for moderate values of $\Gamma_n$ is indicated by the numerical calculations of Section 6-5C, which show that the velocity and pressure fields are relatively unaffected by the radiative transfer.

Let us now consider values of $\Gamma_n$ so small that Eq. (6-5.35) can be
expanded with powers of $\Gamma_n$ higher than the first being neglected. The temperature profile becomes

$$T \simeq 1 + \Gamma_n \ln Z \simeq 1 + \Gamma_n \ln (z/\delta).$$  \hspace{1cm} (6-5.37)

Simplified results for $\delta$ and $F_w$ are obtained from Eqs. (6-5.28) and (6-5.36), respectively:

$$\delta/\Delta \simeq 1 - \frac{3}{2} \Gamma_n$$  \hspace{1cm} (6-5.38)

and

$$F_w \simeq 1 - (\beta - n + 3 + \frac{1}{2}) \Gamma_n.$$  \hspace{1cm} (6-5.39)

It is evident that Eq. (6-5.35) will give inaccurate temperature values very near the body, namely when $(\alpha - \beta - n - 3 + \frac{1}{2}) \Gamma_n \ln Z$ becomes of order unity. For small values of $\Gamma_n$, however, this boundary region of inaccurate temperature will be so thin that no appreciable errors are introduced into the calculation of $\delta$ and $F_w$. An accurate treatment of the flow near the body should include the viscous boundary layer; however, we shall defer discussion of viscous effects until Section 6-6.

In the purely inviscid problem, the gas flowing along the stagnation streamline will take an infinite length of time to reach the stagnation point, where the gas must accordingly be in radiative equilibrium with its surroundings. If the radiation flux from the optically thin shock layer is completely negligible, then the gas at the stagnation point will have the temperature $T_w/(2)^{1/4}$, where $T_w$ is the wall (body surface) temperature. On the other hand, if the flux from the optically thin shock layer is comparable to that from the wall, both fluxes must be included in the calculation of the stagnation gas temperature. In either case the calculated gas temperature at the stagnation point will be a small fraction of the temperature directly behind the shock front.

Chin and Hearne\textsuperscript{70} consider stagnation flow, as well as conical flow for a transparent gas. They approximate the velocity field by the incompressible, potential flow values. The radiative and thermodynamic properties are assumed to have the following power-law variations at constant pressure: \(k T^4 \equiv \kappa \rho T^4 \propto h^{\beta-1}\) and \(\rho \propto h^{-1}\). In order to account for the variable density, the Howarth-Dorodnitsyn variable \(dZ_a \equiv (\rho/\rho_s) dZ\) is used, nondimensionalized as \(d\eta \equiv dZ_a/\Delta_a\), where \(\Delta_a \equiv \int_0^1 (\rho/\rho_s) dZ\); cf. Eqs. (6-5.27) and (6-5.28). For stagnation flow the results are

$$h \equiv h/h_s = [1 + (1 - \beta) \Gamma_n \ln \eta]^{1/(1-\beta)}$$  \hspace{1cm} (6-5.40)

and

$$F_w = \int_0^1 h^\beta d\eta.$$  \hspace{1cm} (6-5.41)
The above equations have the same form as Eqs. (6-5.35) and (6-5.36). Let us compare these equations for a perfect gas: \( \frac{h}{h_s} = \frac{T}{T_s} \) or \( \tilde{\alpha} = 0 \); thus the temperature variations (6-5.30) through (6-5.32) show that \( \beta - n + 4 \) is equivalent to \( \beta \). Accordingly, Goulard’s equations (6-5.35) and (6-5.36) give the above results, but with \( (1 - \beta) \) in Eq. (6-5.40) replaced by \( \left( \frac{3}{2} - \beta \right) \), the symbol \( \beta \) in Eq. (6-5.41) replaced by \( \left( \beta - \frac{1}{2} \right) \), and \( \eta \) replaced by \( Z \). These differences will not be important as far as calculated enthalpy and heat flux values are concerned. Figure 6-5.4 shows that the stagnation point radiant-heat flux calculated using Eq. (6-5.41) agrees quite well with the numerical calculations of Wilson and Hoshizaki\textsuperscript{71} and Howe and Viegas,\textsuperscript{72} which will be discussed in Sections 6-5C and 6-6C, respectively.

Kennet\textsuperscript{73} has calculated radiative cooling effects in a transparent spherical shock layer, including cooling along streamlines which are quite far from the stagnation streamline, as shown in Fig. 6-5.5. The enthalpy distribution in the radiation cooled shock layer was obtained by integration of the energy equation along streamlines, using Lighthill’s (constant-density) solution for the velocity field past a sphere in hypersonic flow. The calculation involves a numerical integration in the final formula for the enthalpy distribution. This analysis does not include the change in detachment distance and shock front shape produced by radiative cooling.
So far only transparent gas solutions have been discussed. For shock layers of finite optical depth Goulard approximately accounts for self-absorption by replacing the $T^4(t)$ in the integrand of Eq. (6-5.33) by the local temperature $T^4(\tau)$. If radiative cooling is small, this approximation should be good, except very near the body surface. Carrying out the integral over $E_1(|t - \tau|)$, the right-hand side of
Eq. (6-5.33) becomes $\frac{1}{2} \Gamma_\eta [E_d(\tau) + E_d(\tau_\delta - \tau)]$, which is approximated by $\Gamma_\eta E_d(\frac{1}{2} \tau_\delta)$ to within 5% for $k_\delta \Delta \approx \tau_\delta < 0.3$. Therefore, for moderately self-absorbing gases with small amounts of radiative cooling, the transparent gas results given above are applicable, but with $\Gamma_\eta$ replaced by $\Gamma_\eta E_d(\frac{1}{2} \tau_\delta)$.

Goulard discusses the case of an optically thick shock layer, for which radiative losses occur in two boundary layers, adjacent to the shock front and wall, respectively, as sketched in Fig. 6-5.6. The constant temperature $T_2$ of the interior region can be estimated from the energy equation (6-4.6) with $\mu = \lambda = 0$ and $u_1 = -V_\infty$. In the interior region (subscript 2) the net flux is zero, and at a point upstream from the shock front (subscript $\infty$) the flux is approximately $\sigma T_2$. Therefore,

$$h_2 + \frac{1}{2} V_2^2 = h_\infty + \frac{1}{2} V_\infty^2 - \frac{\sigma T_2^4}{\rho_\infty V_\infty^3}.$$  \hspace{1cm} (6-5.42)

For a strong shock $\frac{1}{2} V_2^2 \ll h_2$, so $T_2$ can be obtained directly from the above equation [$h_2 = h_2(T_2)$, which becomes $h_2 = c_p T_2$ for a perfect gas]. When the strong-shock assumption cannot be made, Eq. (6-5.42) can be combined with the conservation of mass and momentum equations to obtain an effective Rankine-Hugoniot relation for $T_2$. The term $\sigma T_2^4$ in Eq. (6-5.42) underestimates the radiation lost upstream through the shock front since the gas near the shock front will be at a temperature higher than $T_2$; see Fig. 6-5.6. However, as discussed in Section 6-4A,
\( T_s \) becomes appreciably larger than \( T_2 \) only for strong shocks with large rates of radiative cooling, for which the thickness of the cooling region is much less than a photon mean free path. Thus the flux remains nearly equal to \( \sigma T_2^4 \), and Eq. (6-5.42) should give an estimate for \( T_s \) which is only slightly above the real value. Since we can assume that the shock front is approximately normal in this stagnation flow problem, the shock boundary layer can be approximately described by the profiles of flow variables behind the infinite one-dimensional shock considered in Section 6-4. Therefore, for this optically thick case we need to consider only the boundary layer at the wall (body surface).

If the width of this wall "radiation" boundary layer is not appreciably larger than the viscous boundary layer, one should consider the radiating viscous boundary layer and the radiating viscous shock layer problems, as described in Sections 6-6B and 6-6C, respectively. Only the inviscid problem is described here. When the thickness of the wall boundary is small (less than a few photon mean free paths), the radiation flux to the wall is approximately \( \sigma T_2^4 \). On the other hand, when the thickness of the boundary layer is large (a number of photon mean free paths), the Rosseland diffusion approximation (6-2.2) may be used to determine the temperature profile and radiant-heat transfer. By studying the thermal boundary layer using the diffusion approximation, an estimate can be obtained for the condition of transition from the diffusion to the blackbody limit. This problem is similar to the transient problem considered in Section 6-2F.

Replacing the right-hand side of Eq. (6-5.29) by the diffusion approximation, and transforming from \( z \) to \( Z \) by means of Eq. (6-5.27), the following expression is obtained:

\[
\frac{\rho \alpha V \infty}{\Delta} \left( \frac{\rho}{\rho_s} \right)^{1/2} c_p Z \frac{\partial T}{\partial Z} = - \left( \frac{\rho}{\rho_s} \right)^{1/2} \frac{\partial}{\partial Z} \left[ \frac{16 \sigma T^3}{3 k_R} \left( \frac{\rho}{\rho_s} \right)^{1/2} \frac{\partial T}{\partial Z} \right].
\]  
(6-5.43)

Goulard obtained a simple solution for \( T \) in the constant density problem by assuming \( k_R \propto T^3 \). In the varying density problem above we achieve the same result by considering \( k_R \propto T^3 \rho^{1/2} \). For this special form of \( k_R \), Eq. (6-5.43) reduces to

\[
Z \frac{\partial T}{\partial Z} = - \Delta^2 \Gamma_k \frac{\partial^2 T}{\partial Z^2},
\]  
(6-5.44)

where

\[
\Gamma_k = \frac{16 \sigma T^4}{3 \rho \infty V \alpha h_b} \frac{1}{k_{R,s} \Delta} = \frac{16 Bo^{-1}}{3 \tau_{\Delta}}.
\]  
(6-5.45)
The following solution of Eq. (6-5.44) gives the temperature profile from the wall value $T_w$ to the value $T_2$ in the shock layer interior region:

$$
\frac{T - T_2}{T_w - T_2} = \text{erfc} \left( \frac{Z}{\sqrt{2} \Gamma_k} \right), \quad (6-5.46)
$$

where erfc denotes the complementary error function,

$$
erfc(x) = 1 - \text{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-t^2) \, dt.
$$

The radiant-heat flux to the wall becomes

$$
F_w = -\frac{16\sigma T_2^4}{3k_{R,s}} \left( \frac{\partial T}{\partial Z} \right)_{Z=0} = -\frac{32\sigma T_2^4 T_2}{3k_{R,s}} \frac{1}{\delta^*} 
$$

$$
= -\frac{32\sigma T_2^4}{3k_{R,s}} \left( \frac{\rho_2}{\rho_s} \right)^{1/2} \frac{1}{\delta^*}. \quad (6-5.47)
$$

The “thermal boundary layer thickness” $\delta^*$ is given by

$$
\delta^* = \frac{\Delta(2\pi\Gamma_k)^{1/2}}{[1 - (T_w/T_2)]}, \quad (6-5.48)
$$

which reduces to $\Delta(2\pi\Gamma_k)^{1/2}$ for $T_w = 0$. From Eq. (6-5.47) it is seen that the flux increases to the blackbody value $\sigma T_2^4$ as the optical depth $k_{R,s} \delta^*$ decreases to about $10(\rho_2/\rho_s)^{1/2}$. Therefore, Eq. (6-5.47) should be replaced by $F_w \simeq \sigma T_2^4$ for $k_{R,s} \delta^* \lesssim 10(\rho_2/\rho_s)^{1/2}$. Although the above analysis was for the restrictive condition $k_R \propto T^{3/2}$, it should describe the essential features of the temperature profile near the stagnation point in an optically thick, inviscid shock layer. Goulard utilized the above type of analysis to calculate the radiation regimes shown on the altitude-velocity plot in Fig. 6-1.2.

In a recent paper, Olstad75 discusses stagnation-point solutions which he has obtained for several limiting cases. He obtained a uniformly valid small perturbation solution by using the PLK perturbation of coordinate procedure to eliminate the singularity at the wall [which caused the divergence of the solution given above in Eq. (6-5.35)]. By using the differential approximation (substitute kernel procedure) to substitute for the radiative transfer term in the energy equation, Olstad also studied (1) the slightly absorbing case, (2) the optically thick case, and (3) the radiation-depleted shock layer. The solution for the slightly absorbing shock layer was obtained by retaining only the first
two terms in the expansion of the exponentials. For the optically thick case, conventional boundary layer analyses were carried out near the wall and near the shock, with solutions being obtained by an integral method. The method of matched inner and outer expansions was used to obtain a solution for the radiation-depleted shock layer (in this limit of large radiative cooling, the Boltzmann number is the small parameter used in the expansions).

6-5C Numerical solutions for blunt bodies. Here we consider the numerical calculations which have been carried out for radiating, inviscid shock layers of blunt bodies. All of these calculations assume the shock layer to be optically thin.

The first calculation was carried out by Bird, who modified his earlier adiabatic calculation to account for radiant-energy losses. In this simple computation, performed on a desk calculator, Bird starts with a prescribed shock wave and by a step-by-step procedure constructs a network of streamlines and flow normals behind the shock, using only the elementary forms of the equations of motion. Radiation is included as an energy loss term in the one-dimensional energy equation, which is applied along each streamtube. Results are described for a parabolic shock wave of 5-ft nose radius traveling at a Mach number of 20 in air at 220 K and 1 atm. For these conditions no detectable changes in the velocity or pressure fields are found, but significant changes in the temperature occur near the body surface. These results are a justification for the approximate analytical calculations described in the preceding section, since little or no departure from the nonradiating velocity or pressure field was assumed in each analytical study.

The relative independence of the pressure and velocity distributions on radiation losses permits the use of an approximate streamtube calculation. Hearne et al. have utilized an approximate streamtube method, obtaining results which are in good agreement with the more exact calculations of Wilson and Hoshizaki, discussed later in this section. In terms of the streamtube coordinates \((s, y)\) shown in Fig. 6-5.7, the conservation equations for a transparent gas are

\[
\rho u v dy = dm, \quad (6-5.49)
\]

\[
\rho u \frac{du}{ds} = - \frac{dp}{ds}, \quad (6-5.50)
\]

and

\[
\rho u \frac{d(h + \frac{1}{2}u^2)}{ds} = -4k\alpha T^4, \quad (6-5.51)
\]
where $dm$ is the mass flow rate in the streamtube, and $\nu = 0$ and 1 for two-dimensional and axisymmetric flows, respectively.

Hearne et al. introduce the following approximations: (1) the local velocity vector is parallel to the surface, (2) the normal pressure variation across the shock layer is small, (3) the surface pressure distribution is given by the adiabatic (nonradiating) flow solution, and (4) the shock layer is thin compared with the body radius. For these approximations, Eqs. (6-5.49) through (6-5.51) may be integrated to give

\[
\int_0^y \rho u \, dy = \frac{\rho_\infty V_\infty}{\nu^*} \int_0^{r_1} r_1 \, dr_1 = \frac{\rho_\infty V_\infty}{(\nu + 1)} \frac{r_1^{\nu+1}}{r^{\nu}}, \tag{6-5.52}
\]

\[
u^2 - u_1^2 = -2 \int_{r_1}^{S_1} \frac{1}{\rho} \, dp \, ds, \tag{6-5.53}
\]

and

\[
h - h_1 = -\int_{r_1}^{s} \frac{1}{\rho} \left[ \frac{4k_0 T_4}{u} - \frac{dp}{ds} \right] \, ds. \tag{6-5.54}
\]

The initial conditions (subscript i) for each streamtube at the shock front are obtained from the Rankine-Hugoniot relations. The Planck mean absorption coefficient $k$ may be given as a function of $T$ and $\rho$, and an equation of state may be used to express $T$ and $\rho$ in terms of $h$ and $\rho$. Accordingly, the following power-law relations are used:

\[
\frac{\rho}{\rho_1} = \left( \frac{p}{p_1} \right)^{\gamma_1} \frac{h_1}{h}, \tag{6-5.55}
\]

and

\[
kT_4^4 \frac{k_1 T_4^4}{k_1 T_4^4} = \left( \frac{p}{p_1} \right)^{\gamma_1} \left( \frac{h}{h_1} \right)^{\beta_1}, \tag{6-5.56}
\]
where $\alpha_i$, $\beta_i$, and $\gamma_i$ are assumed to be constant along a given streamline, but are functions of $h_i$ and $p_i$. Since the pressure distribution is assumed to be that for the adiabatic case, and therefore known, Eq. (6-5.55) may be used to express integrand of Eq. (6-5.53) in terms of only one unknown, $h$. Similarly, Eqs. (6-5.55) and (6-5.56) may be used to express the integrand in Eq. (6-5.54) in terms of the two unknowns $h$ and $u$. As only two unknowns are involved, Eqs. (6-5.53) and (6-5.54) can be integrated simultaneously by forward finite differences to obtain $h$ and $u$ along a streamline. The density $\rho$ and emission $kT^4$ are then determined from Eqs. (6-5.55) and (6-5.56), respectively. The continuity equation (6-5.52) is used with the $\rho$ and $u$ distribution along the streamline to determine the streamline paths $y = y(r, r_i)$, with $y(r_i, r_i)$ giving the shock layer thickness at $r_i$.

The radiant-heat flux calculated by this approximate streamtube method of Hearne et al. agrees well with the results of more detailed calculations; see Fig. 6-5.8, which compares the computed flux distribution over a hemispherical body with that calculated by Wilson and Hoshizaki using the integral method described below. Hearne et al. also carried out calculations for a body of 1-ft nose radius with a conical afterbody of 35° half-angle. Figure 6-5.9 shows that the gas velocity will not change by much from the adiabatic case, whereas radiative cooling will appreciably decrease the enthalpy in the inner part of shock layer. Profiles of radiant-energy emission across the shock layer are shown in Fig. 6-5.10 for several distances from the stagnation point. The hotter, inner part of the shock layer is comprised of gas which passed through the strong, curved portion of the shock front. The reduction in shock layer thickness produced by the radiative cooling is illustrated in Fig. 6-5.11.

Wilson and Hoshizaki have used the integral method of Maslen and Moeckel to calculate the inviscid, radiating flow in the shock layer of a blunt body. The integral method involves integrating the equations of motion across the shock layer, and solving the resulting total differential equations. The profiles of flow variables across the shock layer are described by polynomials, in a manner similar to the Kármán-Pohlhausen method for boundary layers. As in the streamtube calculation described above, this calculation is restricted to transparent shock layers of blunt bodies traveling at speeds great enough to produce a large increase in density across the shock front. The resulting small density ratio $\epsilon = \rho_x/\rho_s$ produces a shock layer thickness which is small compared with the nose radius. The shock layer thickness is exaggerated in Fig. 6-5.12, which illustrates the symbols and coordinate system used in this analysis.
Conditions:
Flight velocity 50,000 ft/sec
Altitude 180,000 ft
Nose radius 5 ft
Stagnation point radiative flux
$F_w(0) = 13,000$ BTU/ft$^2$-sec

Fig. 6-5.8. Radiant-heat flux distribution over a spherical body, from Hearne et al. 

Fig. 6-5.9. Tangential velocity and static enthalpy distributions across the shock layer of a blunted cone of 35° half-angle, from Hearne et al.
Fig. 6-5.10. Distributions of radiant-energy emission across the shock layer of a blunted cone of 35° half-angle, from Hearne et al. 78

Fig. 6-5.11. Distribution of shock layer thickness around a blunted cone, from Hearne et al. 78
First, the dimensional quantities (barred) are normalized according to the following relations:

\[ x = \frac{\delta}{L}, \quad y = \frac{\beta}{L}, \quad u = \frac{u}{V_\infty}, \quad v = \frac{v}{V_\infty}, \]

\[ p = \frac{\rho}{\rho_0}, \quad \rho = \frac{\rho}{\hat{\rho}}, \quad H = \frac{2[R]}{V_\infty^2}, \quad h = \frac{2[\hat{r}]}{V_\infty^2}, \]

\[ K = \frac{K}{L}, \quad \delta = \frac{\delta}{L}, \quad r = \frac{r}{L}, \]

\[ \kappa = \frac{\kappa}{\kappa_{\text{ref}}}, \quad \text{and} \quad T = \frac{T}{T_{\text{ref}}}. \quad (6-5.57) \]

Here \( \bar{H} \) is the total enthalpy, \( \bar{K} = \bar{K}(x) \) is the local surface curvature, \( \kappa \equiv \bar{k}/\bar{\rho} \) is the mass absorption coefficient (Planck mean), and the subscripts \((\delta, 0)\) denote the value just behind the shock front \((\delta)\) on the stagnation streamline \((0)\). In terms of the above nondimensional variables, the conservation equations are

mass:

\[ \frac{\partial (r \rho u)}{\partial x} + \frac{\partial (\bar{K} r \rho v)}{\partial y} = 0, \quad (6-5.58) \]

\( x \)-momentum:

\[ \rho u \frac{\partial u}{\partial x} + \bar{K} \rho u \frac{\partial u}{\partial y} + \epsilon \bar{K} \rho u v - \epsilon \frac{\partial p}{\partial x} = 0, \quad (6-5.59) \]

\( y \)-momentum:

\[ \epsilon \rho u \frac{\partial v}{\partial x} + \epsilon \bar{K} \rho u \frac{\partial v}{\partial y} - \bar{K} \rho u^2 + K \frac{\partial v}{\partial y} = 0, \quad (6-5.60) \]
energy:
\[
\rho u \frac{\partial H}{\partial x} + K \rho u \frac{\partial H}{\partial y} + \beta_0 K \rho k T^4 = 0. \tag{6-5.61}
\]

In these equations, \( K = 1 + eK_y \), and \( \beta_0 = 8L \kappa_\text{tot} T^4_{\text{ref}}/V_\infty^3 \) (\( \beta_0 \) becomes equivalent to \( \Gamma_n \) if reference quantities are taken just behind the shock front, with \( L \) taken as the body nose radius \( \tilde{R} \approx \delta/e \)).

For application of the integral method, forms of the conservation equations integrated across the shock layer must be obtained. By equating the incoming mass flux within a radius \( r_b \) to the shock layer mass flux flowing by the corresponding \( x \)-coordinate, the following integrated form of the continuity equation is obtained:
\[
r_b^{\nu+1} = (\nu + 1) \int_0^\delta \rho ur^{\nu} dy, \tag{6-5.62}
\]

which reduces to Eq. (6-5.52) if \( r^\nu \) is assumed to be constant across the shock layer (here we are interested in the axisymmetric case, \( \nu = 1 \)). Since \( r^\nu \) is retained under the integral of Eq. (6-5.62), the Dorodnitsyn variable \( \eta \) is now defined by
\[
\eta = \frac{\int_0^\delta \left( \frac{r}{r_w} \right)^\nu \left( \frac{\rho}{\rho_\infty} \right) dy}{\int_0^\delta \left( \frac{r}{r_w} \right)^\nu \left( \frac{\rho}{\rho_\infty} \right) dy} = \frac{1}{\delta} \int_0^\delta \left( \frac{r}{r_w} \right)^\nu \left( \frac{\rho}{\rho_\infty} \right) dy, \tag{6-5.63}
\]

where \( \delta(x) \) is the transformed shock layer thickness. Now dividing Eq. (6-5.62) by \( \rho_\infty u_s r_w^\nu \) and transforming to the variable \( \eta \), the following result is obtained:
\[
\left( \frac{1}{\nu + 1} \right) \left( \frac{r_b}{r_w} \right)^{\nu+1} \left( \frac{r_b}{\rho_\infty u_s} \right) = I_0, \tag{6-5.64}
\]

where
\[
I_0 = \delta \int_0^1 \left( \frac{u}{u_s} \right) d\eta. \tag{6-5.65}
\]

The conservation equations for \( y \)-momentum and mass are used to put the \( x \)-momentum equation into an appropriate form for integrating across the shock layer. First the pressure term in the \( x \)-momentum equation is written as
\[
\frac{\partial p}{\partial x} = \frac{\partial}{\partial y} \left( y \frac{\partial p}{\partial x} \right) - y \frac{\partial}{\partial x} \left( \frac{\partial p}{\partial y} \right), \tag{6-5.66}
\]

where the last term on the right is obtained by differentiating the
The $y$-momentum equation (6-5.60) with respect to $x$. In order to obtain an equation which is good to order $\epsilon$, we need retain only terms of $O(1)$ in Eq. (6-5.60), i.e., only the pressure term $K \partial p/\partial y$ and the centrifugal force term $-K \rho u^2$. The continuity equation (6-5.58) may be expressed in the form

$$
\frac{\partial}{\partial x} \left[ \left( \frac{r}{r_w} \right)^\nu \rho u \right] + \frac{\partial}{\partial y} \left[ K \left( \frac{r}{r_w} \right)^\nu \rho v \right] + \nu \left( \frac{r}{r_w} \right)^\nu \rho u \frac{dr_w}{dx} = 0. \tag{6-5.67}
$$

Combining the above equation with the $x$-momentum equation (6-5.59), and evaluating the pressure term according to Eq. (6-5.66), the following result is obtained:

$$
\frac{\partial}{\partial x} \left[ \left( \frac{r}{r_w} \right)^\nu \rho u^2 \right] + \frac{\partial}{\partial x} \left[ K^2 \left( \frac{r}{r_w} \right)^\nu \rho uv \right] + \left( \frac{r}{r_w} \right)^\nu \rho u^2 \left[ \frac{\nu K}{r_w} \frac{dr_w}{dx} - \epsilon y \frac{dK}{dx} \right] + \epsilon \frac{\partial}{\partial y} \left( y \frac{\partial p}{\partial x} \right) = 0. \tag{6-5.68}
$$

Integration across the shock layer gives, to order $\epsilon$,

$$
dI_1 = \frac{2I_1}{u_b} \frac{du_b}{dx} - \left( \frac{r_\delta}{r_w} \right)^\nu \frac{d\delta}{dx} - \epsilon \frac{dK}{dx} \int_0^1 \left( \frac{u}{u_b} \right)^2 \left[ \int_0^{\eta} \left( \frac{\rho}{\rho_b} \right) \eta \right] d\eta \\
+ K^2 \left( \frac{r_\delta}{r_w} \right)^\nu \frac{u_b}{u_b} + \nu \frac{dr_w}{dx} \int_0^1 K \left( \frac{u}{u_b} \right)^2 d\eta + \epsilon \delta \frac{\partial p}{\partial x} = 0, \tag{6-5.69}
$$

where

$$
I_1 = \delta \int_0^1 \left( \frac{u}{u_b} \right)^2 d\eta. \tag{6-5.70}
$$

Similarly, combining the continuity equation (6-5.67) with the energy equation (6-5.61) yields

$$
\frac{\partial}{\partial x} \left[ \left( \frac{r}{r_w} \right)^\nu \rho u H \right] + \frac{\partial}{\partial y} \left[ \left( \frac{r}{r_w} \right)^\nu K \rho v H \right] \\
+ \nu \left( \frac{r}{r_w} \right)^\nu \rho u H \frac{dr_w}{dx} + \left( \frac{r}{r_w} \right)^\nu K \beta_\rho \kappa T^4 = 0. \tag{6-5.71}
$$

Integration gives

$$
dI_2 + I_2 \left[ \frac{1}{u_b} \frac{du_b}{dx} + \nu \frac{dr_w}{dx} \right] - \left( \frac{r_\delta}{r_w} \right)^\nu \frac{d\delta}{dx} \\
+ \left( \frac{r_\delta}{r_w} \right)^\nu K_\delta \frac{v_b}{u_b} + \beta_0 \delta \int_0^1 K \kappa T^4 d\eta = 0, \tag{6-5.72}
$$
where

\[ I_2 = \delta \int_0^1 \left( \frac{u}{u_b} \right) \left( \frac{H}{H_b} \right) d\eta. \quad (6-5.73) \]

In order to solve the above equations, the appropriate boundary conditions must be given at the shock front and at the body surface. For a given shock shape, the values of the flow variables just behind the shock front are obtained from the Rankine-Hugoniot relations. Accordingly,

\begin{align*}
    u_b &= \sin \varphi + \xi \cos \varphi, \quad (6-5.74) \\
    v_b &= - \cos \varphi + (\xi/\epsilon) \sin \varphi, \quad (6-5.75) \\
    p_b &= (1 - \epsilon) \cos^2 \varphi + O(\epsilon^2), \quad (6-5.76)
\end{align*}

where the angle \( \xi \) shown in Fig. 6-5.12 is given by

\[ \xi = \Theta - \varphi = \tan^{-1} \left[ \frac{\epsilon}{(1 + \epsilon K \delta)} \frac{d\delta}{dx} \right] \]

\[ = \epsilon \frac{d\delta}{dx} + O(\epsilon^2). \quad (6-5.77) \]

In addition, the following expression is obtained from the energy equation (6-5.61), using the fact that the Rankine-Hugoniot relations require the total enthalpy to be constant along the shock front:

\[ \frac{dH}{d\eta} \bigg|_{\eta=1} = - \frac{(\cos \xi) \beta_0 \delta T^4}{(\cos \varphi)(r_b/r_w)^v} \quad (6-5.78) \]

where the relation \( \delta d\eta = (r_b/r_w)^v dy \) at \( y = \delta \) has been used. As discussed in the preceding section, the temperature at the stagnation point will go to zero. Since no energy absorption processes are considered, the static enthalpy will remain zero along the entire body streamline; therefore

\[ h \bigg|_{\eta=0} = 0. \quad (6-5.79) \]

For this integral method, the \( x \)-velocity and total enthalpy distributions across the shock layer are approximated by polynomials in \( \eta \). The velocity is found to be well represented by the linear profile,

\[ u/u_b = a_0 + a_1 \eta, \quad (6-5.80) \]

with the condition \( u/u_b = 1 \) at \( \eta = 1 \) providing the relation \( a_0 + a_1 = 1 \). More terms are required for the polynomial representation of the total
enthalpy profile, as it will vary considerably across the shock layer. In order to estimate the accuracy of polynomial representations, Wilson and Hoshizaki use the following two polynomials:

\[
\frac{H}{H_\delta} = b_0 + b_1 \eta + b_2 \eta^2 + b_3 \eta^3, \quad (I)
\]

\[
\frac{H}{H_\delta} = b'_1 \eta^{1/4} + b'_2 \eta^{1/3} + b'_3 \eta^{1/2} + b_0, \quad (II)
\]

with \( \frac{H}{H_\delta} = 1 \) at \( \eta = 1 \).

The integral method involves using the integrated equations of motion (6-5.64), (6-5.69), and (6-5.72) to determine the integrals \( I_0 \), \( I_1 \), and \( I_2 \) as functions of \( x \). The above polynomials are then substituted into Eqs. (6-5.65), (6-5.70), and (6-5.73) to obtain relations which, together with the boundary conditions, determine the values of the \( a_i \), \( b_i \), and \( \delta \) from the \( I_i \) at each station \( x \) about the body. Equations (6-5.69) and (6-5.72) are first-order equations which can be solved as initial value problems starting at the stagnation point. This situation is the result of the effectively parabolic nature of the governing equations when terms of \( O(e^2) \) are dropped from the \( y \)-momentum equation. At the stagnation point Eqs. (6-5.69) and (6-5.72) become singular since \( u_\delta \rightarrow 0 \) as \( x \rightarrow 0 \). However, symmetry arguments show that \( dI_1/dx \), \( dI_2/dx \), and \( d\delta/dx \) must be zero at the stagnation point. These conditions with a given shock curvature at \( x = 0 \) are sufficient to determine all stagnation point quantities. In these calculations an assumed shock shape is used, so that the quantities \( d\delta/dx \), \( (\partial p/\partial x)_\delta \), \( u_\delta \), and \( v_\delta \) are known. The term involving \( \int_0^1 (\rho_\delta/\rho_\delta') d\eta' \) in Eq. (6-5.69) is very small and a suitable approximation can be used for the purposes of the calculation (see Maslen and Moeckel). The above discussion outlines the calculation of the \( u/u_\delta \) and \( H/H_\delta \) profiles. The \( y \)-momentum equation may be integrated using the known \( u/u_\delta \) profile to determine the \( p/p_\delta \) profile, which, together with \( H/H_\delta \) and \( u/u_\delta \) profiles determines \( \rho/\rho_\delta \) as a function of \( \eta \). Then Eq. (6-5.63) can be inverted to obtain \( y(\eta) \) and the physical shock thickness \( \delta \).

An iteration procedure was used by Wilson and Hoshizaki to obtain the shock shape. Their iteration scheme is based on a convergence of the shock angle, as the solution is more strongly dependent on the shock angle than on the standoff distance \( \delta \). Figure 6-5.13 illustrates the rapid convergence of the iteration procedure.

Wilson and Hoshizaki carried out numerical calculations for the flow about a hemisphere and a 30° hemisphere-cone at flight velocities between 30,000 and 60,000 ft/sec for altitudes of 190,000 and 200,000 ft. The calculated reduction in the stagnation point radiation flux produced
by radiative cooling is shown in Fig. 6-5.4. Also, the calculated distribution of radiation flux around a hemispherical body is shown in Fig. 6-5.8. This normalized distribution was found to be relatively insensitive to the amount of radiative cooling. Just as for the solutions discussed above, it was found that radiative cooling produced little change in the normalized velocity distribution, but significantly changed the total and static enthalpy distributions. Figure 6-5.14 shows the velocity and enthalpy distributions calculated using lower and upper emissivity estimates, and the two profiles for $H/H_b$ given by Eqs. (6-5.81). It was found that profile II gives the best results for moderate radiative cooling rates ($T_n \sim 0.1$), whereas profile I gives the best results for large cooling rates ($T_n \sim 1$). The application of the integral method to a viscous shock layer is discussed below in Section 6-6C.

6-5D **PERTURBATION SOLUTIONS FOR WEDGES AND CONICAL BODIES.**

Most of this section deals with the general perturbation treatment for supersonic wedge flow given by Olfe. At the end of the section we consider perturbed conical flow for a transparent shock layer only, as presented in the appendix of Olfe. Radiative cooling in the shock layers of wedges and conical bodies is also considered in Section 6-5E, where a thin shock layer analysis is used to obtain solutions for arbitrary amounts of radiative transfer.

In the perturbation analysis, the radiative transfer term in the energy equation is assumed to be a perturbation which is determined by the
unperturbed (radiationless) flow field. The results obtained here are therefore strictly applicable only for very small amounts of radiative cooling. By considering this perturbation limit, the problem of radiating wedge flow can otherwise be solved in a general manner; i.e., we can include non-gray absorption coefficients, real-gas effects, precursor heating, and surface radiation. As shown below, this perturbation treatment reduces the equations of motion to wave equations with source terms determined by the radiative transfer. The resulting linearized equations of motion are simpler than those for the acoustic waves considered in Section 6-3, since here the radiative transfer is determined by the unperturbed quantities, rather than by the first-order perturbations. On the other hand, the boundary conditions are more complicated
in this problem, since the appropriate relations between the perturbed quantities at the shock front must be satisfied, as well as the tangential velocity condition at the wedge surface.

By substituting \( p = p_0 + p' \), \( \rho = \rho_0 + \rho' \), etc., the conservation equations (6-3.1) through (6-3.3), are linearized for steady, two-dimensional flow,

\[
\begin{align*}
\rho_0 \frac{\partial u'}{\partial x} + \rho_0 \frac{\partial v'}{\partial y} + u_0 \frac{\partial p'}{\partial x} &= 0, \\
\rho_0 u_0 \frac{\partial u'}{\partial x} &= - \frac{\partial p'}{\partial x}, \\
\rho_0 u_0 \frac{\partial v'}{\partial x} &= - \frac{\partial p'}{\partial y}, \\
\rho_0 u_0 \frac{\partial h'}{\partial x} - u_0 \frac{\partial p'}{\partial x} + \rho_0 e_0 &= 0,
\end{align*}
\]

where the subscript zero denotes unperturbed quantities, and the primes denote the perturbation quantities; the coordinate system is illustrated in Fig. 6-5.15a. The quantity \( e_0 \equiv \nabla \cdot \mathbf{F} / \rho_0 \) is the net rate of energy emission per unit mass (emission minus absorption), which is to be computed using the unperturbed quantities, \( e_0 = e_0\{T_0, \rho_0; x, y\} \). Although real-gas effects such as ionization can be easily included, as shown in Olfe, here we shall restrict ourselves to a perfect gas. For a perfect gas the static enthalpy perturbation \( h' \) is given by

\[
h' = \gamma R T' / (\gamma - 1),
\]
where the temperature perturbation $T'$ may be expressed in terms of $p'$ and $\rho'$ by means of the equation of state,

$$\frac{T'}{T_0} = \frac{p' - \rho'}{\rho_0}, \quad \text{with} \quad p_0 = \rho_0 RT_0.$$  

(6-5.87)

With the above perfect-gas relations, the energy equation (6-5.85) becomes

$$u_0 \frac{\partial p'}{\partial x} = \frac{u_0}{a_0^2} \frac{\partial \rho'}{\partial x} + \rho_0 \varepsilon,$$  

(6-5.88)

where $a_0 = (\gamma \rho_0 h_0)^{1/2}$ is the isentropic speed of sound, and $\varepsilon = (\gamma - 1) \rho_0 e_0 / \rho_0 = e_0 / h_0$ is the ratio of the net emission rate $e_0 = \nabla \cdot F / \rho_0$ to the unperturbed enthalpy $h_0$. Combining the above equation with Eqs. (6-5.82) through (6-5.84) gives

$$a^2 (M^2 - 1) \frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 p'}{\partial y^2} = -\rho_0 u_0 \frac{\partial \varepsilon}{\partial x},$$  

(6-5.89)

and

$$a^2 (M^2 - 1) \frac{\partial^2 \rho'}{\partial x^2} - \frac{\partial^2 \rho'}{\partial y^2} = \frac{\partial \varepsilon}{\partial y}.$$  

(6-5.90)

where $M = u_0 / a_0$ is the Mach number of the flow behind the shock wave. Here the supersonic case is considered, $M > 1$; therefore Eqs. (6-5.89) and (6-5.90) are wave equations with source terms $-\rho_0 u_0 \partial \varepsilon / \partial x$ and $\partial \varepsilon / \partial y$, respectively, which are determined by the unperturbed flow field. An alternative formulation consists of introducing a potential $\varphi$ defined by $p = -\rho_0 u_0 \partial \varphi / \partial x$ and $\rho' = \partial \varphi / \partial y$ (cf. Section 6-3), so that Eq. (6-5.84) is directly satisfied. Then the remaining equations of motion yield the wave equation for $\varphi$ with the source term $\varepsilon$.

The requirement of tangential velocity at the wedge surface provides the boundary condition

$$v'(x, 0) = 0.$$  

(6-5.91)

The conservation equations across the (transparent) shock front may be utilized to obtain the following relations between the perturbations at the shock front (subscript $s$):

$$\theta'_s = (v'_s / u_0) \Psi + A,$$  

(6-5.92a)

$$\rho'_s / \rho_0 = (v'_s / u_0) \Gamma + B,$$  

(6-5.92b)

$$\rho'_s / \rho_0 = (v'_s / u_0) \Delta + C,$$  

(6-5.92c)

$$u'_s / u_0 = (v'_s / u_0) \Omega + D,$$  

(6-5.92d)
where $\theta'_0$ is the perturbation of the shock angle. For a perfect gas,

$$\Omega = \frac{2(\gamma M^4 \sin^4 \alpha_0 + 1) \cot \alpha_0}{2\gamma M^4 \sin^4 \alpha_0 - (3\gamma - 1) M^4 \sin^2 \alpha_0 - (3 - \gamma) M^2 + 2},$$  

(6-5.93a)

$$\Gamma = -\left[\frac{\gamma M^2 \sin \alpha_0 \cos \alpha_0}{(\gamma - 1) M^2 \sin^2 \alpha_0 + 2}\right] [(2 + (\gamma + 1) M^2 - 2M^2 \sin^2 \alpha_0) \Omega \tan \alpha_0$$
$$- 2(1 - M^2 \sin^2 \alpha_0)],$$  

(6-5.93b)

$$\Delta = \frac{2(1 + \gamma M^2 \sin^2 \alpha_0) \cot \alpha_0 - [2(1 + \gamma M^2 \sin^2 \alpha_0) - (\gamma + 1) M^2] \Omega}{(\gamma - 1) M^2 \sin^2 \alpha_0 + 2}$$  

(6-5.93c)

$$\Psi = -\left(\frac{\gamma + 1}{2}\right) \frac{M^2 \sin^2 \alpha_0}{1 - M^2 \sin^2 \alpha_0} \left(1 + \Omega \cot \alpha_0\right),$$  

(6-5.93d)

where $\alpha_0$ is the unperturbed angular thickness of the shock layer, $\alpha_0 \equiv \theta_{s,0} - \theta_{x}$. Equations (6-5.93) reduce to $\Omega = -2\gamma \alpha_0/(\gamma - 1)$, $\Gamma = (\gamma - 1)/\alpha_0$, $\Delta = 0$, and $\Psi = (\gamma + 1)/2$ in the limit of a strong shock [$M^2 \sin^2 \alpha_0 = (\gamma - 1)/2\gamma$] and a thin shock layer ($\alpha_0 \to 0$). The quantities $A$, $B$, $C$, and $D$ depend on the perturbations ahead of the shock front, and are zero if there are no upstream perturbations. As in the other shock layer calculations discussed above, we shall neglect these upstream perturbations (precursor heating), although the quantities $A$, $B$, $C$, and $D$ have been retained in the general solutions given in Olfe.80

The general solutions of the wave equations (6-5.89) and (6-5.90) are

$$p'(x, y) = -\frac{\rho_0 u_0}{2(M^2 - 1)^{1/2}} \int \int \frac{\partial \epsilon(\xi, \eta)}{\partial \xi} d\xi d\eta$$  

(6-5.94)

and

$$v'(x, y) = \frac{1}{2(M^2 - 1)^{1/2}} \int \int \frac{\partial \epsilon(\xi, \eta)}{\partial \eta} d\xi d\eta,$$  

(6-5.95)

where the integration is taken over the upstream Mach triangle shown in Fig. 6-5.16. As in supersonic thin-wing theory, the boundary conditions are satisfied by placing sources on or outside the boundaries. The condition $v'(x, 0) = 0$ is satisfied by placing sources of strength $\epsilon(\xi, \eta) = \epsilon(\xi, -1 \eta)$ in the negative $\eta$ region. The relation between $p'$ and $v'$ at the shock front is satisfied by placing line sources (appearing as delta functions in the above equations) along the shock front $\eta = \xi \tan \alpha_0$ and its reflection $\eta = -\xi \tan \alpha_0$, with the strength $\sigma(\xi)$ of the line sources determined by the requirement that Eq. (6-5.92b)
be satisfied. Carrying out the $\xi$ integration in Eq. (6-5.94) and the $\eta$ integration in Eq. (6-5.95), then changing the $\eta$ integration in Eq. (6-5.94) to a $\xi$ integration, the following results (including the line sources) are obtained:

\[
\frac{p'}{p_0} = -\left(\frac{\rho_0 u_0}{2p_0}\right) \frac{1}{\mu^2} \left[ \int_{\xi_2}^{\infty} e^{\left\{ \xi, \frac{1}{\mu} (x - \xi) \right\}} d\xi + \int_{\xi_1}^{\xi_2} e^{\left\{ \xi, y - \frac{1}{\mu} (x - \xi) \right\}} d\xi - \int_{\xi_1}^{\xi_2} e^{\left\{ \xi, y + \frac{1}{\mu} (x - \xi) \right\}} d\xi \right] + \left(\frac{\mu}{1 + K}\right) \left( \sigma'(\xi_2) + \sigma'(\xi_1) \right) \quad (6-5.96)
\]

and

\[
\frac{v'}{u_0} = \frac{1}{2u_0\mu} \left[ \int_{\xi_2}^{\infty} e^{\left\{ \xi, y + \frac{1}{\mu} (x - \xi) \right\}} d\xi - \int_{\xi_1}^{\xi_2} e^{\left\{ \xi, y - \frac{1}{\mu} (x - \xi) \right\}} d\xi \right] + \left(\frac{\mu}{1 + K}\right) \left( \sigma'(\xi_2) - \sigma'(\xi_1) \right), \quad (6-5.97)
\]

where

\[
\mu = (M^2 - 1)^{1/2}, \quad K = (M^2 - 1)^{1/2} \tan \alpha_0, \\
\xi_1 = \left( \frac{x - \mu y}{1 + K} \right), \quad \xi_2 = \left( \frac{x + \mu y}{1 + K} \right), \\
\sigma'(\xi_{1,2}) = \sigma(\xi_{1,2}) - \sigma(0), \quad (6-5.98)
\]

with $\xi_{1,2}$ denoting either $\xi_1$ or $\xi_2$.

At the shock front $y = x \tan \alpha_0 = xK/\mu$; $\xi_1 = x(1 - K)/(1 + K)$;
and \( \xi_2 = x \); therefore substitution of Eqs. (6-5.96) and (6-5.97) into the boundary condition (6-5.92b) gives

\[
\sigma'(x) = \left( \frac{\Gamma' - 1}{\Gamma' + 1} \right) \sigma' \left( \frac{1 - K}{1 + K} \right)^{x} \\
+ \frac{1}{\mu} \left( 1 + K \right) \left( \frac{\Gamma' - 1}{\Gamma' + 1} \right) \int_{x_0}^{x} \frac{y}{x_0^{1-K}/(1+K)} \left( \frac{1 - K}{1 + K} \right)^{x_0} d\xi,
\]

where \( \Gamma' = \mu \Gamma / \gamma M^2 \). The above relation must be satisfied for all values of \( x \). Therefore, the following expression is obtained by iteration considering the points \( x = [(1 - K)/(1 + K)]^{x_1} \) (which correspond to the reflection points on the shock front \( \eta = \xi \tan \alpha_0 \) and on \( \eta = -\xi \tan \alpha_0 \) for a Mach wave running upstream from the point \( \xi_2 \) on \( \eta = \xi \tan \alpha_0 \) or from \( \xi_1 \) on \( \eta = -\xi \tan \alpha_0 \)):

\[
\sigma'(\xi_1,2) = \frac{1}{\mu} \left( 1 + K \right) \sum_{n=1}^{\infty} \left( \frac{\Gamma' - 1}{\Gamma' + 1} \right)^{n} \\
\times \left( \frac{1 - K + 1}{(1 - K)/(1 + K)} \right)^{x_1,2} \epsilon \left\{ \xi, \frac{1}{\mu} \left[ \frac{(1 - K)^{n}}{(1 + K)^{n-1}} \right] \xi_1,2 - \xi \right\} d\xi.
\]

With this equation for \( \sigma'(\xi_1,2) \), Eqs. (6-5.96) and (6-5.97) constitute solutions for \( p' \) and \( v' \), for a general distribution \( \epsilon(\xi, \eta) \) of radiant-energy loss throughout the shock layer.

The following expressions for the perturbations \( u' \) and \( \rho' \) are obtained by integrating Eqs. (6-5.83) and (6-5.88) with the conditions (6-5.92b) through (6-5.92d):

\[
\frac{u'}{u_0} = -\frac{1}{\gamma M^2} \frac{p'(x, y)}{p_0} + \left( \frac{1}{\gamma M^2} + \frac{\Omega}{\Gamma} \right) \frac{p'(y \cot \alpha_0, y)}{p_0}
\]

and

\[
\frac{\rho'}{\rho_0} = \frac{1}{\gamma} \frac{p'(x, y)}{p_0} - \left( \frac{1}{\gamma} - \frac{\Delta}{\Gamma} \right) \frac{p'(y \cot \alpha_0, y)}{p_0} + \frac{1}{u_0} \int_{y \cot \alpha_0}^{\infty} \epsilon(\xi, y) d\xi.
\]

The temperature perturbation is then determined from the equation of state (6-5.87), and the shock angle perturbation \( \theta' \) may be computed from the velocity perturbation \( v' \) at the shock front by means of Eq. (6-5.92a). The general solution for the unsteady normal shock problem considered in Olfe may be obtained by direct transformation from the above supersonic wedge solution, as discussed in Olfe.80
The integrals appearing in the above general solution have been evaluated for gases of various absorbing properties. First, consider a transparent gas, for which the emission is given by Eq. (6-2.1); i.e.,

\[ \epsilon = \frac{\nabla \cdot \mathbf{F}}{\rho_0 h_0} = 4 \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{k_p \sigma T_0^4}{\rho_0} \right), \]  

(6-5.103)

where \( k_p \) is the Planck mean absorption coefficient. Here emission from the wedge surface is neglected, as the surface will generally be appreciably cooler than the gas; see Olfe\(^80\) for a discussion of surface emission and reflection effects. Since the unperturbed flow variables are constant throughout the shock layer, the transparent gas radiation term \( \epsilon \) is constant. Thus integration of Eqs. (6-5.96), (6-5.97), and (6-5.100) for constant \( \epsilon \) yields the transparent gas results,

\[ \frac{p'}{P_0} = -4(\gamma - 1) \left( \frac{M^2}{M^2 - 1} \right) \left( \frac{\Gamma'K}{1 + \Gamma'K} \right) \left( \frac{\sigma T_0^4}{\rho_0 \mu_0} \right) k_p x \]  

(6-5.104)

and

\[ \frac{v'}{\mu_0} = -4 \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{1}{1 + \Gamma'K} \right) \left( \frac{\sigma T_0^4}{\rho_0 \mu_0} \right) k_p y. \]  

(6-5.105)

These formulae agree with those of Zhigulev et al.,\(^82\) where only the special solution for a transparent gas was obtained. The \( x \)-velocity, density, and temperature perturbations may be computed from Eqs. (6-5.101), (6-5.102), and (6-5.87), respectively. The resulting temperature perturbation may be expressed in the form

\[ \left( \frac{\rho_0 \mu_0}{\sigma T_0^4} \right) \left( \frac{\tan \alpha_0}{k_p y_s} \right) \frac{T'}{T_0} = -4 \left( \frac{\gamma - 1}{\gamma} \right) \left[ 1 + (\gamma - 1) \left( \frac{M^2}{M^2 - 1} \right) \left( \frac{\Gamma'K}{1 + \Gamma'K} \right) \right] \]

\[ - \left[ 1 - \left( \frac{M^2}{M^2 - 1} \right) \left( \frac{\Gamma'K}{1 + \Gamma'K} \right) \right] \frac{y}{y_s}, \]  

(6-5.106)

where \( y_s = x \tan \alpha_0 \) is the unperturbed shock layer thickness at a distance \( x \) from the wedge vertex. The transparent gas perturbations are plotted in Fig. 6-5.17 for the limit of a strong shock \([M^2 \sin^2 \alpha_0 = (\gamma - 1)/2\gamma]\) and a thin shock layer \((\alpha_0 \to 0)\). As expected for this limit, the velocity perturbations are of higher order [for \( \gamma \to 1 \) the pressure perturbation also becomes of higher order than the density and temperature perturbations; see Eqs. (6-5.123)]. The parameter \((\sigma T_0^4/\rho_0 \mu_0)\) appearing in the above equations is equal to \(\gamma/(\gamma - 1)\) times the reciprocal of the Boltzmann number, \( Bo^{-1} \equiv \sigma T_0^4/\rho_0 \mu_0 h_0 \).
We may easily calculate the reduction in radiant-heat transfer to the wedge surface produced by the above flow field perturbations. Using the notation of Section 6-5B, $\vec{R}_p \propto \rho^{n+1}T^\beta$, and integrating across the shock layer from $y = 0$ to $y_s = x \tan \alpha_0 + \int_0^x \dot{\theta}_s(x') \, dx' \simeq x \tan \alpha_0 [1 + (\dot{\theta}_s(x)/2\alpha_0)]$, for small $\alpha_0$,

$$F_w(x) = \frac{F_w(x)}{F_{w,0}(x)} = \int_0^{1+\theta_c(x)/2\alpha_0} \left(1 + \frac{T'}{T_0}\right) \left(1 + \frac{\rho'}{\rho_0}\right)^{n+1} d\eta$$

$$= 1 + \frac{1}{2} \frac{\theta_c(x)}{\alpha_0} + (n + 1) \frac{\rho'(x)}{\rho_0} + (\beta - n + 3) \frac{T'(x, \eta = \frac{1}{2})}{T_0},$$

(6-5.107)

where $\eta = y/x \tan \alpha_0$. For a strong shock and thin shock layer, the above expression becomes

$$F_w = 1 - \left[\beta + 4 + \frac{(\gamma - 1 - 2n)}{2(2\gamma - 1)}\right] \gamma \Gamma(x)$$

![Diagram](image)

**Fig. 6-5.17.** The perturbations across a wedge shock layer in the limit of a strong shock and a thin shock layer ($\alpha_0 \rightarrow 0$). Computed for a transparent, perfect gas with $\gamma = 1.67$; from Olfe.59
and

\[ F_w \rightarrow 1 - (\beta - \eta + 4) \Gamma_n(x) \quad \text{for} \quad \gamma \rightarrow 1. \quad (6-5.108) \]

Here the definition of the radiation parameter \( \Gamma_n(x) = (4\sigma T_0^4/\rho_0 u_0 k_0)(\bar{k}_p x/2) \) is based on the mass flow rate \( \rho_0 u_0 \) behind the shock and on the average distance \( x/2 \) traveled in the shock layer by the gas at station \( x \). The reduced heat transfer \( F_w \) for the transparent shock layer of a wedge is thus given by an expression (6-5.108), which is nearly identical in form to the relation (6-5.39) for stagnation flow.

Since only relatively thin shock layers are of interest, say \( \alpha_0 < 25^\circ \), it can be shown that the gray-gas radiative transfer term \( \varepsilon \) is accurately represented by the one-dimensional expression (\( k \equiv \text{gray absorption coefficient} \)),

\[ \varepsilon = 2 \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{k_0 T_0^4}{\rho_0 u_0} \right) [E_2(k \gamma) + E_2(k(x \tan \alpha_0 - y))]. \quad (6-5.109) \]

Substitution of this \( \varepsilon \) function into Eqs. (6.5.96), (6.5.97), and (6.5.100) gives the following gray-gas perturbations:

\[ \frac{p'}{p_0} = - (\gamma - 1) M^2 \left( \frac{\sigma T_0^4}{\rho_0 u_0} \right) \frac{1}{\mu} \left[ \left( \frac{2 + K}{1 + K} \right) - E_3 \left( \frac{K}{1 + K} k \left( \frac{x}{\mu} + y \right) \right) \right] \\
- E_3 \left( \frac{K}{1 + K} k \left( \frac{x}{\mu} - y \right) \right) + \left( \frac{2K}{1 - K^2} \right) E_3 \left( K \left( \frac{x}{\mu} - y \right) \right) \\
- \left( \frac{2}{1 - K^2} \right) E_3 \left( K k \left( \frac{x}{\mu} - y \right) \right) \\
+ \sum_{n=1}^{\infty} \left( \frac{n - 1}{n + 1} \right) \left[ 2 \left( \frac{2 - K}{1 - K^2} \right) - E_3 \left( K \left( \frac{1 - K^n}{1 + K^{n+1}} \right) k \left( \frac{x}{\mu} + y \right) \right) \right] \\
- E_3 \left( K \left( \frac{1 - K^n}{1 + K^{n+1}} \right) k \left( \frac{x}{\mu} - y \right) \right) - E_3 \left( K \left( \frac{1 - K^{n-1}}{1 + K^n} \right) k \left( \frac{x}{\mu} + y \right) \right) \\
- E_3 \left( K \left( \frac{1 - K^n}{1 + K^{n+1}} \right) k \left( \frac{x}{\mu} - y \right) \right) \\
- \left( \frac{2}{1 - K^2} \right) E_3 \left( K \left( \frac{1 - K^n}{1 + K} \right) k \left( \frac{x}{\mu} + y \right) \right) \\
+ E_3 \left[ K \left( \frac{1 - K^n}{1 + K} \right) k \left( \frac{x}{\mu} - y \right) \right] \right]. \quad (6-5.110) \]
\[
\frac{v'}{u_0} = - \left( \frac{\gamma - 1}{\gamma} \right) \left( \frac{\sigma T_0^4}{p_0 u_0} \right) \left\{ 1 - 2E_3(\kappa y) + E_3 \left( \frac{K}{1 + K} \right) k \left( \frac{x}{\mu} + y \right) \right\} \\
- E_3 \left( \frac{2}{1 - K^2} \right) \left[ E_3 \left( K \frac{x}{\mu} - y \right) \right] + \left( \frac{2}{1 - K^2} \right) \left[ E_3 \left( K \frac{x}{\mu} - y \right) \right] - E_3 \left( Kk \left( \frac{x}{\mu} - y \right) \right] \\
+ \sum_{n=1}^{\infty} \left( \frac{l^n - 1}{l^n + 1} \right) \left[ E_3 \left( K \frac{(1 - K)^n}{(1 + K)^{n+1}} \right) k \left( \frac{x}{\mu} + y \right) \right] \\
- E_3 \left( K \frac{(1 - K)^n}{(1 + K)^{n+1}} \right) k \left( \frac{x}{\mu} - y \right) \right\} \\
+ E_3 \left( K \frac{(1 - K)^{n-1}}{1 + K} \right) k \left( \frac{x}{\mu} + y \right) \right\} - E_3 \left( K \frac{(1 - K)^{n-1}}{1 + K} \right) k \left( \frac{x}{\mu} - y \right) \right\} \\
+ \left( \frac{2}{1 - K^2} \right) \left( E_3 \left( K \frac{1 - K}{1 + K} \right) k \left( \frac{x}{\mu} + y \right) \right] \\
- E_3 \left( K \frac{1 - K}{1 + K} \right) k \left( \frac{x}{\mu} - y \right) \right\]. \\
(6-5.111)
\]

These expressions reduce to the transparent gas results (6-5.104) and (6-5.105) when the shock layer becomes transparent, i.e., when \( Kkx/\mu \equiv ky_s \to 0 \). Figure 6-5.18 shows that the pressure perturbation builds up linearly with increasing \( x \) and is independent of \( y \) for small \( x \) (transparent shock layer), whereas at very large distances \( x \) from the wedge vertex (optically thick shock layer) the pressure perturbation is constant except within a distance of about two photon mean free paths of the shock front. The \( E_3 \{ k[K(\chi/\mu) - y] \} \equiv E_3 \{ k(y_s - y) \} \) term in Eq. (6-5.110) produces this profile at the shock front. The \( y \)-velocity perturbation shown in Fig. 6-5.19 increases linearly with \( y \), independent of \( x \), in the transparent regime (dashed line), whereas when \( x \) is large, \( v' \) is constant except in boundary regions at the wedge surface and shock front. It is seen from Eq. (6-5.111) that the wedge boundary profile is given by the \( E_3 \{ ky \} \) term, whereas the shock boundary profile is given by the \( E_3 \{ k[K(\chi/\mu) - y] \} \equiv E_3 \{ k(y_s - y) \} \) term.

Since the shock angle perturbation \( \theta' \) is proportional to the value of \( v' \) at the shock front [Eq. (6-5.92a)], reference to Fig. 6-5.19 shows that \( -\theta'_s \) will increase linearly with \( y_s \) for small \( ky_s \) and will level off to a constant value when \( ky_s \gtrsim 5 \), i.e., when the shock layer thickness...
becomes greater than about five photon mean free paths. The perturbations $u'$, $\rho'$, and $T'$ for a gray gas may be determined from Eqs. (6-5.101), (6-5.102), and (6-5.87), respectively. Figure 6-5.20 shows the perturbations across the shock layer for a gray gas in the optically thick limit. The perturbation profiles remain constant, except for the temperature
and density perturbation profiles near the wedge surface, which increase linearly with $x$ according to

$$\frac{T'}{T_0} = -\frac{\rho'}{\rho_0} = -2 \left(\frac{\gamma - 1}{\gamma}\right) \left(\frac{\sigma T_0^4}{\rho_0 u_0}\right) k x E_2(\text{ky}).$$  \hfill (6-5.112)

This linear change in $T'$ and $\rho'$ is produced by the last term of Eq. (6-5.102), which accounts for radiation losses at constant pressure. In contrast to the above perturbation near the wedge surface, the temperature perturbation near the shock front depends on the shock wave properties, and in the optically thick limit is given by

$$\left(\frac{\rho_0 u_0}{\sigma T_0^4}\right) \tan \alpha_0 \frac{T'}{T_0} = -\left(\frac{\gamma - 1}{\gamma}\right) \left(\frac{M^2}{1 - M^2}\right) \left(1 - \frac{\Delta}{\Gamma}\right) \left(\frac{K^2}{1 - K^2}\right)$$

$$+ \frac{\gamma}{\gamma - 1} \left(\frac{2 - K^2}{1 - K^2}\right)$$

$$- 2 \left(1 - (\gamma - 1) \left(\frac{M^2}{1 - M^2}\right) \left(\frac{K^2}{1 - K^2}\right) \right) E_3(h(y_0 - y)).$$ \hfill (6-5.113)

The analysis in Olfe\textsuperscript{80} also considers radiation from nonoverlapping, collision-broadened lines which are appreciably self-absorbed. The resulting perturbed variables have profiles which are similar at different $x$-stations, with the perturbations being proportional to $\sqrt{x}$ (as compared with the transparent gas case in which the perturbations are proportional to $x$).

It should be noted that the perturbation results for different absorption models and different absorption values may be added to give results for a more general non-gray spectrum. For this purpose, the above expressions may be converted to formulae for finite bands by replacing $\sigma T_0^4$ by the integral of the blackbody radiancy $R_0 = \pi B_0$ over the frequency bands of interest. In this manner the gray-gas formulae approximate bands of continuum radiation or bands of well-overlapped lines. Thus combining the results for different gray bands with the results for different bands of nonoverlapping lines, a fairly general spectrum of continuum and line radiation may be represented.

We shall now consider radiation-perturbed flow in a transparent shock layer of a conical body. In spherical coordinates (see Fig. 6-5.15b), the conservation equations for steady, axisymmetric flow of a radiating, transparent gas are

$$\rho \left(u \frac{\partial h}{\partial r} + \frac{v}{r} \frac{\partial h}{\partial \theta}\right) - \left(u \frac{\partial p}{\partial r} + \frac{v}{r} \frac{\partial p}{\partial \theta}\right) = -4k_0 \sigma T,$$ \hfill (6-5.114)
FIG. 6-5.20. The perturbations across a wedge shock layer in the limit of a strong shock and a thin shock layer ($s_0 \rightarrow 0$). Computed for an optically thick, perfect gas with $\gamma = 1.67$; from Olfe.80

\[
\frac{\partial}{\partial r} (\rho u^2 \sin \theta) + \frac{\partial}{\partial \theta} (\rho u v \sin \theta) = 0, \quad (6-5.115)
\]

\[
\rho \left( u \frac{\partial u}{\partial r} + \frac{v}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\rho v^2}{r} = - \frac{\partial p}{\partial r}, \quad (6-5.116)
\]

and

\[
\rho \left( u \frac{\partial v}{\partial r} + \frac{v}{r} \frac{\partial v}{\partial \theta} \right) + \frac{\rho u v}{r} = - \frac{1}{r} \frac{\partial p}{\partial \theta}, \quad (6-5.117)
\]

where $u$ and $v$ now denote velocities in the $r$ and $\theta$ directions, respectively. For linearization the flow variables may be written in the form $p = p_0 + p'$, $\rho = \rho_0 + \rho'$, etc.; however, now the quantities for the nonradiating flow (subscript zero) are functions of $\theta$. Linearization of the above conservation equations yields partial differential equations for the perturbation variables. These equations are reduced to ordinary differential equations by writing the perturbations as linear functions.
of \( r \); e.g., \( p' = P(\theta) r \) and \( \rho' = R(\theta) r \). Including the perfect-gas
equations (6-5.86) and (6-5.87), the resulting equations become

\[
\rho_0(u_0H + v_0H') - (u_0P + v_0P') + v_0h'\rho = -4k_{P,0}\sigma T_0^4,
\]

\[
3\rho_0U + \rho_0V' + (\rho_0 + \rho_0 \cot \theta) V + v_0R' + (3u_0 + v_0 + v_0 \cot \theta) R = 0,
\]

\[
v_0U' + u_0U + (u_0 - 2v_0) V + \frac{1}{\rho_0} P = 0,
\]

\[
v_0U + v_0V' + (v_0 + 2u_0) V + \frac{1}{\rho_0} P' - \frac{\rho_0'}{\rho_0} R = 0,
\]

and

\[
\frac{R}{\rho_0} = \frac{P}{\rho_0} - \frac{H}{\rho_0}.
\]

(6-5.118)

where the prime now denotes differentiation with respect to \( \theta \).

Now radiative transfer will be important only for strong shock waves,
which yield a small density ratio \( \epsilon \equiv \rho_\infty/\rho_s \). This small density ratio
produces a thin shock layer, since \( \theta_s - \theta_c \approx \frac{1}{2} \epsilon \tan \theta_c \) for a nonradiating
conical shock layer. The unperturbed variables may be determined as
functions of \( \theta \) from the radiationless conservation equations. In terms
of power series in \( \epsilon \),

\[
u_0/\nu_s = 1 + \frac{1}{4} \epsilon^2 \tan^2 \theta_c (1 - \varphi^2) + \cdots,
\]

\[
v_0/\nu_s = -\epsilon \tan \theta_c \varphi + \cdots,
\]

\[
\rho_0/\rho_s = 1 + \frac{1}{4} \epsilon (1 - \varphi^2) + \cdots,
\]

\[
\rho_0/\rho_s = 1 + \frac{1}{4} \epsilon (1 - \varphi^2) + \cdots,
\]

\[
h_0/h_s = T_0/T_s = 1 + \frac{1}{4} \epsilon^2 (1 - \varphi^2) + \cdots,
\]

(6-5.119)

where \( \varphi \equiv (\theta - \theta_c)/\frac{1}{2} \tan \theta_c \); i.e., \( \varphi = \gamma/\gamma_s \) to lowest order in \( \epsilon \).

Substitution of Eqs. (6-5.119) into Eqs. (6-5.118) gives five equations
for the perturbation quantities \( P, R, H, U, \) and \( V \).

At the shock front the conservation equations give the following
strong shock relations to lowest order in \( \epsilon \) [cf. Eqs. (6-5.92)]:

\[
R_s = 0, \quad \frac{P_s}{\rho_0} = \frac{5}{3} \cot \theta_c \frac{V_s}{u_0},
\]

\[
\frac{U_s}{u_0} = -\frac{2}{3} \tan \theta_c \frac{V_s}{u_0}, \quad \theta_s' = \frac{dy_s}{dr} - \frac{1}{2} \epsilon \tan \theta_c = \frac{2}{3} \left( \frac{V_{0'}}{u} \right).
\]

(6-5.120)

(6-5.121)

At the cone surface

\[ V(\varphi = 0) = 0. \]
The solution for each perturbation quantity has the form of a power series in $\epsilon$ and $\varphi^{1/2}$; e.g., $P = \sum_m \sum_n \rho_{m,n} \epsilon^m \varphi^{n/2}$, where the coefficients $\rho_{m,n}$ are calculated by substitution into Eqs. (6-5.118) through (6-5.121) and solving to each order in $\epsilon$. The following perturbation solutions are obtained by solving the equations to first (lowest) order, and, in some cases, second order in $\epsilon$ [the boundary conditions (6-5.120) are sufficient for the determination of the solutions given below; however, higher-order solutions would require the inclusion of higher-order terms in the boundary conditions]:

\[
\frac{h'}{h_s} = \frac{T'}{T_s} = -\frac{\rho'}{\rho_s} = -2\Gamma_n(1 - \varphi^{1/2}), \quad \frac{\varphi'}{u_s} = -\epsilon\Gamma_n \tan \theta_c \varphi
\]

\[
\frac{\rho'}{\rho_s} = -\epsilon\Gamma_n \left(\frac{16}{15} + \varphi^2 - \frac{2}{5} \varphi^{5/2}\right), \quad \frac{u'}{u_s} = \frac{2}{3} \epsilon\Gamma_n \tan^2 \theta_c \varphi^{1/2}
\]

\[
\bar{y}_s = \frac{y_s}{y_{s,0}} \sim \left(\frac{\theta_s - \theta_c}{\frac{1}{2} \epsilon \tan \theta_c}\right) = 1 - \frac{2}{3} \Gamma_n, \quad \frac{F_w}{F_{w,0}} = \frac{F_w}{F_{w,0}} = 1 - \frac{2}{3}(\beta - n + 4) \Gamma_n
\]

where

\[h_s = \frac{\gamma R T_s}{\gamma - 1} = \frac{1}{2} V_\infty^2 \sin^2 \theta_c, \quad \rho_s = \frac{\rho_\infty}{\epsilon}, \quad u_s = V_\infty \cos \theta_c,
\]

\[\rho_s = \rho_\infty V_\infty^2 \sin^2 \theta_c.
\]

Here the definition of $\Gamma_n$ involves the values at the shock front, $\Gamma_n \equiv (4\sigma T_s^4 / \rho_s u_s h_s) (k_p s)^{n/2}$. The radiant-heat transfer has been evaluated as in Eq. (6-5.107) by assuming $k_p \propto \rho^{n+1} T^\beta$ (this variation of the absorption coefficient will also affect higher orders of the other perturbations, since the emission term is constant across the shock layer only to lowest order).

Equations (6-5.122) show that the velocity and pressure perturbations are of higher order in $\epsilon$ than the temperature and density perturbations. The corresponding perturbations in a thin, transparent shock layer of a wedge are

\[
\frac{h'}{h_s} = \frac{T'}{T_s} = -\frac{\rho'}{\rho_s} = -2\Gamma_n(1 - \varphi), \quad \frac{\varphi'}{u_s} = -2\epsilon\Gamma_n \tan \theta_w \cdot \varphi,
\]

\[
\frac{\rho'}{\rho_s} = -4\epsilon\Gamma_n, \quad \frac{u'}{u_s} = 2\epsilon\Gamma_n \tan^2 \theta_w \cdot \varphi,
\]

\[
\bar{y}_s = 1 - 2\Gamma_n, \quad \frac{F_w}{F_{w,0}} = 1 - (\beta - n + 4) \Gamma_n
\]

(6-5.123)
where

\[ h_s = \frac{\gamma R T_s}{\gamma - 1} = \frac{V_\infty^2}{\epsilon} \sin^2 \theta_w, \quad \rho_s = \frac{\rho_\infty}{\epsilon}, \quad u_s = V_\infty \cos \theta_w, \]

\[ p_s = \rho_\infty V_\infty^2 \sin^2 \theta_w. \]

The above expression for \( F_w \) agrees with Eq. (6-5.108), and the other perturbations in Eq. (6-5.123) may be obtained from the previous wedge analysis by considering the limit of a strong shock wave \([M^2 \sin^2 \alpha_0 = (\gamma - 1)/2\gamma]\) and a thin shock layer produced by a small value of \( \epsilon[\alpha_0 = \epsilon \tan \theta_w \ll 1 \) produced by \( \epsilon \ll 1; \) i.e., \( \gamma \approx 1 \) and \( (\gamma - 1) \approx 2\epsilon \). The perturbations in flow variables given above for the wedge and cone cases are plotted on the left-hand sides of Figs. 6-5.21 and 6-5.22, respectively.

![Fig. 6-5.21](image)

Fig. 6-5.21. Radiation-produced changes in the flow variables across the thin, transparent shock layer of a wedge, calculated (a) for vanishingly small values of the radiation parameter \( \Gamma_n \to 0 \), and (b) for the relatively large value, \( \Gamma_n = 1 \); from Olfe.81

In the next section we start with the thin shock layer assumption and then carry out solutions for arbitrary amounts of radiative transfer.

6-5E THIN SHOCK LAYER SOLUTIONS FOR WEDGES AND CONICAL BODIES. Here we follow the treatment given by Olfe81 for thin, trans-
parent shock layers of wedges and cones. A more general treatment for pointed and blunt bodies has been given recently by Wang,\textsuperscript{83} who has carried out calculations for a wedge, cone, and sphere. Wang has also considered the effects of absorption.\textsuperscript{84}

In our analysis, the following equation of state is used:

\[
\left(\frac{\rho}{\rho_s}\right) = \left(\frac{\rho}{\rho_s}\right)^m \left(\frac{h}{h_s}\right)^{-1}
\]  

\text{(6-5.124)}

where again the subscript \( s \) denotes values just behind the shock front. For \( m = 1 \), Eq. (6-5.124) reduces to the perfect-gas relation. A \(-1\) power has been used for the enthalpy in Eq. (6-5.124); however, the use of a more general \(-n\) power would not change the calculation procedure given below, but it would yield more complicated formulae. The emission is assumed to follow the power law

\[
\frac{kT^4}{k_s T_s^4} = \left(\frac{\rho}{\rho_s}\right)^\beta \left(\frac{h}{h_s}\right)^\alpha.
\]  

\text{(6-5.125)}

The exponent \( \alpha \) is related to the exponents \( \beta \) and \( \bar{\beta} \) given previously by \( \alpha = \beta - 1 = \bar{\beta} - n + 3 \).
As mentioned in the preceding section, strong shock waves are required to produce sufficient radiation to affect the flow. The density ratio $\epsilon = \rho_w/\rho_s$ across a strong shock is small, yielding thin shock layers of angular thicknesses $\theta_s - \theta_w = \epsilon \tan \theta_w$ and $\theta_s - \theta_c = \frac{1}{2} \epsilon \tan \theta_c$ in the nonradiating wedge and cone cases, respectively (radiation will decrease the shock layer thicknesses further). The small density ratio $\epsilon$ may be used as an expansion parameter for the flow variables in the shock layer, viz.,

\[
\begin{align*}
  h &= h_0 + \epsilon h_1 + \epsilon^2 h_2 + \cdots, \\
  \rho &= \rho_0 + \epsilon \rho_1 + \epsilon^2 \rho_2 + \cdots, \\
  p &= p_0 + \epsilon \rho_1 + \epsilon^2 p_2 + \cdots, \\
  u &= u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots, \\
  v &= \epsilon v_1 + \epsilon^2 v_2 + \cdots.
\end{align*}
\]

In the above expansions the lowest-order terms for the pressure and velocity are assumed to be the constant, Newtonian values:

\[
\begin{align*}
  p_0 &= \rho_0^2 V_\infty \sin^2 \theta_{w,c}, \\
  u_0 &= V_\infty \cos \theta_{w,c}, \\
  v_0 &= 0,
\end{align*}
\]

where $\theta_{w,c}$ represents the wedge or cone angle, and $u$ and $v$ are the corresponding velocity components shown in Fig. 6-5.15. The use of Newtonian values for the pressure and velocity in the lowest-order approximation is standard shock layer procedure, and gives consistent results in this radiating flow problem. This invariance (to lowest order) of pressure and velocity to the radiative transfer is observed in the other shock layer calculations described in this chapter.

Let us now consider the wedge case, which is described by the steady, two-dimensional forms of conservation equations (6-3.1) through (6-3.3). Since the shock layer angular thickness is of order $\epsilon$, we replace $y$ by $\eta = y/\epsilon$ in order to have independent variables $x$ and $\eta$ which are the same order of magnitude in the shock layer. The expansions (6-5.126) are then substituted into the conservation equations, and the coefficient of each power of $\epsilon$ is set equal to zero. The lowest-order terms ($\epsilon^0$) in the energy equation, combined with the lowest-order terms of Eqs. (6-5.124) and (6-5.125), give

\[
\left( \frac{\partial}{\partial x} + \frac{v_1}{u_0} \frac{\partial}{\partial \eta} \right) \left( \frac{h_0}{h_s} \right) = - c \left( \frac{h_0}{h_s} \right)^\alpha, \quad \text{(6-5.127)}
\]

where $c \equiv (4\sigma T^\alpha_s/\rho_s u_s h_s) k_s \equiv 4Bo^{-1}k_s$. The $h_0$ and $v_1$ terms are determined by solving the energy equation (6-5.127) together with the
continuity equation (the momentum equations affect only higher-order terms in the enthalpy and \( y \)-velocity distributions). Since we are considering the strong shock limit, \( M_\infty \sin \theta_w \gg 1 \), the shock relations yield a constant density \( \rho_s = \rho_\infty / \epsilon \) along the entire shock front, and to lowest order in \( \epsilon \) the enthalpy will also be constant along the shock front, \( h_s = V_\infty^2 \sin^2 \theta_w / 2 \).

Continuity is conveniently satisfied by introducing the stream function \( \psi \),

\[
\begin{align*}
  u &= \frac{\rho_s}{\rho} \frac{\partial \psi}{\partial y}, \\
  v &= -\frac{\rho_s}{\rho} \frac{\partial \psi}{\partial x}.
\end{align*}
\]  

In order to integrate Eq. (6-5.127), a von Mises transformation of the independent variables is carried out, \((x, \eta) \rightarrow (x, \psi)\). The derivatives transform in the following manner:

\[
\left( \frac{\partial}{\partial x} \right)_\eta + \frac{v_1}{u_0} \left( \frac{\partial}{\partial \eta} \right)_x = \left( \frac{\partial}{\partial x} \right)_\psi
\]

and

\[
\frac{\rho_s}{\rho_0} \frac{1}{\epsilon \eta_0} \left( \frac{\partial}{\partial \eta} \right)_x = \left( \frac{\partial}{\partial \psi} \right)_x.
\]

Using Eq. (6-5.129), the energy equation (6-5.127) may be integrated to give

\[
\left( \frac{h_0}{h_s} \right)^{-\alpha} = 1 + \alpha \eta (x - x_s)
\]

where the shock position \( x_s \) (cf. Fig. 6-5.15a) must now be determined. In this paper we assume \( \alpha > 1 \), which does not limit the usefulness of this analysis as \( \alpha \) is a fairly large positive number for air radiation in the temperature range of interest; the values \( \alpha = 4 \) and \( 8 \) will be used later for representative calculations.

Conservation of mass may be utilized to relate the shock position \( x_s \) to the \( \psi \) value of the streamline under consideration. To lowest order, we equate the mass flux through the shock front between 0 and \( x \) to the mass flux in the shock layer at the station \( x \) (see Fig. 6-5.15a):

\[
\rho_\infty V_\infty x_s \sin \theta_w = \int_0^{\psi_s} \rho u dy = \rho_s \psi
\]

or

\[
x_s = \frac{\psi}{\epsilon V_\infty \sin \theta_w}.
\]  

(6-5.132)
Now a relation between $x$, $y$, and $\psi$ is obtained by integrating Eq. (6-5.128) for $u$ to lowest order,

$$ y = \frac{\rho_s}{u_0} \int_0^\psi \rho_0^{-1}(x, \psi) \, d\psi = \frac{1}{u_0} \int_0^\psi \frac{h_0}{h_s} \, d\psi $$

$$ = \frac{\epsilon \tan \theta_w}{c(x - 1)} \left(1 + c\alpha x^{(a-1)/\alpha} - \left[1 + c\alpha \left(\frac{y}{\epsilon \tan \theta_w} \right)^{(a-1)/\alpha}\right]\right). $$

$$ \left(6-5.133\right) $$

Using Eqs. (6-5.132) and (6-5.133) to substitute for $x_s$ in Eq. (6-5.131) the enthalpy distribution becomes

$$ \frac{h_0}{h_s} = \left[1 + c\alpha x^{(a-1)/\alpha} - c(x - 1) \left(\frac{y}{\epsilon \tan \theta_w}\right)^{(a-1)/\alpha}\right]^{-1}/\alpha. $$

$$ \left(6-5.134\right) $$

The shock front position is determined by setting $h_0/h_s = 1$ in Eq. (6-5.134),

$$ y_s = \frac{y_s}{x_0 \tan \theta_w} = \frac{(1 + 2\alpha \Gamma_n)^{(a-1)/\alpha} - 1}{2(\alpha - 1) \Gamma_n}, $$

$$ \left(6-5.135\right) $$

where $\Gamma_n$ is the radiation parameter given in the preceding section, $\Gamma_n = (4\sigma T_0^4/\rho_0 u_s h_s)(\kappa_p u_0 x/2) = 1/cx$, and $x_0 \tan \theta_w$ is the shock position $y_{s0}$ for a nonradiating gas ($\Gamma_n = 0$). Equation (6-5.134) may now be written in terms of $\Gamma_n$ and the fractional distance $\varphi = y/y_s$ across the shock layer,

$$ \frac{h_0}{h_s} = (1 + 2\alpha \Gamma_n)^{-1/\alpha} \left[1 - \left[1 - (1 + 2\alpha \Gamma_n)^{-1/\alpha}\right] \varphi_j \right]^{-1/(\alpha-1)}. $$

$$ \left(6-5.136\right) $$

To lowest order $\rho_0/\rho_s = (h_0/h_s)^{-1}$.

The pressure and velocity components at the shock front may be determined through order $\epsilon$ by applying the conservation equations across the shock front, which has a slope $\theta_w + (dy_s/dx)$ determined from Eq. (6-5.135),

$$ \frac{p_s}{p_0} = 1 + \epsilon [2(1 + 2\alpha \Gamma_n)^{-1/\alpha} - 1], $$

$$ \left(6-5.137a\right) $$

$$ \frac{u_s}{u_0} = 1 - \epsilon \tan^2 \theta_w (1 + 2\alpha \Gamma_n)^{-1/\alpha}, $$

$$ \left(6-5.137b\right) $$

$$ \frac{v_s}{u_0} = \epsilon \tan \theta_w [(1 + 2\alpha \Gamma_n)^{-1/\alpha} - 1], $$

$$ \left(6-5.137c\right) $$
where \( p_0 \) and \( u_0 \) are the Newtonian values, \( \rho_\infty V_\infty^2 \sin^2 \theta_w \) and \( V_\infty \cos \theta_w \), respectively. Solving Eq. (6-5.133) for \( \psi(x, y) \), \( v(x, y) \) is obtained by differentiating \( \psi \) with respect to \( x \) and substituting for \( \rho_s/\rho_0 = h_0/h_s \) using Eq. (6-5.134). For simplicity we express the result in terms of the enthalpy distribution (6-5.136),

\[
\frac{v}{u_0} = -\epsilon \tan \theta_w \left( \frac{h_0}{h_s} - \frac{h_w}{h_s} \right).
\] (6-5.138)

The quantity \( h_w \) denotes the enthalpy at the wedge surface: \( h_w/h_s = (1 + 2\alpha \Gamma_r)^{-1/\alpha} \). It is noted that boundary condition (6-5.137c) is satisfied, as well as the condition \( v = 0 \) on the wedge surface.

As one would expect from the derivation of Eq. (6-5.133) for the stream function, substitution of \( \psi \) into Eq. (6-5.128) gives \( u \) only to lowest order, \( u = u_0 \). In order to determine \( u \) through order \( \epsilon \), the momentum equations must be utilized. First, the pressure distribution is calculated by considering the \( y \)-momentum equation to lowest order

\[
\rho_0 u_0 \left( \frac{\partial}{\partial x} + \frac{v_1}{u_0} \frac{\partial}{\partial \eta} \right) v_1 = -\frac{1}{\epsilon} \frac{\partial p_1}{\partial \eta}. 
\] (6-5.139)

Multiplying by \( \rho_0/\rho_0 u_0 \) and carrying out transformations (6-5.129) and (6-5.130), this equation becomes \( (\partial p_1/\partial \psi)_x = -\rho_s(\partial v_1/\partial x)_x \). Substituting \( v_1 \) as a function of \( x \) and \( \psi \), differentiating with respect to \( x \), and integrating with respect to \( \psi \) gives

\[
\frac{p_1}{p_0} = P(x) - \left( 1 + \epsilon \alpha \left[ x - \left( \frac{\psi}{\epsilon u_0 \tan \theta_w} \right) \right] \right)^{-1/\alpha} \left[ 1 + \epsilon \alpha (1 + \epsilon \alpha) \right], 
\] (6-5.140)

The function \( P(x) \) of integration is determined from the boundary condition (6-5.137a) at the shock front, where \( \psi = x \epsilon u_0 \tan \theta_w \). Expressed in terms of the enthalpy distribution, the resulting pressure distribution is

\[
\frac{p_1}{p_0} = 1 + \epsilon \left[ 2 \left( \frac{h_w}{h_s} \right) - \left( \frac{h_0}{h_s} \right) - \frac{1}{\alpha} \left( \frac{h_w}{h_s} \right)^{(\alpha+1)} \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - 1 \right] \right].
\] (6-5.141)

The \( x \)-momentum equation to lowest order is

\[
\rho_0 u_0 \left( \frac{\partial u_1}{\partial x} + \frac{v_1}{u_0} \frac{\partial u_1}{\partial \eta} \right) = -\frac{\partial p_1}{\partial x},
\]
or

\[
\frac{1}{u_0} \left( \frac{\partial u_1}{\partial x} \right)_x = -\epsilon \tan^2 \theta_w \left( \frac{h_0}{h_s} \right) \frac{1}{p_0} \left( \frac{\partial p_1}{\partial x} \right)_y. 
\] (6-5.142)
For the pressure distribution (6-5.141) the right-hand side of Eq. (6-5.142) is seen to be of higher order. Therefore, \( u_1/u_0 \) is a function of \( \psi \) only, which is determined by substituting \( x = \psi/\varepsilon u_0 \tan \theta_w \) into the boundary condition (6-5.137b). The resulting velocity distribution is

\[
\frac{u}{u_0} = 1 - \varepsilon \tan^2 \theta_w \left[ 1 + \left( \frac{h_\psi}{h_0} \right)^{-\alpha} - \left( \frac{h_0}{h_\psi} \right)^{-\alpha} \right]^{-1/\alpha}. \tag{6-5.143}
\]

Let us now compute the ratio \( F_w \) of the radiant-heat transfer with radiative cooling to that for no cooling (at station \( x \)),

\[
F_w = 2 \int_0^{u_0(x)} \frac{k T^4 dy}{k_s T_s^4 2 \varepsilon \tan \theta_w} = \frac{1}{2 \Gamma_n} \left[ 1 - (1 + 2 \alpha \Gamma_n)^{-1/\alpha} \right] \tag{6-5.144}
\]

where the radiant-energy emission has been expressed in terms of the enthalpy distribution by means of Eq. (6-5.125).

The above calculations for the flow variables and heat transfer have been carried out for an arbitrary level of radiative cooling, characterized by the radiation parameter \( \Gamma_n \). The results obtained should be valid for all values of \( \Gamma_n \) since the assumptions of a thin shock layer and Newtonian pressure and velocity values to lowest order will not be violated for any value of \( \Gamma_n \) as observed from Eqs. (6-5.135), (6-5.138), (6-5.141), and (6-5.143). In the limit of small radiative cooling, \( \Gamma_n \rightarrow 0 \), and the expressions derived above reduce to the perturbations given in Eqs. (6-5.123) for a perfect gas.

Profiles of the perturbations are shown in Fig. 6-5.21a, whereas Fig. 6-5.21b gives the changes in flow variables produced by a finite amount of radiation, \( \Gamma_n = 1 \). For small \( \Gamma_n \) the profiles vary linearly with \( \Gamma_n \) and are independent of \( \alpha \), whereas for larger values of \( \Gamma_n \) the changes in flow variables vary less rapidly with \( \Gamma_n \), and decrease in magnitude with increasing \( \alpha \). The dependence on \( \alpha \) may be explained as follows: \( \Gamma_n \) fixes the radiant-energy emission at the shock front, and a larger value of \( \alpha \) results in a faster reduction of the emission from a fluid element as it cools during its flow away from the shock front. In the limit \( \alpha \rightarrow \infty \) for fixed \( \Gamma_n \) (as well as for \( \Gamma_n \rightarrow 0 \) at fixed \( \alpha \)) the radiant-energy loss from a fluid element will approach zero, and the changes in the flow variables will accordingly approach zero. In the perturbation limit \( \Gamma_n \rightarrow 0 \), the cooling is too small to affect the emission, so the changes produced are independent of \( \alpha \) (this result is obvious from the perturbation calculation, where the emission is determined by the unperturbed shock layer enthalpy).
Enthalpy profiles are shown in Fig. 6-5.23 for various values of the radiative cooling parameter $\Gamma_n$. For small $\Gamma_n$ the enthalpy decreases linearly with $\Gamma_n$, with the rate of decrease becoming much less at large $\Gamma_n$ values. Figure 6-5.24 shows the decreases in heat transfer $\tilde{F}_w$ and

![Diagram showing enthalpy profiles for various values of radiative cooling parameter $\Gamma_n$.](image)

**Fig. 6-5.23.** Enthalpy distributions produced by various values of radiative cooling $\Gamma_n$ in the thin, transparent shock layer of a wedge; from Olfe. 81

![Diagram showing nondimensional radiant-heat transfer $\tilde{F}_w$ and shock layer thickness $\bar{y}_s$ as functions of the radiative cooling $\Gamma_n$.](image)

**Fig. 6-5.24.** The nondimensional radiant-heat transfer $\tilde{F}_w$ and shock layer thickness $\bar{y}_s$ as functions of the radiative cooling $\Gamma_n$ in the thin, transparent shock layer of a wedge; from Olfe. 81
shock layer thickness $\tilde{y}_s$ with increasing radiative cooling $\Gamma_n$. The radiant-heat transfer decrease is greater than the decrease in shock layer thickness. In this connection we note that for the perturbation limit [Eq. (6-5.123) for $F_w$, where $(\beta - n + 4) = (\alpha + 1)$], the heat transfer reduction is made up of a $-\alpha \Gamma_n$ term contributed by the change in enthalpy and a $-\Gamma_n$ term contributed by the change in shock layer thickness, whereas the fractional change in shock layer thickness is equal to $-2\Gamma_n$. Therefore, for the representative values $\alpha = 4$ and 8, the reduction in shock layer thickness $\tilde{y}_s$ accounts for 20 and 11% of the total reduction in radiant-heat transfer $F_w$ as $\Gamma_n \rightarrow 0$. Although the absolute reduction in $F_w$ produced by the change in $\tilde{y}_s$ increases with $\Gamma_n$, the fraction of the $F_w$ reduction produced by the change in $\tilde{y}_s$ decreases with increasing $\Gamma_n$ from the values given above.

Although the detailed calculations are more laborious, the treatment for radiating flow over a cone follows the procedure given above for the wedge. For the cone problem, the expansions (6-5.126) are substituted into the conservation equations for axisymmetric flow [Eqs. (6-5.114) through (6-5.117)]. The solution gives the following equation for the enthalpy:

$$
\varphi = \frac{(1 + 2\alpha \Gamma_n)^{(2\alpha - 1)/\alpha} - \left(\frac{2\alpha - 1}{\alpha}\right)(1 + 2\alpha \Gamma_n) \left(\frac{h_0}{h_s}\right)^{-(\alpha - 1)} + \left(\frac{\alpha - 1}{\alpha}\right) \left(\frac{h_0}{h_s}\right)^{-(2\alpha - 1)}}{(1 + 2\alpha \Gamma_n)^{(2\alpha - 1)/\alpha} - \left(\frac{2\alpha - 1}{\alpha}\right)(1 + 2\alpha \Gamma_n) + \left(\frac{\alpha - 1}{\alpha}\right)}
$$

(6-5.145)

where $\varphi \equiv (\theta - \theta_c)/(\theta_s - \theta_c)$, or $\varphi = y/y_s$ to lowest order. By calculating $\varphi$ as a function $h_0/h_s$, the enthalpy profiles given in Fig. 6-5.25 were determined. The shape of the shock front is given by

$$
\tilde{y}_s \approx \frac{(\theta_s - \theta_c)}{\frac{1}{2} \epsilon \tan \theta_c} = \frac{2}{(2\alpha \Gamma_n)^{\alpha}} \left[\left(\frac{\alpha}{\alpha - 1}\right) \left(\frac{\alpha}{2\alpha - 1}\right)(1 + 2\alpha \Gamma_n)^{(2\alpha - 1)/\alpha} \right.

- \left(\frac{\alpha}{\alpha - 1}\right) \left(1 + 2\alpha \Gamma_n + \left(\frac{\alpha}{2\alpha - 1}\right)\right].
$$

(6-5.146)

In terms of the enthalpy distribution (6-5.145), the conical flow analysis gives the following expression for the velocity and pressure distributions:

$$
\frac{v}{u_0} = -\epsilon \tan \theta_c \varphi - \epsilon \frac{\tan \theta_c}{2\alpha \Gamma_n} \left[\left(\frac{1}{\alpha - 1}\right) \left(\frac{h_0}{h_s}\right)^{-(\alpha - 1)} + (1 + 2\alpha \Gamma_n) \left(\frac{h_0}{h_s}\right) \right.

- \left(\frac{\alpha}{\alpha - 1}\right) \left(1 + 2\alpha \Gamma_n\right)^{(\alpha - 1)/\alpha}]\right.,
$$

(6-5.147)
\[
\frac{\dot{u}}{u_0} = 1 - \epsilon \tan^2 \theta_c \left[ \left( 1 + 2 \alpha \Gamma_n \right) - \left( \frac{h_0}{h_s} \right)^{-\alpha} \right]^2 \\
\times \left\{ \left( \frac{\alpha}{2\alpha - 1} \right)^{2(\alpha - 1)/\alpha} \left[ \left( 2 \alpha + \alpha \Gamma_n \right) - \left( \frac{h_0}{h_s} \right)^{-\alpha} \right] \\
- \left( \frac{\alpha}{\alpha - 1} \right) \left[ \left( 2 \alpha + \alpha \Gamma_n \right) - \left( \frac{h_0}{h_s} \right)^{-\alpha} \right] ^{(\alpha - 1)/\alpha} + \left( \frac{\alpha}{\alpha - 1} \right) \left( \frac{\alpha}{2\alpha - 1} \right) \right\},
\]

(6-5.148)

\[
\frac{\dot{p}}{p_0} = 1 - \epsilon \left( 1 - \frac{2}{(2\alpha \Gamma_n)^2} \right) \left( \frac{\alpha}{2\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right)^{(2\alpha - 1)/\alpha} \\
- \left( \frac{\alpha}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right)^{(\alpha - 1)/\alpha} + \left( \frac{\alpha}{\alpha - 1} \right) \left( \frac{\alpha}{\alpha - 1} \right) \left[ \left( \frac{2}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right) - 1 \right] \\
\times \left( \frac{2(2\alpha \Gamma_n)^2}{(2\alpha \Gamma_n)^2} \right) \left[ \left( \frac{1}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right) - \left( \frac{2}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right) - 1 \right] \\
\times \left( 1 + 2 \alpha \Gamma_n \right)^{-1/\alpha} \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - \left( \frac{h_0}{h_s} \right)^{-\alpha} \right]^2 - (2\alpha \Gamma_n)^2 \\
+ 2(1 + 2 \alpha \Gamma_n)^2 (2\alpha \Gamma_n)^2 \left( 1 - \left( \frac{h_0}{h_s} \right) \right) + \left( \frac{4\alpha}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right) \\
\times \left( 1 + (\alpha - 2) \Gamma_n \right) (2\alpha \Gamma_n) \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - 1 \right] \\
- \left( \frac{2\alpha}{2\alpha - 1} \right) \left[ \left( \frac{3\alpha - 1}{\alpha} \right) \left( 1 + 2 \alpha \Gamma_n \right)^2 \\
+ \frac{2}{\alpha} \left( \frac{1}{\alpha - 1} \right) \left( 1 + 2 \alpha \Gamma_n \right) - \left( \frac{\alpha + 1}{\alpha} \right) \right] \\
\times \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - 1 \right] \\
+ \left( \frac{4\alpha}{3\alpha - 1} \right) \left[ \left( \frac{3\alpha - 1}{2\alpha - 1} \right) (1 + 2 \alpha \Gamma_n) + \left( \frac{1}{\alpha - 1} \right) \right] \\
\times \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - 1 \right] \\
- 4 \left( \frac{\alpha}{2\alpha - 1} \right) \left( \frac{\alpha}{4\alpha - 1} \right) \left[ \left( \frac{h_0}{h_s} \right)^{-\alpha} - 1 \right] \right). 
\]

(6-5.149)

The radiation-induced changes in the flow variables plotted in Fig. 6-5.22 were calculated from the above expressions, after subtracting the radiationless values given in Eq. (6-5.119). For small radiative transfer
1.0

0.8

0.6

0.4

0.2

0.2 0.4 0.6 0.8 1.0

Φ = y/\gamma_s

FIG. 6-5.25. Enthalpy distributions produced by various values of radiative cooling \( \Gamma_n \) in the thin, transparent shock layer of a cone; from Olfe.\(^8\)

\( \Gamma_n \rightarrow 0 \), the changes in flow variables reduce to the perturbation given by Eqs. (6-5.122).

The radiant heat transfer ratio \( F_w \) at station \( r \) on the cone surface is given to lowest order by

\[
F_w = \frac{1}{\Gamma_n} \left\{ 1 - \frac{1}{2(\alpha - 1)} \frac{1}{\Gamma_n} \left[ (1 + 2\alpha \Gamma_n)^{(\alpha - 1)/\alpha} - 1 \right] \right\} \quad (6-5.150)
\]

Figure 6-5.26 shows the decrease in heat transfer and shock layer thickness with increasing \( \Gamma_n \) for this cone case. It should be noted that the enthalpy and radiant-heat transfer expressions given above are equivalent to those given by Chin and Hearne,\(^70\) who assumed constant pressure and unperturbed velocities, with the density variations being accounted for by the use of the Howarth-Dorodnitsyn variable \( dy_a = (\rho/\rho_s) dy \).

6-5F SIMILARITY SOLUTIONS FOR SLENDER BODIES. Hypersonic flow over a slender body may be determined from “small-disturbance” theory, provided the velocity and pressure perturbations are small compared to the free-stream velocity \( U_\infty \) and dynamic pressure \( \rho_\infty U_\infty^2 \), respectively. Defining \( \tau \) as the maximum inclination angle of the body, the hypersonic similarity parameter \( M_\infty \tau \) may be introduced. Since the values of \( M_\infty \tau \) considered are generally not small, the velocity and pressure perturbations are not small compared to the free-stream sound velocity and static pressure, respectively; thus small-disturbance theory is essentially a nonlinear theory.
The equations of motion (6-3.1) through (6-3.3) may be written in the following form for steady flow:

\( u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial r} + \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial r} + \nu \frac{\rho v}{r} = 0. \) (6-5.151)

\( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} = 0, \) (6-5.152)

\( \rho u \frac{\partial h}{\partial x} + \rho v \frac{\partial h}{\partial y} - u \frac{\partial p}{\partial x} - v \frac{\partial p}{\partial r} + \nabla \cdot \mathbf{F} = 0. \) (6-5.153)

In Eq. (6-5.151), \( \nu = 0 \) and 1 for the planar and axisymmetric cases, respectively, with \( r \) representing the Cartesian coordinate \( y \) in the planar case, and the cylindrical radius in the axisymmetric case.

![Graph](image)

**FIG. 6-5.26.** The nondimensional radiant-heat transfer \( F_w \) and shock layer thickness \( \bar{y}_s \) as functions of the radiative cooling \( \Gamma_n \) in the thin, transparent shock layer of a cone; from Olfe.81

Hypersonic flow over a slender body produces a shock angle of order \( \tau \), resulting in velocity perturbations behind the shock which are of order \( U_\infty \tau \) in the lateral direction and of order \( U_\infty \tau^2 \) in the axial direction. Since the axial velocity perturbation is of higher order, the small-disturbance equations of motion are obtained from the above equations by replacing the axial velocity \( u \) by the free-stream velocity \( U_\infty \) (see Hayes and Probstein85 for a more rigorous derivation):

\( U_\infty \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial r} + \rho \frac{\partial v}{\partial r} + \nu \frac{\rho v}{r} = 0, \) (6-5.154)
If \( U_\infty \frac{\partial \rho}{\partial x} \) is replaced by the time derivative \( \frac{\partial \rho}{\partial t} \), the above equations are transformed to the exact equations of motion for unsteady flow in the direction \( r \); i.e., these equations may be used to describe the propagation of plane, cylindrical, and spherical shock waves, for \( \nu = 0, 1, \) and \( 2 \), respectively. "Similarity" or "self-similar" solutions have been obtained for the propagation of plane, cylindrical, and spherical shock waves, with the plane and cylindrical cases also representing hypersonic flow about slender two-dimensional and axisymmetric bodies, respectively.

In this section we consider the similarity solution obtained by Romishevskii,\(^86\) who included transparent gas radiation for the axisymmetric problem. A similarity solution for the radiating spherical shock is given by Elliot\(^87\) in the Rosseland diffusion limit. The plane case has been studied by Wang,\(^88\) who obtains solutions for a gray gas by using the differential approximation discussed in Section 6-2, as well as solutions for the transparent and diffusion limits. Except at these limits, similarity solutions will exist in a gray gas only if the density ahead of the shock front varies in a prescribed manner. In his analysis for a transparent gas, Romishevskii also considers the case of varying density upstream from the shock front; however, here we shall be concerned only with the most common situation of constant upstream density.

As in the preceding section, the transparent gas radiation is approximated by a power-law formula,

\[
\nabla \cdot \mathbf{F} = 4\sigma T^4 k_p \sim C h^{m} \rho^{k}, \tag{6-5.157}
\]

where the use of the exponents \( m \) and \( k \) follows the notation used by Romishevskii.\(^88\) Using the perfect-gas relation \( h = \gamma p/\rho (\gamma - 1) \) to express Eqs. (6-5.156) and (6-5.157) in terms of \( \rho \) and \( p \) only, the following form of the energy equation is obtained:

\[
U_\infty \frac{\rho}{\rho} \frac{\partial \rho}{\partial x} + \frac{\rho}{\gamma} \frac{\partial \rho}{\partial r} - U_\infty \frac{\partial p}{\partial x} - \frac{\gamma}{\gamma - 1} \frac{\partial p}{\partial r} = C \left( \frac{\gamma}{\gamma - 1} \right)^{m-1} \rho^{m} \rho^{k-m}. \tag{6-5.158}
\]
values of \( n \) greater than or equal to \( n^* = 2/(3 + \nu) \), the value \( n^* \) corresponding to the constant energy or "blast wave" case. For \( n > 1 \), the body slope \( dr_b/dx = n\lambda_b x^{n-1} \) increases along the body length from the value of zero at \( x = 0 \) (pointed nose). Therefore the body length must be less than a critical value in order that the maximum body slope be small enough for a valid application of small-disturbance theory. At the surfaces of these concave bodies, \( \rho \to \infty \) and \( T \to 0 \). For axisymmetric flow, \( \nu = 1 \), so that for \( 1/2 < n < 1 \) there exist similarity solutions which yield convex bodies (decreasing slope) with blunt noses. For these convex bodies, small-disturbance theory will give accurate results away from the nose region and away from the fluid adjacent to the body surface which passed through the blunt nose region. The similarity solutions for convex bodies give \( \rho \to 0 \) and \( T \to \infty \) at the body surfaces.

As trial similarity solutions of Eqs. (6-5.154), (6-5.155), and (6-5.158), consider the functions

\[
\rho = \rho_\infty \left( \frac{\gamma + 1}{\gamma - 1} \right) R(\lambda), \quad v = \left( \frac{2nU_\infty}{\gamma + 1} \right) \frac{r}{x} V(\lambda), \quad \rho = \left( \frac{2n^2 \rho_\infty U_\infty^2}{\gamma + 1} \right) \frac{r^2}{x^2} P(\lambda),
\]

where \( \lambda = r/x^n \). Since the (small) shock angle at station \( x \) is approximately \( nr_{\lambda}/x \), the values of the above variables at the shock front will equal the appropriate Rankine-Hugoniot strong shock values if we require

\[
R(\lambda_s) = V(\lambda_s) = P(\lambda_s) = 1. \quad (6-5.160)
\]

Substitution of Eqs. (6-5.159) into Eqs. (6-5.154) and (6-5.155) gives

\[
\lambda \left[ V - \left( \frac{\gamma + 1}{2} \right) \right] R' + \lambda RV' + 2RV = 0 \quad (6-5.161)
\]

and

\[
n\lambda \left[ V - \left( \frac{\gamma + 1}{2} \right) \right] V' + \left[ nV - \left( \frac{\gamma + 1}{2} \right) \right] V + \left( \frac{\gamma - 1}{2} \right) \frac{n}{R} (\lambda P' + 2P) = 0,
\]

which are total differential equations in the variable \( \lambda \), with the primes denoting differentiation with respect to \( \lambda \).

Substitution of Eqs. (6-5.159) into the energy equation (6-5.158) yields a total differential equation in \( \lambda \) only if the following relation holds between \( n \) and \( m \):

\[
m = \frac{2n - 3}{2n - 2}. \quad (6-5.163)
\]
Therefore, for a given body shape described by the value of \( n \), a similarity solution will exist only for a particular enthalpy (temperature) dependence of the Planck mean absorption coefficient. For the above value of \( m \) the energy equation becomes

\[
\frac{dV}{d(\lambda/\lambda_0)} = \left( \frac{\gamma}{n\lambda/\lambda_0} \right) \left[ \frac{\Phi x_1}{\gamma} + 2n(\gamma - 1) \left( \frac{PV}{R} \right) - (\gamma^2 - 1) \alpha_2 \right], \quad (6-5.165)
\]

\[
\frac{dR}{d(\lambda/\lambda_0)} = \left( \frac{R}{n\Phi /\lambda_0} \right) \left[ \frac{\gamma(\gamma^2 - 1) \alpha_2 - 4nV\Phi^2 - \Phi x_1}{2\Phi^2 - \gamma(\gamma - 1) \left( P/R \right)} \right], \quad (6-5.166)
\]

\[
\frac{dP}{d(\lambda/\lambda_0)} = \left( \frac{\gamma}{n\lambda/\lambda_0} \right) \left[ \frac{2(\gamma + 1) R\Phi x_2 - 4nP\Phi - P\alpha_1}{2\Phi^2 - \gamma(\gamma - 1) \left( P/R \right)} \right]. \quad (6-5.167)
\]

In these equations

\[ \Phi \equiv V - \left( \frac{\gamma + 1}{2} \right), \]

\[ \alpha_1 \equiv (\gamma + 1) V - 2nV^2 - 2n(\gamma - 1) \frac{P}{R} \]

and

\[ \alpha_2 \equiv - W \left( \frac{\lambda}{\lambda_0} \right)^{2n-2} P^{mR^{k-m-1}} + \frac{P}{\gamma R} - \frac{2nP}{\gamma(\gamma + 1)} R \]

with

\[ W \equiv \left( \frac{4\gamma I_g^4}{\rho_s U_{\infty} h_s} \right) \left( \frac{k}{P_s s^X} \right) = \left[ 2^{m-2} n^{2m-2} C \left( \frac{\gamma + 1}{\gamma - 1} \right)^{k+2m-1} \right] \rho_0^{k-1} U_{\infty}^{2m-3} \lambda_{\infty}^{2m-2}. \]

Since the gas velocity behind the shock is nearly equal to \( U_\infty \) for a slender body, the radiation parameter \( W \) is equivalent to the parameter \( \Gamma_n \) introduced in the preceding section.

Equations (6-5.165), (6-5.166), and (6-5.167) may be numerically integrated, starting at the shock front \( (\lambda/\lambda_0 = 1) \) where each of the variables takes on the value of unity, as prescribed by Eq. (6-5.160). The body surface is reached when the velocity becomes tangential to
the surface; i.e., when \( \nu / U_\infty = [2n/(\gamma + 1)](r/x) V \) equals the body slope \( nr/x \). This condition may be written as \( \Phi = V - [(\gamma + 1)/2] = 0 \), which produces an infinite slope for \( R \) [see Eq. (6-5.166)] and thus an infinite or zero density at the surface, as mentioned above. Romishevskii has carried out calculations for (1) the values \( \gamma = 1.13, m = 3.5, \) and \( k = 2 \) corresponding to high temperature air, and (2) the values \( \gamma = 1.67, m = \frac{1}{2}, \) and \( k = 2 \) corresponding to ionized hydrogen. The similarity condition (6-5.163) requires the convex body \( n = \frac{4}{3} \) for the air values, and the concave body \( n = \frac{4}{3} \) for the hydrogen values. The computed profiles of density, pressure, and velocity across the shock layer are shown in Figs. 6-5.27, 6-5.28, and 6-5.29, respectively.

Radiative cooling produces an appreciable increase in density (temperature decrease), resulting in a reduction of the shock layer thickness. The pressure and velocity are changed very little.

6-6 Radiating viscous flows

So far in this chapter the flows have been treated as inviscid, except for a brief discussion in Section 6-4 of viscous effects within a shock front. Since the presence of viscous effects generally implies that heat conduction is important, we shall be considering energy transfer by heat conduction, as well as by viscous dissipation, radiation, and convection. The simplest viscous problem, Couette flow, will be discussed first, then boundary layers and viscous shock layers will be described.
6-6 Radiating Viscous Flows

6-6A COUETTE FLOW. Couette flow is defined as steady parallel flow between a bottom surface at rest and a top surface moving with the constant velocity $U$. Choosing a Cartesian coordinate system with $x$-axis along the direction of flow and $y$-axis perpendicular to the boundary surfaces, we shall consider the steady-state limit for which there are no flow variations in the $x$-direction. The $x$-momentum equation for steady, viscous, compressible flow is (see, e.g., Schlichting⁸⁹),
\[
\begin{align*}
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} &= -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left\{ \mu \left( 2 \frac{\partial u}{\partial x} - \frac{2}{3} \left[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right] \right) \right\} \\
& \quad + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x} \right) \right] + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial u}{\partial x} \right) \right],
\end{align*}
\]

where \( \mu \) is the coefficient of viscosity. For Couette flow, \( v = w = 0 \), and there are no variations in the \( x \)-direction; therefore the above equation reduces to

\[
\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = 0.
\]  

(6-6.2)

Here we shall consider constant \( \mu \), so that integration of Eq. (6-6.2) yields

\[
u = U y / y_w,
\]  

(6-6.3)

where \( y_w \) is the distance between the walls. Thus a linear velocity profile is obtained independently of the energy equation, and therefore independently of the radiation, heat conduction, and viscous dissipation. On the other hand, for \( \mu \) given as a function of temperature, the energy equation must be coupled with the momentum equation (6-6.2) to determine the temperature and velocity profiles.

The energy equation for a steady, viscous, compressible flow with radiation is

\[
\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} + \rho c_p w \frac{\partial T}{\partial z} = u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z} + \frac{\partial}{\partial x} \left( \lambda \frac{\partial T}{\partial x} \right) \\
+ \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( \lambda \frac{\partial T}{\partial z} \right) - \mathbf{\nabla} \cdot \mathbf{F} + \mu \Phi,
\]  

(6-6.4)

where \( \lambda \) is the coefficient of thermal conductivity, and the viscous dissipation term \( \Phi \) is given by

\[
\Phi = 2 \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] + \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right)^2 \\
+ \left( \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x} \right)^2 - \frac{2}{3} \left[ \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)^2. \right.
\]  

(6-6.5)

For Couette flow \( v = w = 0 \), no variations in the \( x \)-direction, the above equations reduce to

\[
\frac{d}{dy} \left( \lambda \frac{dT}{dy} \right) - \frac{dF}{dy} + \mu \left( \frac{dw}{dy} \right)^2 = 0,
\]  

(6-6.6)
where the temperature $T$, radiation flux $F$, and velocity $u$ are functions of $y$ only. Equation (6-6.6) shows that a fluid element will remain at a constant temperature in this steady-state limit by means of a balance of heat conduction and radiation energy losses with the energy produced by viscous dissipation.

In this section we shall follow Greif, who extended the analysis of Lick to include the viscous dissipation term. All three coefficients (thermal conductivity, viscosity, and radiation absorption) are assumed to be constant. Therefore, the heat conduction term in Eq. (6-6.6) becomes $\lambda \frac{d^2T}{dy^2}$, and the viscous dissipation term takes on the constant value $\mu U^2/y_w^2$ since the velocity is given by the linear profile Eq. (6-6.3). Therefore,

$$\lambda \frac{d^2T}{dy^2} - \frac{dF}{dy} + \frac{\mu U^2}{y_w^2} = 0.$$  \hspace{1cm} (6-6.7)

The radiation term $dF/dy$ is treated by the substitute kernel method in which the $E_2(\tau)$ function is replaced by $me^{-n\tau}$, as discussed in Section 6-2C. This method yields the approximate differential equation (6-2.23) for a gray gas, which for constant absorption coefficient $k$ becomes

$$\frac{d^2F}{dy^2} = 4kam \frac{dT}{dy} + k^2n^2F.$$ \hspace{1cm} (6-6.8)

The radiation flux $F$ is eliminated between the above equations by first differentiating Eq. (6-6.7) with respect to $y$ and using Eq. (6-6.8) to substitute for $d^2F/dy^2$; then this combined equation is differentiated, with Eq. (6-6.7) being used to eliminate the resulting $dF/dy$ term. The resulting fourth-order differential equation is integrated twice to give

$$\epsilon \frac{d^2T}{d\xi^2} - 8T - T^4 = -\alpha - \beta \tau_w \xi + \frac{1}{2} \varphi \tau_w \xi^2.$$ \hspace{1cm} (6-6.9)

Here nondimensional quantities are defined by

$$\xi = \frac{y}{y_w}, \quad \tilde{T} = \frac{T}{T_0}, \quad \tilde{\tau} = \frac{n\tau}{ny}, \quad \tilde{\tau}_w = \frac{nky_w}{ny},$$

$$\epsilon = \left(\frac{n^2}{4m}\right) \left(\frac{\lambda k}{\alpha T_0^3 y_w^2} \right), \quad \delta = \epsilon \tilde{\tau}_w^2, \quad \varphi = \left(\frac{n}{4m}\right) \left(\frac{\mu U^2}{\alpha T_0^4 y_w^2} \right),$$ \hspace{1cm} (6-6.10)

with the values $m = \frac{3}{4}$ and $n = \frac{3}{4}$ being used by Greif. In order to evaluate the constants of integration, $\alpha$ and $\beta$, we first put Eq. (6-6.7)
into nondimensional form and eliminate $\epsilon \frac{d^2 T}{d\xi^2}$ by means of Eq. (6-6.9), obtaining

$$
\left( \frac{1}{4k_0 T_0^4} \right) \frac{dF}{dy} = \delta \frac{dT}{d\tau} + T^4 - \alpha + \frac{\varphi}{\tilde{T}_w} - \beta \tilde{T}_w \xi + \frac{\varphi}{2} \tilde{T}_w^2. \quad (6-6.11)
$$

For $\frac{dF}{dy}$ we use the following integral expression, which includes radiation from the black walls at temperatures $T_1$ and $T_w$ at $\tau = 0$ and $\tau = \tilde{\tau}_w$, respectively [cf. Eq. (6-2.22)]:

$$
\frac{dF}{dy} = 2k_0 \left[ 2T^4 - n \int_0^\tau T^4 e^{-n(\tau - t)} dt - n \int_\tau^{\tilde{\tau}_w} T^4 e^{-n(\tilde{\tau}_w - t)} dt \right],
$$

where the exponential function $E_2(\tau)$ has been replaced by $m e^{-n\tau}$. The above integrals are evaluated using Eq. (6-6.9) for $T^4$, and substitution into Eq. (6-6.11) yields the following expressions for $\alpha$ and $\beta$:

$$
\alpha = \left( \frac{1}{2 + \tilde{T}_w} \right) \left[ \tilde{T}_w^4 + (1 + \tilde{T}_w) T_1^4 + \frac{\varphi}{2\tilde{T}_w} (2 + \tilde{T}_w)^2 \right. \left. + \delta \left[ \tilde{T}_w + (1 + \tilde{T}_w) T_1 + \frac{dT}{d\tau} \right]_{\tilde{T}_w} - (1 + \tilde{T}_w) \frac{dT}{d\tau} \right]_{T_1},
$$

and

$$
\beta = \left( \frac{1}{2 + \tilde{T}_w} \right) \left[ \tilde{T}_w^4 - T_1^4 + \frac{\varphi}{2} (2 + \tilde{T}_w) + \delta \left[ \tilde{T}_w - T_1 + \frac{dT}{d\tau} \right]_{\tilde{T}_w} + \frac{dT}{d\tau} \right]_{T_1}.
$$

The nondimensional heat flux $\tilde{q}$ includes both conduction and radiation,

$$
\tilde{q} = \frac{q}{\sigma T_0^4} = \frac{1}{\sigma T_0^4} \left( -\lambda \frac{dT}{dy} + F \right) = -\left( \frac{4m}{n} \right) \frac{\delta}{\tilde{T}_w} \frac{dT}{d\xi} + \left( \frac{F}{\sigma T_0^4} \right).
$$

Again differentiating Eq. (6-6.7) and using Eq. (6-6.8) to substitute for $\frac{d^2 F}{dy^2}$, the nondimensionalized form of the resulting differential equation is then subtracted from the derivative of Eq. (6-6.9) to obtain

$$
-\tilde{q} = \left( \frac{4m}{n} \right) (\beta - \varphi \xi)
\quad = 2(\beta - \varphi \xi) \quad \text{for} \quad m = \frac{3}{4} \quad \text{and} \quad n = \frac{3}{4}.
$$

(6-6.16)
The differential equation (6-6.9) may be easily solved for various limiting cases. First, if \( \epsilon \ll 1 \) then conduction is important only in "boundary layers" adjacent to each wall. Furthermore, if the channel is not optically thick, then \( \delta = \epsilon \tau_{w} \ll 1 \), and away from the wall boundary layers Eq. (6-6.9) reduces to

\[
T^4 = \alpha^{(0)} + \beta^{(0)} \tau_{w} \xi - \frac{1}{2} q \tau_{w} \xi^2, \tag{6-6.17}
\]

where \( \alpha^{(0)} \) and \( \beta^{(0)} \) are the values of \( \alpha \) and \( \beta \) computed for \( \delta = 0 \). For application near the walls, a boundary layer equation is obtained by stretching the coordinate in the manner \( \xi = \xi / \epsilon^{1/2} \), so that the \( \epsilon \alpha^2 \xi^2 \) term in Eq. (6-6.9) becomes of the same order of magnitude as the largest term in the rest of the equation. Therefore, Eq. (6-6.9) becomes transformed to

\[
\frac{d^2 T}{d \xi^2} - T^4 = -\alpha^{(0)} - \beta^{(0)} \tau_{w} \xi^2_a + \frac{1}{2} q \tau_{w} \xi^2_a = -f^4_a, \tag{6-6.18}
\]

where the subscript \( a \) denotes \( 1 \) at the lower wall (\( \xi_w = 0 \)) and \( w \) at the upper wall (\( \xi_w = 1 \)). We note that the \( \frac{d^2 T}{d \xi^2} \) term may be written as \( (dT/d\xi)(d(dT/d\xi)/d\xi) \), thus Eq. (6-6.18) may be directly integrated to give

\[
\delta \left( \frac{dT}{d\tau} \right) = \delta^{1/2} \left[ (T^5 - f^5_a) - 2f^4_a(T - f_a) \right]^{1/2}. \tag{6-6.19}
\]

Numerical integration of this equation gives the temperature distribution. The determination of \( \tilde{q} \) from Eqs. (6-6.16) and (6-6.14) requires only the temperature derivatives at the walls, which may be directly obtained from Eq. (6-6.19).

Consider now the opposite limit \( \epsilon \gg 1 \), for which conduction is dominant. It is evident from Eq. (6-6.7) that pure conduction produces a linear temperature profile, and viscous dissipation adds a quadratic term. Thus for \( \epsilon \gg 1 \) the temperature profile should be adequately described by the power series

\[
\tilde{T} = a_0 + a_1 \xi + a_2 \xi^2 + a_3 \xi^3, \tag{6-6.20}
\]

where the coefficients are determined by substitution of this series into Eq. (6-6.9). This solution corresponds to that for the nonradiating problem, plus terms proportional to \( 1/\epsilon \).

The third limiting solution to be considered is that for large optical depth, \( \tau_{w} \gg 1 \). Neglecting terms of higher order in \( 1/\tau_{w} \), Eq. (6-6.9) becomes

\[
\delta \tilde{T} + \tilde{T}^4 = \alpha^* + \beta^* \tau_{w} \xi - \frac{1}{2} q \tau_{w} \xi^2, \tag{6-6.21}
\]
where

\[ \alpha^* = \delta T_1 + T_1^4 \] (6-6.22)

and

\[ \beta^* = \delta \frac{(T_w - T_1)}{\tau_w} + \frac{(T_w^4 - T_1^4)}{\tau_w} + \frac{c}{2}. \] (6-6.23)

These relations for \( \alpha^* \) and \( \beta^* \) may be obtained from Eqs. (6-6.13) and (6-6.14) for \( \tau_w \) large, or may be obtained directly from Eq. (6-6.21) by requiring that the gas temperature approach \( T_1 \) and \( T_w \) at the lower and upper walls, respectively. These optically thick gas results may also be derived by directly using the Rosseland diffusion approximation (6-2.2) in the energy equation (6-6.7), as carried out by Viskanta and Grosh. The heat flux is determined by substituting the above expression for \( \beta^* \) into Eq. (6-6.16):

\[ -q = 28 \frac{(T_w - T_1)}{\tau_w} + 2 \frac{(T_w^4 - T_1^4)}{\tau_w} + \varphi(1 - 2\xi). \] (6-6.24)

An approximate extension of the above thick-gas results to all values of optical depth may be achieved by introducing the "radiation slip" concept discussed in Section 6-2F. Greif has added the viscous dissipation term to Probstein's formula (which assumes the separability of the heat conduction and radiation fluxes), obtaining the expression,

\[ -\tilde{q} = \frac{28(T_w - T_1)}{\tau_w} + \frac{2(T_w^4 - T_1^4)}{(2 + \tau_w^3)} + \varphi(1 - 2\xi). \] (6-6.25)

Using the methods described above, Greif computed the curves of nondimensional heat transfer \(-\tilde{q}\) versus \( \delta \) shown in Fig. 6-6.1 for various values of \( \tau_w \equiv \tau_w/n, \xi, \) and \( \varphi \). As expected from the above discussion, good results are obtained (1) for the boundary layer analysis when \( \delta \) and \( \epsilon \equiv \delta/\tau_w^3 \) are both small, (2) for the power-series formula when \( \epsilon \equiv \delta/\tau_w^3 \) is large, and (3) for the diffusion analysis when \( \tau_w \) is large. The radiation slip plus conduction formula for \(-\tilde{q}\) is seen to give good results for all values of \( \tau_w \) and \( \delta \), although the limiting formulae obtained above from the substitute kernel formulation will generally be somewhat better in their ranges of applicability (the radiation-slip temperature profiles also will be slightly inferior to those calculated by the substitute kernel method).

The interaction of radiation and heat conduction has been investigated in the following studies for conditions where viscous dissipation can be neglected: Greif has extended Lick's substitute kernel analysis to include power-law temperature dependences for the coefficients of...
6-6 BOUNDARY LAYERS. The flow of a gas around a body generally may be divided into two parts: an outer flow region, where viscosity may be neglected, and a thin boundary layer adjacent to the body surface, where viscosity reduces the velocity from the inviscid value to zero at the surface. For reentry bodies, the boundary layer concept is...
useful at lower altitudes, but it breaks down at higher altitudes where the reduced density extends viscous effects across the entire shock layer. Therefore, the viscous shock layer problem described in the next section will be more widely applicable to the reentry problem, although the laminar, compressible boundary layer does represent one of the most fundamental and important flow fields.

Howe's analysis\textsuperscript{98} of a transparent laminar boundary layer in the stagnation region of an axisymmetric body is discussed in this section. An estimate for the effect of radiation in a transparent boundary layer was first made by Smith,\textsuperscript{99} who neglected convection terms in the energy equation in order to obtain the simple conduction plus radiation problem. The transparent boundary layer has also been studied by Scala and Sampson\textsuperscript{42} for axisymmetric flow, and by Koh and DeSilva\textsuperscript{100} for flow over a flat plate. The following authors have used the Rosseland diffusion approximation to carry out calculations for optically thick boundary layers: Viskanta and Grosh,\textsuperscript{101} Scala and Sampson,\textsuperscript{42} Romishevskii,\textsuperscript{102} and Rumynskii.\textsuperscript{103} Rumynskii has also used the differential approximation in an investigation\textsuperscript{104} of the boundary layer for a gray gas. In addition to his transparent boundary layer analysis,\textsuperscript{98} Howe was studied\textsuperscript{105} the effect of injecting an absorbing gas into the boundary layer. Here we consider only the transparent case, but self-absorption will be included in the next section when Howe's viscous shock layer calculations are described.

The boundary layer equations are obtained from the conservation equations by assuming that the boundary layer thickness is small compared with the linear dimension along the body (the boundary layer equations are asymptotic equations for very large Reynolds numbers, $Re \equiv \rho VL/\mu$). Accordingly, the conservation of mass equation for steady boundary layer flow is obtained from Eq. (6-5.58) by noting that the $v$-coordinate in the boundary layer is small, giving

\begin{equation}
\frac{\partial}{\partial x} \left( \rho v^r \right) + \frac{\partial}{\partial y} \left( \rho v^r \right) = 0,
\end{equation}

where $v = 0$ and 1 for two-dimensional and axisymmetric flow, respectively. Howe considers flow near the stagnation point of an axisymmetric body ($v = 1$), with $r \simeq x$ as shown in Fig. 6-6.2.

Since the boundary layer is thin, the momentum equation in the $y$-direction gives $\partial p/\partial y \simeq 0$. The $x$-velocity $u$ varies across the thin boundary layer of width $\delta$ from the inviscid flow value at the outer edge of the boundary layer to the value zero at the body surface, therefore
the term \( \partial(u \partial u / \partial y) / \partial y \) will be much larger than the other viscous terms in Eq. (6-6.1). Accordingly, the \( x \)-momentum equation becomes

\[
\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right). \tag{6-6.27}
\]

The pressure gradient \( dp/dx \) may be expressed in terms of the velocity gradient at the outer edge of the boundary layer by means of the inviscid momentum equation, \( dp/dx = -\rho e u_e \frac{du_e}{dx} \).

In a similar manner the viscous dissipation term \( \Phi \) given by Eq. (6-6.5) may be shown to equal \( (\partial u / \partial y)^2 \) in the boundary layer approximation, and the energy equation becomes

\[
\rho c_p u \frac{\partial T}{\partial x} + \rho c_p v \frac{\partial T}{\partial y} = u \frac{dp}{dx} + \frac{\partial}{\partial y} \left( \lambda \frac{\partial T}{\partial y} \right) + \mu \left( \frac{\partial u}{\partial y} \right)^2 - \nabla \cdot \mathbf{F}. \tag{6-6.28}
\]

Substituting for the pressure gradient \( dp/dx \) from the momentum equation (6-6.27), and introducing the total enthalpy, \( dH \equiv d(h + \frac{1}{2}u^2) = c_p dT + u du \), the following form of the energy equation is obtained:

\[
\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \left( \frac{\mu}{Pr} \frac{\partial H}{\partial y} \right) + \frac{\partial}{\partial y} \left[ \mu \left( 1 - \frac{1}{Pr} \right) \frac{\partial}{\partial y} \left( \frac{u^2}{2} \right) \right] - \nabla \cdot \mathbf{F}, \tag{6-6.29}
\]

where \( Pr \equiv c_p \mu / \lambda \) is the Prandtl number. This form of the energy equation is valid for a reacting gas if the Lewis number \( Le \equiv c_p \rho D / \lambda \) is unity, where \( D \) is the diffusion coefficient (below we shall consider a dissociating gas in equilibrium).

The radiation term \( \nabla \cdot \mathbf{F} \) in Eq. (6-6.29) is equal to \( 4\sigma T^4 k_p \), as given
by the transparent gas relation (6-2.1). By empirically correlating the Planck mean absorption coefficient for air, Howe obtains the expression

\[ \nabla \cdot \mathbf{F} = cT^d \left( \frac{\rho}{\rho_0} \right)^{\alpha T_s} \]

where \( \rho_0 \) is a reference density, and \( c, d, n, \) and \( a \) are constants appropriately chosen for the regime of interest.

In viscous flow, as the wall (body surface) is approached the velocity goes to zero, and the gas temperature becomes equal to the wall temperature, which is assumed to be constant in the stagnation region. Therefore, we have the following boundary conditions at the wall (\( y = 0 \)):

\[ \text{at } y = 0: \quad u = 0, \quad v = 0, \quad H = H_w. \quad (6-6.31) \]

At the outer edge of the boundary layer (taken as \( y = \infty \) in the boundary layer equations), the variables \( u \) and \( H \) must approach the inviscid values. The appropriate inviscid value for \( u_e \) is obtained from the modified Newtonian velocity in the stagnation region,

\[ \text{for } y \to \infty : \quad u \to u_e = \beta x, \quad \text{with } \beta \equiv \frac{1}{R} \left[ \frac{2(p_s - p_w)}{(p_e)s} \right]^{1/2}, \quad (6-6.32) \]

where \( R \) is the body nose radius; the subscript \( e \) denotes values at the outer edge of the boundary layer, and the subscript \( s \) denotes stagnation quantities. The second condition at the outer edge of the boundary layer is obtained from the inviscid energy equation (since \( H \) approaches the inviscid value), with radiation included:

\[ \text{for } y \to \infty : \quad \rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} \to - \nabla \cdot \mathbf{F}. \quad (6-6.33) \]

The conservation equations (6-6.26), (6-6.27), and (6-6.29) are to be solved subject to the boundary conditions (6-6.31), (6-6.32), and (6-6.33). First, the independent variables \( x \) and \( y \) are transformed to \( s \) and \( \eta \) by means of the Levy transformation,

\[ s = \int_0^x \rho_e u_e \mu_e r^{2u} dx \]

\[ \eta = \frac{u_e r^u}{(2sC)^{1/2}} \int_0^y \rho dy, \]

where \( C \equiv \rho \mu / \rho_e \mu_e \) is assumed to be constant. Furthermore, it is assumed
that $\rho_{e}\mu_{e} = (\rho_{e}\mu_{e})_s$. Since $r \approx x$ and $u_e = \beta x$, the above equations become

$$s = \frac{\beta \rho_{e}\mu_{e} x^{2(\nu+1)}}{2(\nu + 1)}$$  \hspace{1cm} (6-6.36)

and

$$\eta = \left[\frac{(\nu + 1) \beta}{C_{\rho_{e}\mu_{e}}}\right]^{1/2} \int_0^y \rho \, dy.$$  \hspace{1cm} (6-6.37)

The partial derivatives are accordingly transformed by

$$\frac{\partial}{\partial y} = \left[\frac{(\nu + 1) \beta}{C_{\rho_{e}\mu_{e}}}\right]^{1/2} \frac{\partial}{\partial \eta}$$  \hspace{1cm} (6-6.38)

and

$$\frac{\partial}{\partial x} = \rho u_{e}\mu_{e} x^{2\nu} \left[\frac{\partial}{\partial s} + \frac{\partial \eta}{\partial s} \frac{\partial}{\partial \eta}\right].$$  \hspace{1cm} (6-6.39)

A stream function $\psi$ is defined so that conservation of mass, Eq. (6-6.26), is satisfied:

$$\frac{\partial \psi}{\partial y} = \rho u r^\nu, \quad -\frac{\partial \psi}{\partial x} = \rho v r^\nu.$$  \hspace{1cm} (6-6.40)

The following quantities are also defined:

$$g(\eta) = H/H_0$$  \hspace{1cm} (6-6.41)

and

$$f(\eta) = \psi/(2sC)^{1/2}.$$  \hspace{1cm} (6-6.42)

These relations assume a similarity solution, which is shown to exist by the reduction of the conservation equations to the total differential equations given below. By differentiating Eq. (6-6.42) and utilizing Eqs. (6-6.36), (6-6.38), and (6-6.40), we obtain

$$f'(\eta) = u/u_e,$$  \hspace{1cm} (6-6.43)

where the prime denotes differentiation with respect to $\eta$.

The following differential equations (for constant $Pr$) are obtained by transforming the conservation equations (6-6.27) and (6-6.29) to the independent variables $s$ and $\eta$, using the above definitions:

$$f'' + f f'' = \frac{2s}{u_e} \frac{du_e}{ds} \left( f' r^2 - \frac{\rho_e}{\rho} \right)$$  \hspace{1cm} (6-6.44)
and
\[
\frac{g''}{Pr} + fg' - \left[ c \frac{\rho}{(v + 1) \beta H_0 \rho} \right] \left[ \frac{p l_D}{\rho_0 RT (l_D + H_0 g - c_p T)} \right]^{nT^a - 1} = 0,
\]
\[ \text{(6-6.49)} \]

It can be shown that the right-hand side of Eq. (6-6.44) may be neglected because the surface temperature is much lower than \( T_e \).\(^{98}\) The right-hand side of Eq. (6-6.45) may be neglected since \( u_e^2 \ll H_e \) in the stagnation region of a body traveling at the hypersonic speeds of interest.

For a perfect, nonreacting gas, the temperature and density appearing in the radiation term of Eq. (6-6.45) may easily be expressed in terms of \( g(\eta) \) by means of (1) the perfect-gas relation for \( H \): \( H \simeq h = c_p T = H_0 g(\eta) \), since \( u_e^2 \ll H \), and (2) the equation of state: \( \rho = \rho_0 RT = \rho_0 c_p / R H_0 g(\eta) \). Howe, on the other hand, considers the gas to be made up of air molecules and air atoms, with the degree of dissociation being given by the equilibrium value. Therefore, the relative concentration of molecules and atoms is determined by the equilibrium constant \( K(T) = p_{\text{atom}}^{-1} p_{\text{molecule}} \) for dissociation. Since the specific heats \( c_p \) of the molecules and atoms do not differ by much in the temperature range of interest, the temperature and density are approximately given by the following relations\(^{98}\)

\[
T = \frac{1}{c_p} \left( H_0 g(\eta) - l_D \left[ 1 + \frac{4 \rho}{K(T_0)} \exp \left( - \frac{l_D}{R} \left( \frac{1}{T_0} - \frac{1}{T} \right) \right)^{-1/2} \right] \right),
\]
\[ \text{(6-6.46)} \]

and
\[
\rho = \frac{p l_D}{[l_D + H_0 g(\eta) - c_p T] RT}.
\]
\[ \text{(6-6.47)} \]

In the above equations \( R \) represents the gas constant for the molecules, and \( l_D \) is the heat of dissociation. The pressure \( \rho \) may be treated as constant in the stagnation region.

The differential equations (6-6.44) and (6-6.45) are reduced to the following expressions if the small terms on the right-hand sides are omitted, and Eq. (6-6.47) is used to eliminate \( \rho \):

\[
f'''' + ff'' = 0
\]
\[ \text{(6-6.48)} \]

with \( T \) being related to \( g(\eta) \) by means of Eq. (6-6.46). Transformation of the boundary conditions (6-31) at the wall yields

\[
at \ \eta = 0: \quad f = 0, \quad f' = 0, \quad g = g_w.
\]
\[ \text{(6-6.50)} \]
At the outer edge of the boundary layer Eq. (6-6.32) gives
\[ f' \rightarrow 1. \tag{6-6.51} \]
The inviscid energy equation (6-6.33) may be transformed to
\[ \frac{c T^d}{(v + 1) \beta H_e \rho_0} \left[ \frac{p l_0}{\rho_0 RT (l_D + H_e g - c_p T)} \right]^{n T^a - 1} \tag{6-6.52} \]
Therefore, combining this result with Eq. (6-6.49) at the outer edge of the boundary layer, we obtain the condition
\[ g'' \rightarrow 0. \tag{6-6.53} \]

To the approximation of this analysis, radiation does not enter into either the differential equation (6-6.48) or the boundary conditions (6-6.50) and (6-6.51). Accordingly, the velocity profile \( u/u_e = f' \) is the same as that for the nonradiating stagnation boundary layer. This velocity profile may be obtained from the Blasius function \( F(\xi) \), which is the solution for the flat plate boundary layer, by means of the transformation \( \xi = \eta/\sqrt{2} \) and \( f(\eta) = F(\xi)/\sqrt{2} \), giving \( f'(\eta) = F'(\eta/\sqrt{2})/\sqrt{2} \).

Profiles of the total enthalpy ratio \( g(\eta) \) are obtained from the simultaneous numerical integration of Eqs. (6-6.48) and (6-6.49), with boundary conditions (6-6.50), (6-6.51), and (6-6.53). In order to start the numerical integration, the value of \( g' \) is determined by trial and error so that the boundary condition (6-6.53) is satisfied. The temperature and density profiles are calculated from \( g(\eta) \) by means of Eqs. (6-6.46) and (6-6.47). Howe has carried out calculations for conditions corresponding to the flight of axisymmetric bodies of nose radii up to 10 ft, traveling at a speed of 31,000 ft/sec at an altitude of 165,000 ft, and with a wall temperature of 2000°K. The calculated temperature profiles are given in Fig. 6-6.3, where the thickness of the thermal boundary layer is seen to increase with increasing body nose radius.

The heat transfer to the wall \(-q_w\) is obtained (for \( Le = 1 \)) by adding the conduction and radiation terms,
\[ -q_w = \lambda \frac{\partial T}{\partial y} \bigg|_w + \frac{1}{2} \int_0^y c T^d \left( \frac{\rho}{\rho_0} \right)^{n T^a} dy, \tag{6-6.54} \]
where the factor \( \frac{1}{2} \) appears in the radiation term because just half of the emitted radiation reaches the wall in this transparent gas limit. It is evident from the wall temperature gradient shown in Fig. 6-6.3 that the convective heat transfer (conduction term) is essentially unaffected by
the radiation for the conditions considered. The interaction of the convective and radiative heat transfer is investigated more thoroughly below in the viscous shock layer analysis, where a wide range of conditions is considered, and self-absorption is included.

6-6C Viscous Shock Layers. For the viscous shock layer problem, we shall first describe Howe's analysis,\textsuperscript{72,106} which is similar to his boundary layer calculation discussed in the preceding section; then a brief description will be given of the calculations by Hoshizaki and Wilson,\textsuperscript{28,107} who use the integral method described in Section 6-5C. The analysis by Burggraf\textsuperscript{108} is also noted.

In Ref. 72 Howe and Viegas consider a shock layer of thickness $\delta$ small compared to the body nose radius $R$, with calculations being carried out only in the stagnation region, $x = O(\delta)$; see Fig. 6-6.2. Since the shock layer is not as thin as a normal boundary layer, the curvature term $K = 1 - f(y/R)$ must be retained in the conservation equations (although $\bar{K}$ is near unity, the derivative of $\bar{K}$ is not negligible). Conservation of mass is therefore given by

$$
\frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\bar{K} \rho v v^2) = 0.
$$

(6-6.55)

Howe and Viegas justify replacing the $y$-momentum equation by $\partial p/\partial y = 0$, which is the same result as in the boundary layer analysis. Accordingly, $p = p(x)$, and $x$-momentum equation becomes

$$
\rho u \frac{\partial u}{\partial x} + \bar{K} \rho v \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} + \bar{K} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right).
$$

(6-6.56)
In the energy equation it is assumed that the radiative transfer term is given by gray-gas radiation in a plane-parallel shock layer, with the body surface being black. Since the kinetic energy \( \frac{1}{2}(u^2 + v^2) \) is much less than the total enthalpy \( H = h + \frac{1}{2}(u^2 + v^2) \) in the stagnation region, the energy equation may be written in the form

\[
\rho u \frac{\partial H}{\partial x} + kev \frac{\partial H}{\partial y} = \frac{d}{dx}\left( \frac{\mu}{Pr} \frac{\partial H}{\partial y} \right) + \hat{k}k \left[ 2\sigma \int_0^\tau T^4(t) E_1(|t - \tau|) dt - 4\sigma T^4(\tau) + 2\sigma T_w^4E_2(\tau) \right],
\]

(6-6.57)

where \( \tau = \int_0^y k dy \) is the gray-gas optical depth. The effects of diffusion of dissociated and ionized species are included in the above equation by requiring that \( Pr \) represent an effective Prandtl number which includes a "reaction conductivity" as well as the ordinary thermal conductivity.

At the body surface we have the conditions

\[
\text{at } y = 0: \quad u = 0, \quad v = v_w, \quad H = H_w,
\]

(6-6.58)

which allow for the possibility of a nonzero normal velocity \( v_w \) produced by mass addition. If the injected gas is not air, the distribution of foreign species across the shock layer must be determined by coupling the appropriate diffusion equation with the conservation equations. The absorption coefficient \( k \) is then given as the sum of the air and foreign species contributions (see Howe and Viegas\(^{72} \) for this development). In the hypersonic limit of interest here, the shock relations yield the following relations for the flow variables at the shock front (subscript \( s \)):

\[
\text{at } y = \delta: \quad u_s = U \frac{x}{R}, \quad v_s = -\epsilon U \left(1 - \frac{x^2}{2R^2}\right),
\]

\[
p_s = \rho_\infty U^2(1 - \epsilon) \left(1 - \frac{x^2}{R^2}\right), \quad H_s = \frac{1}{2} U^2,
\]

(6-6.59)

where \( \epsilon \equiv \rho_\infty/\rho_s \). Since \( p = p(x) \), the above expression for \( p_s(x) \) may be used to substitute for the \( dp/dx \) term in Eq. (6-6.56).

As in the boundary layer analysis of the preceding section, new independent and dependent variables are defined. Here \( \hat{K} \) is included in the definition of the independent variables \( s \) and \( \eta \),

\[
s = \int_0^x \rho_0 u_s \mu_s \left(\frac{r}{\hat{K}}\right)^{2\nu} dx
\]

(6-6.60)
and

\[ \eta = \frac{u_s}{(2s)^{1/2}} \left( \frac{r}{R} \right)^v \int_0^r \frac{\dot{K}_p}{\rho} \, dy. \] (6-6.61)

In order to satisfy the continuity equation (6-6.55) the stream function \( \psi \) is defined by

\[ \frac{\partial \psi}{\partial y} = pur^v, \quad -\frac{\partial \psi}{\partial x} = \dot{K}_p \rho v r^v. \] (6-6.62)

The functions \( g(\eta) \) and \( f(\eta) \) are now defined by

\[ g(\eta) = \frac{H}{H_s} \] (6-6.63)

and

\[ f(\eta) = \frac{\psi}{(2s)^{1/2}}, \] (6-6.64)

which assume that similarity exists. Equations (6-6.61), (6-6.62), and (6-6.64) give

\[ f'(\eta) = \frac{u}{u_s}. \] (6-6.65)

We further define

\[ T = \frac{T}{T_s}, \quad \theta = \frac{\rho \mu \dot{K}_p}{\rho_s \mu_s \dot{K}_{p_s}}, \] (6-6.66)

with \( \rho_s \mu_s \) being assumed constant in the stagnation region.

Transformation of the conservation equations (6-6.56) and (6-6.57) yields the following total differential equations, showing that a similarity solution exists:

\[ \left( \frac{\theta f''}{Pr} \right)' + ff'' = \left( \frac{1}{v + 1} \right) \left[ f'' - 2 \frac{p_s}{\rho} (1 - \epsilon) \right] \] (6-6.67)

and

\[ \left( \frac{\theta}{Pr} g' \right)' + fg' = - \left( \frac{4}{v + 1} \right) \left( \frac{\sigma T_s^4}{\rho_s U_s^3} \right) \left( \frac{p_s}{\rho} \right) kR \left[ \int_0^\infty T^4(t) E_1(t - \tau) \, dt - 2T^4 \right. \]

\[ + \left. \left. T_s^4 E_2(\tau) \right] . \right. \] (6-6.68)

In these equations \( \theta, Pr, \rho, T, \) and \( k \) are known functions of the enthalpy ratio \( g(\eta) \), as determined by the equilibrium real-gas properties. The defining relation for the optical depth \( \tau \) may be transformed to give the following approximate relation between \( \tau \) and \( \eta \):

\[ \tau = \left[ \frac{\rho_s \mu_s R}{(v + 1) U_s} \right]^{1/2} \int_0^\eta \frac{k}{\rho} \, d\eta. \] (6-6.69)
The transformed boundary conditions Eqs. (6-6.58) and (6-6.59) become

at $\eta = 0$:

$$f_w = -\rho_w v_w \left[ \frac{R}{(v + 1) \rho_s \mu_s U} \right]^{1/2}, \quad f'_w = 0, \quad g_w = \frac{2h_w}{U^2}; \quad (6-6.70)$$

at $\eta = \eta_s$:

$$f_s = \rho_s \left[ \frac{RU}{(v + 1) \rho_s \mu_s} \right]^{1/2}, \quad f'_s = 1, \quad g_s = 1. \quad (6-6.71)$$

The shock front coordinate $\eta_s$ is determined by the value of $\eta$ at which the above equations for $f_s$ and $f'_s$ are simultaneously satisfied. The shock standoff distance $\delta$ is accordingly given by

$$\delta = \left[ \frac{\rho_s \mu_s R}{(v + 1) U} \right]^{1/2} \int_0^{\eta_s} \frac{d\eta}{\rho}. \quad (6-6.72)$$

The convective heat transfer is determined from

$$-q_c = \left( \frac{\mu}{Pr} \frac{\partial h}{\partial y} \right)_w = \left( \frac{\varphi}{Pr} \frac{U^5}{2} \right) \left[ \frac{(v + 1) \rho_s \mu_s}{R} \right]^{1/2} g'(0), \quad (6-6.73)$$

and the radiant-heat transfer to the (black) surface is

$$-q_r = -F_w = \sigma T_s^4 \left[ 2 \int_0^{\eta_s} T_4(t) E_2(t) \, dt - T_w^4 \right]. \quad (6-6.74)$$

Howe and Viegas have carried out numerical solutions of Eqs. (6-6.67) and (6-6.68) by first assuming profiles of $\theta$, $Pr$, $\rho$, and $Q$ as functions of $\eta$, where $Q$ is the entire right-hand side of Eq. (6-6.68). Equation (6-6.67) is then integrated numerically, with iteration being used to find the value of $f''(0)$ which yields a solution $f(\eta)$ satisfying the boundary conditions (6-6.71) at the shock front. This solution $f(\eta)$ is used in Eq. (6-6.68) to determine $g(\eta)$ directly by integration, with the value $g'(0)$ chosen so that the boundary condition $g(\eta_s) = 1$ is satisfied. The resulting $g(\eta)$ and $g'(\eta)$ profiles are then used to calculate new profiles of $\varphi$, $Pr$, $\rho$, $T$, and $Q$. Except for the use of the transparent gas expression when the shock layer is optically thin, $Q$ is calculated by using a finite series approximation for the gray-gas integrals appearing in Eq. (6-6.68). The above numerical procedure is repeated until $f''(0)$ does not change from one major iteration to the next. Howe and Viegas give a more complete description of this calculation. From the calculated flow field, the shock standoff distance, convective heat transfer, and radiant-heat transfer are computed from Eqs. (6-6.72), (6-6.73), and (6-6.74), respectively.
Figure 6-6.4 shows the computed enthalpy profiles, which are compared with the profiles determined by the one-dimensional analysis of Yoshikawa and Chapman\textsuperscript{2} discussed in Section 6-5A. Although both analyses give good results for the radiation-cooled outer half of the shock layer, Howe's profiles show lower enthalpy and larger enthalpy gradients produced by convective transport in the viscous, heat-conducting region near the body surface. Figure 6-6.5 shows flow-field

![Figure 6-6.4](image1)

**Fig. 6-6.4.** Comparison of enthalpy profiles with those of Ref. 2, from Howe.\textsuperscript{106}

![Figure 6-6.5](image2)

**Fig. 6-6.5.** Flow-field profiles for $U = 32,000$ ft/sec at 110,000 ft altitude, $p_s = 10$ atm, and $R = 5$ ft; from Howe and Viegas.\textsuperscript{72}
profiles at a flight velocity (32,000 ft/sec), which is low enough for ionization to be negligible and the shock layer to be transparent. For the conditions given, the flow field is seen to be comprised of an inviscid shock layer attached to a boundary layer at the wall. As shown in Fig. 6-6.6, the effect of an increased flight velocity is to yield (1) a broadened boundary layer produced by ionization and (2) increased radiation, which reduces the enthalpy ratio. The boundary layer becomes further smeared out as the pressure is reduced.

The coupling of radiation and convection is illustrated in Fig. 6-6.7, where $Nu$ is the Nusselt number, $Nu = -q_w c_{pw}/\lambda_w (H_s - h_w)$, and $Re_s$ is the Reynolds number, $Re_s = \rho_w u_w c_{pw}/\mu_w$. Radiative cooling is seen to appreciably reduce the convective heat transfer, with the reduction increasing with increasing flight velocity and body radius (the reduction is very small at the flight velocity of 31,000 ft/sec considered in the boundary layer analysis of Section 6-6B). The ratio $q_r/q_{r,0}$ of actual radiant-heat transfer to that calculated for a transparent shock layer with no radiative cooling is plotted versus body nose radius in Fig. 6-6.8 (points were also given in Fig. 6-5.4). The ratio $q_r/q_{r,0}$ decreases more rapidly at 40,000 ft/sec than at 50,000 ft/sec as the result of real-gas properties. The effect of mass addition has been considered by Howe (especially in Howe106). Mass addition will appreciably reduce the convective heat transfer, and will slightly increase the radiant-heat transfer because the shock standoff distance will be slightly increased.

Fig. 6-6.6. Flow-field profiles for $U = 50,000$ ft/sec at 190,000 ft altitude, $p_a = 1$ atm, and $R = 5$ ft; from Howe and Viegas.72
The radiation will be further increased if the injected gas is a stronger emitter than air; however, this increase will not be large, as the injected gas will remain in the relatively cool portion of the flow field near the body.
Howe's analysis considers only the stagnation region. Hoshizaki and Wilson, on the other hand, have calculated radiative cooling effects throughout the entire shock layer, considering hemispheres and hemisphere-cones. They use the integral method of Maslen and Moeckel, extended to include viscous terms and the radiative transfer term. As a detailed description of the integral method was given in Section 6-5C in connection with the inviscid shock layer calculation of Wilson and Hoshizaki, only a very brief description of the application to the viscous shock layer is given here.

First, we shall describe Hoshizaki and Wilson's transparent gas calculation. The conservation equations are simplified by neglecting convective terms of $O(\epsilon^2)$ and higher ($\epsilon \equiv \rho_x/\rho_o$), and by neglecting viscous terms of $O(1/Re)$ and higher. The resulting equations are equivalent to the boundary layer equations plus curvature terms. As in Howe's analysis, the mass diffusion of species is included in a total conductivity along with the ordinary thermal conduction. The integral method involves integration of the $x$-momentum and energy equations across the shock layer, with the continuity equation being used to evaluate the $y$-component of velocity, and the $y$-momentum equation being used to change the pressure gradient term into a form suitable for integration. The large profile variations in this viscous shock layer case are accounted for by representing the $x$-velocity and total enthalpy profiles by fifth- and sixth-order polynomials, respectively, as compared with the first- and third-order polynomials given in Eqs. (6-5.80) and (6-5.81) for the inviscid case. As in the inviscid calculation, the shock shape is assumed, and the integrated momentum and energy equations are solved at each station $x$ about the body, starting at the stagnation point. The shock shape is then calculated and compared with the assumed shape, with the entire calculation being repeated until satisfactory convergence is obtained. The calculated velocity and enthalpy profiles at various stations around a hemisphere-cone are given in Fig. 6-6.9 (computed for the same flight conditions as the inviscid profiles in Fig. 6-5.14). The profiles in the stagnation region agree quite well with those of Howe. The effects of radiative cooling on the distributions of convective and radiant heat transfer around a hemispherical body are shown in Fig. 6-6.10.

In their most recent calculation, Hoshizaki and Wilson included the effects of non-gray absorption and mass addition. Absorption was found to be important for the flight conditions appropriate for an Apollo-type entry.

A recent analysis for the stagnation region has been carried out by Burggraf, who has used the method of matched inner and outer
FIG. 6-6.9. Profiles of the $x$-velocity $f' = u/u_0$, total enthalpy $g = H/H_0$, and nondimensional emission (with $\epsilon/L$ = emissivity per unit length) shown at various stations $\xi = x/L$ and body angles $\theta$ around a hemisphere-cone (30° half-angle). Calculated for $U = 50,000$ ft/sec at 190,000 ft altitude, and $R = 5$ ft; from Hoshizaki and Wilson.107
expansions. The conservation equations in the shock layer are approximated by the boundary layer equations (cf. Section 6-6B). Burggraf finds that the coupling between the velocity and enthalpy produces an inner expansion which is characterized by powers and logarithms of the Reynolds number, rather than the usual square-root dependence. The analytical results are compared with solutions obtained by direct numerical integration of the boundary layer equations.

REFERENCES

6. RADIATION GASDYNAMICS

REFERENCES

6. Radiation Gasdynamics