Chapter 7

ABLATION DURING ATMOSPHERIC ENTRY*;†

In this chapter we present a summary of recent theoretical and experimental work on the ablation of reentry vehicles. After a brief introduction (Section 7-1), we consider first such relatively simple problems as steady-state ablation and ablation in a radiation field (Section 7-2), and then treat successively a series of unsteady problems of increasing complexity, viz., surface melting without study of the detailed motion of the liquid layer (Section 7-3), surface regression controlled by a rate process for time-independent and time-dependent regression rates (Sections 7-4A to 7-4B), constant ablation rate with time-dependent energy flux (Section 7-4C), melting and the coupled motion of the liquid and adjacent gas layers (Sections 7-5 and 7-6), ablation with combustion of gasification products and/or depolymerization and liquid-phase reactions (Section 7-7). In Sections 7-3 to 7-6A, we assume that the ablating solid is completely opaque and, therefore, energy input occurs only at the surface of the ablating material. In Section 7-6B, we allow for coupled conductive and radiative-energy transport within the solid material. In Section 7-7, we present a brief qualitative summary (as of 1965) of our current physical understanding of the important features of ablating materials that undergo chemical reactions of one type or another. Unsteady ablation coupled to a steady, laminar boundary-layer flow is treated in Section 7-8; an interesting aspect of this analysis is the derivation of results in semi-analytical form.

* Chapter 7 is by S. S. Penner.
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7. ABLATION

7-1 Introduction

The problem of ablation during atmospheric entry is of prime interest in connection with a wide spectrum of engineering and scientific studies. The most obvious application relates to the construction of heat shields for the survival of reentering space vehicles and missiles, a problem for which there exist at least three well-defined solutions. These are (1) the use of materials with such large heat capacities that the reentry vehicle may be considered to be a large heat sink, which can absorb all of the heat load without a concomitant fatal rise in temperature; (2) the use of melting, ablating, and charring solid materials which absorb energy on change of phase or decomposition; (3) the use of carefully planned reentry trajectories, in conjunction with finned structures, that are capable of losing rapidly (by radiation) most of the absorbed energy. Of the specified three solutions, the second is the most economical in terms of weight requirements, the most popular especially for missile applications, and the most easily adaptable to a variety of vehicle sizes and mission requirements.

The scientific interest in ablation rates results from quantitative studies of selected problems in space science and cosmology, especially problems connected with meteor, comet, and tektite entry. Detailed study of the aerodynamic history of such bodies, which may be deduced either from direct observations during atmospheric entry or from careful examination of deposited materials, may provide important clues not only about the entry trajectory and velocity but also about their origin. Because of the absence of the type of carefully controlled, albeit exceedingly costly, observational program that characterizes the work on space vehicles and missiles, the quantitative reconstruction of entry dynamics for meteors, comets, and tektites generally involves a more challenging and relatively less well understood set of phenomena.

Many papers have been published recently on ablation rates. We shall review these, to some extent, in connection with the quantitative studies described in succeeding sections. Here it will suffice to mention representative publications that cover the broad range of activities in the field. Landau\textsuperscript{13} (1950) solved the problem of heat conduction in a melting solid, and Sutton and Scala\textsuperscript{2} (1958) first emphasized the interplay between melting and the motion of a molten layer of ablated material during supersonic flow. Adams\textsuperscript{3} (1959), using only relatively crude steady-state approximations for ablation rates, considered conservative requirements for heat-shield design, a program which has been extensively elaborated in more recent reports.\textsuperscript{4–8} Bethe and Adams\textsuperscript{4} presented the first semianalytic solution to the problem of melting with
evaporation, properly coupled to the external gas flow. Chapman and his colleagues\(^9\text{--}^{12}\) have performed especially detailed experimental and theoretical studies on tektite ablation; Ostrach and his colleagues\(^{13}\text{--}^{15}\) have emphasized possible discrepancies in Chapman's interpretation of tektite entry by stressing the potential importance of nonsteady phenomena. Of some relevance is a series of papers dealing with steady\(^{16}\text{--}^{19}\) and unsteady\(^{20},^{21}\) coupled convective and radiative heat transfer in slabs of finite thickness and in a semiinfinite slab, as well as the problem of radiative-energy transport within an ablating body.\(^{22}\) Finally, mention should be made of the extensive experimental programs on ablation of missile and space-vehicle heat shields, some of which have been described in the open literature.\(^{23}\text{--}^{33}\) The fluid dynamics of the external flow is discussed, for example, by Fay \textit{et al.}\(^{34}\) and by Hayes and Probstein.\(^{34a}\) A recent discussion on the effect of Reynolds number on ablation has been given by Gilbert and Goldberg.\(^{34b}\) The analysis of phase-change problems with variable surface temperature has also been extended.\(^{34c}\)

### 7-2 Steady-state ablation rates and ablation with radiant heating\(^1\)

The rate of ablation of a solid material is generally determined by the properties of the ablating solid alone. Thus a solid body will melt when the (surface) temperature exceeds the melting point; it will evaporate (sublime) at all temperatures at a rate that is controlled, in first order, by the surface temperature alone; it will char at a rate that is primarily determined by the physicochemical properties that characterize depolymerization rates and bond rupture. As a consequence of this behavior, the ablation processes and rates are coupled to the external flow field only in so far as these affect the rate of energy input into the ablating solid.\(^*\) It is, therefore, appropriate to consider initially the properties of the solid material alone and to examine both steady and nonsteady ablation, without detailed reference to the external flow fields, the influence of which is made manifest through an appropriately selected surface boundary condition.

#### 7-2A Formulation of the problem without coupled flow\(^1\)

The problem of ablation with simultaneous radiative energy transport

\(^*\) The converse statement is not generally true, i.e., the motion of the liquid layer and the external flow field are extensively influenced by ablation and mass addition rates.
may be formulated at various levels of sophistication, although the steady-state regression rates turn out to be substantially independent of the way the problem is formulated. In the following instructive one-dimensional example, we neglect convective heat transfer as well as scattering and reradiation within the solid. The steady-state ablation rate is simply an energy conservation statement, it is independent of the last two assumptions, and it may be generalized to the case with occurrence of convective heat transfer by simply adding the convective-energy input to the radiative-energy flux term.

We consider an ablating, semiinfinite slab which is fed into the plane \( x^* = 0 \) at such a rate that the surface of the slab coincides with the plane normal to the origin (see Fig. 7-2.1). The required linear convection rate of the slab \textit{in the negative} \( x^*-\text{direction} \) is designated as the positive quantity \( \dot{m}^*/\rho^* \), i.e., the convection rate in the positive \( x^*-\text{direction} \) is \( -\dot{m}^*/\rho^* \).

Neglecting scattering and including only the external radiant-flux term, the equation for conservation of energy becomes

\[
\rho^* c_p^* \left[ \frac{\partial(T^* - T_0^*)}{\partial t^*} - \frac{\dot{m}^*}{\rho^*} \frac{\partial(T^* - T_0^*)}{\partial x^*} \right] = \lambda^* \frac{\partial^2(T^* - T_0^*)}{\partial x^{*2}} + 2q_0^* K^* E_n(K^* x^*),
\]  

(7-2.1)

where \( K^* \) denotes the (assumed constant) absorption coefficient per unit length for radiant energy,

\[
E_n(\gamma) = \int_1^\infty e^{-\gamma y} y^{-n} dy = \int_0^1 y^{n-2} e^{-\gamma/y} dy \quad (n = 1, 2, \ldots),
\]
and we use * to identify dimensional quantities. Let

\[ T = \frac{(T^* - T_0^*)}{T_0^*}, \]

\[ t = \frac{\lambda^* t^* K^* \beta}{\rho^* c_p^*}, \]

\[ v(t) = \frac{\dot{m}^* c_p^*}{\lambda^* K^*}, \] (7-2.2)

\[ \beta = \frac{q_e^*}{\lambda^* T_0^* K^*}, \]

\[ x = K^* x^*, \]

where \((\bar{K}^*)^{-1}\) is a characteristic length of the problem. Equation (7-2.1) becomes now

\[ \frac{\partial T}{\partial t} = v(t) \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2} + 2\beta E(x), \] (7-2.3)

where all quantities are dimensionless and \(\beta\) is a constant for any given ablating system and external flux. The boundary conditions are

\[ T(x, 0) = 0 \quad \text{for} \quad x \geq 0, \]

\[ T(\infty, t) = 0 \quad \text{for} \quad t > 0, \] (7-2.4)

\[ + \left[ \lambda^* \frac{\partial (T^* - T_0^*)}{\partial x^*} \right]_{x^*=0+, t^*} = \dot{m}^* \Delta h^*_{\text{vap}} \quad \text{or} \quad + \left( \frac{\partial T}{\partial x} \right)_{x=0+, t} = v(t) \theta_v, \]

where \(\Delta h^*_{\text{vap}}\) = heat of evaporation and \(\theta_v = \Delta h^*_{\text{vap}}/c_p^* T_0^*\) (\(\Delta h^*_{\text{vap}}\) is intrinsically positive). The boundary condition for the surface-temperature gradient requires some explanation. Without convective heat transfer but a heat sink at the surface (corresponding to the energy absorbed in ablation or melting), it is apparent that energy must reach the surface by heat conduction and that, therefore, the temperature gradient at the surface is positive. In other words, the present formulation demands a temperature profile of the form sketched in Fig. 7-2.2. With this type of temperature profile, the temperature gradient at the ablator surface is intrinsically positive. Hence, the product of the thermal conductivity by the positive surface temperature gradient, which measures the energy transfer through thermal conduction to the surface, is an intrinsically positive quantity, which may be set equal to
the (positive) surface energy absorption associated with melting and/or ablation.

We expect that, for sublimation at the surface temperature, the surface regression rate will be controlled by a surface rate law, which has been used in an earlier approximate treatment by Penner and Olfe. On the other hand, if surface melting precedes ablation, then it is more reasonable to assume that the surface temperature is constant and that the rate of removal of material is now determined as an eigenvalue of the problem which is defined by the amount of material that has been melted at the assumed constant surface temperature. This last problem has been treated by Landau. For the surface rate law, we write

\[ v(t) = v(T(x = 0, t)) = v(T_s(t)) \]

since we expect that, for evaporation into a vacuum,

\[ \rho_{\infty} b_0 R^* \exp \left(- \frac{E^*}{\rho_{\infty} R^* T^*} \right), \] (7-2.5)

where \( b_0, \alpha_1, \alpha_2, \) and \( E^*/R^* \) are empirically determined constants, \( \rho_{\infty} \) denotes the density of the solid phase, and \( p^* \) is the equilibrium vapor pressure of the subliming material. In the Knudsen approximation, \( E^* \) = heat of sublimation, \( \alpha_2 = 0, \) and

\[ \rho_{\infty} b_0 R^* T^* \approx \kappa p_{\infty}^* (W^*/2\pi R^* T^*)^{1/2} \]

where \( p_{\infty}^* \) is the integration constant in the Clausius-Clapeyron equation for vapor pressure, \( \kappa \) is the evaporation coefficient, and \( R^*/W^* \) denotes the gas constant per unit mass.

7-2B Steady-state solution for ablation rates without coupled flow. By definition, \( T(x, t_{ss}) = T_{ss}(x) \) and \( v[T_{ss}(0)] = v_{ss} \) are constants at the steady state. Equation (7-2.3) becomes now

\[ 0 = v_{ss} \frac{dT_{ss}}{dx} + \frac{d^2 T_{ss}}{dx^2} + 2\beta E_a(x) \quad \text{for} \quad t > t_{ss} \] (7-2.3a)
with the surface boundary condition

$$\left( \frac{dT_{ss}}{dx} \right)_{x=0+, t>t_{ss}} = v_{ss} \theta_v = \text{const.}; \quad (7-2.4a)$$

$t_{ss}$ is the dimensionless time required to reach the steady state; and the time derivative of $T_{ss}$ has been set equal to zero. Equation (7-2.3a) may be integrated directly with the result

$$\left[ \left( \frac{dT_{ss}}{dx} \right)_{x=\infty} - \left( \frac{dT_{ss}}{dx} \right)_{x=0+} \right] + v_{ss} \left( T_{ss x=\infty} - T_{ss x=0+} \right) + 2\beta [E_3(x = 0 +) - E_3(x = \infty)] = 0$$

or, since

$$\left( \frac{dT_{ss}}{dx} \right)_{x=\infty} = T_{ss x=\infty} = 0,$$

$$\left( \frac{dT_{ss}}{dx} \right)_{x=0+} = -v_{ss} T_{ss}(x = 0 +) + \beta = +v_{ss} \theta_v ,$$

where we have used Eq. (7-2.4a). Hence

$$\beta = v_{ss} \left[ \theta_v + T_{ss}(x = 0 +) \right]. \quad (7-2.5a)$$

In terms of the dimensional variables, Eq. (7-2.5a) becomes

$$\dot{m}_{ss}^* = \frac{q_e^*}{\Delta h_{vap}^* + c_p^* \left[ T_{ss}^*(x^* = 0 +) - T_0^* \right]}, \quad (7-2.6)$$

which is the well-known steady-state solution for this one-dimensional problem, without consideration of reradiation from the ablator surface.

7-2C Solution of Time-Dependent Radiant-Heating Problems. Analytical or semianalytical solutions are obtainable for the time-dependent problem without surface regression. This problem will be considered first, before we study the moving-boundary problem.

a. No surface regression. Without surface regression, $v(t) = 0$ and the boundary-value problem reduces to the simpler case

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + 2\beta E_3(x) \quad (7-2.3b)$$
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with

\[
T(x, 0) = 0 \quad \text{for } x \geq 0,
\]

\[
T(\infty, t) = 0 \quad \text{for } t \geq 0,
\]

\[
\left( \frac{\partial T}{\partial x} \right)_{t+, t} = 0. *
\]

The solution of Eq. (7-2.3b), subject to the boundary conditions specified in Eq. (7-2.4b), is

\[
\frac{T(x, t)}{\beta} = 2 \left( \frac{t}{\pi} \right)^{1/2} \left[ \exp \left( - \frac{x^2}{4t} \right) \right] - 2E_4(x) - x \left[ \text{erfc} \left( \frac{x}{2\sqrt{t}} \right) \right]
\]

\[
+ \int_{1}^{\infty} \frac{d\tau}{\tau^4} \left[ \exp(\tau^2 t - \tau x) \right] \left[ \text{erfc} \left( \tau \sqrt{t} - \frac{x}{2\sqrt{t}} \right) \right]
\]

\[
+ \int_{1}^{\infty} \frac{d\tau}{\tau^4} \left[ \exp(\tau^2 t + \tau x) \right] \left[ \text{erfc} \left( \frac{x}{2\sqrt{t}} + \tau \sqrt{t} \right) \right].
\]

In connection with the use of this last relation, it is convenient to note the following asymptotic forms:

\[
\frac{T(0, t)}{\beta} = 2 \left( \frac{t}{\pi} \right)^{1/2} - \frac{2}{3} \int_{1}^{\infty} \frac{d\tau}{\tau^4} \left[ \text{erfc} (\tau \sqrt{t}) \right] \left[ \exp(\tau^2 t) \right]
\]

\[
(7-2.7a)
\]

\[
\frac{T(0, t)}{\beta} \approx 2t + \frac{4}{3\sqrt{\pi}} t^{3/2} \ln t - 0.53 t^{3/2} + O(t^2) \quad \text{for } t \ll 1,
\]

\[
(7-2.7b)
\]

\[
\frac{T(0, t)}{\beta} \approx 2 \left( \frac{t}{\pi} \right)^{1/2} - \frac{2}{3} + O(t^{-1/2}) \quad \text{for } t \gg 1,
\]

\[
(7-2.7c)
\]

\[
\frac{T(x, t)}{\beta} \approx 2 \left( \frac{t}{\pi} \right)^{1/2} \left[ \exp \left( - \frac{x^2}{4t} \right) \right] - 2E_4(x) - x \left[ \text{erfc} \left( \frac{x}{2\sqrt{t}} \right) \right] + O(t^{-1/2})
\]

\[
\text{for } \frac{x}{t} \ll 1, \quad \sqrt{t} \gg 1.
\]

(7-2.7d)

Representative results for \( T(x, t)/\beta \), as a function of \( x \) and \( t \), are shown in Fig. 7-2.3.

An analytical solution is readily obtainable for a related, somewhat simpler problem. Thus, if the external radiant-energy source were

* An equivalent boundary condition is provided by the relation \( \int_{0}^{\infty} T \, dx = \beta t \).
coherent rather than incoherent, then the last term in Eq. (7.2.1) should be replaced by the expression

\[ qf^*R^* \exp(-K^*x^*) \]

where \( qf^* \) (cal/cm\(^2\)-sec) is the external flux density of the (laser) source, the factor 2 in Eq. (7.2.1) is replaced by unity because the mean beam length for a semiinfinite slab now equals the geometric thickness (it is twice the geometric thickness for incoherent radiation), and the function \( E_2(K^*x^*) \) has been replaced by the Beer-Bouguer factor \( \exp(-K^*x^*) \).

In the revised problem, we define the reduced flux

\[ \beta_1 = \frac{qf^*}{\lambda^*K^*T_0^*} \]

and then obtain the result

\[
\frac{T(x, t)}{\beta_1} = 2\left(\frac{t}{\pi}\right)^{1/2} \left[ \exp\left(-\frac{x^2}{4t}\right) \right] - x \left[ \text{erfc}\left(\frac{x}{2\sqrt{t}}\right) \right] \\
+ \frac{1}{2} e^t \left[ e^x \left[ \text{erfc}\left(\frac{x}{2\sqrt{t}} + \sqrt{t}\right) \right] + e^{-x} \left[ \text{erfc}\left(-\frac{x}{2\sqrt{t}} + \sqrt{t}\right) \right] \right] - e^{-x}.
\]
b. Constant surface regression rate. For a constant regression rate, \( v(t) = a \), the solution of the boundary-value problem as specified by Eqs. (7-2.3) and (7-2.4) may be shown to be

\[
T(x, t) = -\left(\frac{\theta v}{2}\right) + \left(\frac{\beta}{a^2}\right) (x + \log(1 - a)) \left[ e^{-ax} \text{erfc} \left(\frac{x - at}{2\sqrt{t}}\right) \right]
+ \left(\frac{a^2 + a - 4}{2a(1 - a)}\right) \left[ \log(1 - a) \right] \left[ \text{erfc} \left(\frac{x + at}{2\sqrt{t}}\right) \right]
+ \left(\frac{\beta}{a^2}\right) (x + at) \left[ \frac{2}{a} \right] \left[ \text{erfc} (x + at) \right]
- \left\{ a \theta v + \left(\frac{2\beta}{a^2}\right) (x + \log(1 - a)) \right\} \left(\frac{t}{\tau}\right)^{1/2} \left( \exp \left[ -\frac{(x + at)^2}{2\sqrt{t}} \right] \right)
- \beta \left[ \exp \left( -\frac{ax}{2} \right) \right] \int_1^\infty d\tau \left( \frac{\exp\left\{ x\tau^2 - \alpha \tau + (a^2/4)\right\}^{1/2} + (\tau^2 - \alpha \tau) t}{\tau^2(\tau - a) \left[ \tau^2 - \alpha \tau + (a^2/4)\right]^{1/2} + (a/2)} \right)
\times \left\{ \text{erfc} \left(\frac{x - 2\tau \tau^2 - \alpha \tau + (a^2/4)\right\}^{1/2}}{2\sqrt{t}} \right\}
- \exp\left\{ x\tau^2 - \alpha \tau + (a^2/4)\right\}^{1/2} + (\tau^2 - \alpha \tau) t \right\} \left[ \tau^2 - \alpha \tau + (a^2/4)\right]^{1/2} - (a/2)
\times \text{erfc} \left(\frac{x + 2\tau \tau^2 - \alpha \tau + (a^2/4)\right\}^{1/2}}{2\sqrt{t}} \right\}
+ 2\beta \int_1^\infty d\tau \left[ \frac{\exp\left( -\tau \tau \right)}{\tau^2(\tau^2 - \alpha \tau)} \right] \left[ \exp[(\tau - \alpha \tau) t] - 1 \right], \tag{7-2.8}
\]

where all logarithms are taken to the base \( e \).

For \( a = 0 \), Eq. (7-2.8) reduces to Eq. (7-2.7). A general parametric description of results derived from Eq. (7-2.8) is quite laborious.

7-3 Surface melting of opaque materials

H. G. Landau\(^{1a}\) has solved a special problem in which \( \mathcal{K}^* \) is allowed to increase without limit so that the term \( 2\beta \mathcal{E}_\alpha(x) \) may be deleted. The surface temperature gradient now becomes negative and, if we use the
positive sign for the convection velocity in the positive $x$-direction, the problem may be reformulated as follows in our notation:*  

\[
\frac{\partial T}{\partial t} = v(t) \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2},
\]

\[T(x, 0) = 0 \quad \text{for} \quad x > 0,
\]

\[T(\infty, t) = 0 \quad \text{for} \quad t > 0,
\]

\[\beta = -\left( \frac{\partial T}{\partial x} \right)_{x=0^+} + v(t) \theta_v ,
\]

where

\[v(t) = 0 \quad \text{for} \quad t < t_{cr} , \quad (7-3.2)
\]

\[T(0, t) = T_{cr} \quad \text{for} \quad t > t_{cr} . \quad (7-3.3)
\]

Hence, for $t < t_{cr}$ when $T(0, t) < T_{cr}$ (where $T_{cr}$ is assumed to be a minimum surface temperature for which regression is important), $v(t)$ is thus set equal to zero and the problem reduces to a classical problem in heat conduction. For $t > t_{cr}$, we assume a constant surface temperature $T(0, t) = T_{cr}$ and determine the dimensionless regression rate $v(t)$ as an eigenvalue of the problem. The surface boundary condition corresponds, in terms of the physical variables, to the statement

\[q_e^* = -\lambda^* \left[ \frac{\partial (T^* - T_0^*)}{\partial x^*} \right]_{x^*=0^+} + \dot{m}^* \Delta h_{vap}^* .
\]

For $v(t) = 0$, $t < t_{cr}$, the solution to the boundary-value problem is well known to be

\[T(x, t) = \beta \left\{ \left[ 2 \left( \frac{t}{\pi} \right)^{1/2} \exp \left( -\frac{x^2}{4t} \right) \right] - \left[ x \text{erfc} \left( \frac{x}{2\sqrt{t}} \right) \right] \right\}, \quad (7-3.4)
\]

\[\text{for} \quad T < T_{cr} , \quad t < t_{cr} ,
\]

where, as usual,

\[\text{erfc} y = \frac{2}{\sqrt{\pi}} \int_y^\infty (\exp - \lambda^3) \, d\lambda .
\]

* In nondimensionalizing the parameters for opaque materials, we have conveniently replaced $(\mathcal{R}^*)^{-1}$ by a characteristic vehicle length $R_v^*$ in Eqs. (7-2.2).
In terms of the physical variables, Eq. (7-3.4) becomes

\[
T^* - T_0^* = 2q_0^* \left( \frac{t^*}{\lambda^* \rho^* c_p^*} \right)^{1/2} \left\{ \frac{1}{\sqrt{\pi}} \exp \left[ - \frac{(x^*)^2}{4 t^* (\lambda^*/\rho^* c_p^*)} \right] \right. \\
- \left. \left[ \frac{2 (x^*/\rho^* c_p^*)^{1/2}}{2 (\lambda^*/\rho^* c_p^*)^{1/2}} \text{erfc} \left( \frac{x^*}{2 t^* (\lambda^*/\rho^* c_p^*)^{1/2}} \right) \right] \right\}.
\] (7-3.4a)

From Eq. (7-3.4), it is apparent that

\[ T(0, t_{cr}) = T_{cr} = 2 \beta \left( \frac{t_{cr}}{t_{cr}} \right)^{1/2} \]

and

\[ t_{cr} = \frac{\pi}{4 \beta^2} T_{cr}^2 \quad \text{or} \quad t_{cr}^* = \frac{\pi}{4} \frac{c_p^* \rho^* \lambda^*}{(q_0^*)^2} (T_{cr}^* - T_0^*)^2. \] (7-3.5)

For \( t > t_{cr} \), Landau assumed \( T(0, t) \) to remain constant and equal to \( T_{cr} \). A steady-state regression rate is now not immediately reached because of heat conduction within the interior of the slab which must readjust conditions, for a given energy input, to the presence of an energy sink at the ablator surface. Landau has obtained a numerical solution to the eigenvalue problem, the results of which are reproduced in Fig. 7-3.1. One of the principal difficulties with effecting a simple solution appears to be associated with the fact that the form of \( T(x, t_{cr}) \) is incompatible with the existence of a heat sink at the surface and, furthermore, imposition of the artificial boundary condition \( T(0, t) = T_{cr} \) for \( t > t_{cr} \) implies the necessity of a (molten) layer of growing thickness.

![FIG. 7-3.1. The reduced regression rate \( (\theta_0/\beta) v(t) \) as a function of the reduced time \( (t/t_{cr}) - 1 \) after reaching the melting temperature at the surface. The parameter \( m \) represents the dimensionless quantity \( \frac{1}{2} (\pi)^{1/2} (T_{cr}/\theta_0) \). Reproduced from Landau.][1]
with temperature $T_{cr}$; this last condition does not appear to be unrealistic for the problem of a melting solid treated by Landau but is not obviously applicable to an ablating solid.

In Fig. 7-3.1, we show $(\theta_{v}/\beta)v(t)$ as a function of $(t/t_{cr}) - 1$ for various values of $\frac{1}{2}(T_{cr}/\theta_{v})$. The growth of the molten layer is depicted in Fig. 7-3.2 where $s^{*}/(m^{*}/\rho^{*})t_{cr}^{*} \equiv \delta$ is shown as a function of $(t/t_{cr}) - 1$ for various values of $\frac{1}{2}(T_{cr}/\theta_{v})$

![Graph](image)

**FIG. 7-3.2.** The dimensionless thickness $\delta = s^{*}/(m^{*}/\rho^{*})t_{cr}^{*}$ of the molten layer as a function of the reduced time $(t/t_{cr}) - 1$ after reaching the melting temperature at the surface. The parameter $m$ represents the dimensionless quantity $\frac{1}{2}(T_{cr}/\theta_{v})$. Reproduced from Landau.\textsuperscript{1a}

It should be noted that the calculated physical variables and, especially, the estimated times for reaching steady-state regression are the results of the following critical assumptions made by Landau: (1) existence of a constant surface temperature after melting and surface regression have begun; (2) a temperature profile at $t = t_{cr}$ (that was derived without consideration of a surface heat sink for steady external energy input) which must undergo a drastic readjustment after introduction of the surface heat sink at $t = t_{cr}$.
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7-4 Energy absorption at the ablator surface with the regression rate controlled by a rate law

We consider the following eigenvalue problem as a straightforward extension of Landau’s problem to the case of ablation controlled by a rate process, the numerical value of which is determined by the surface temperature:

\[
\frac{\partial T}{\partial t} = \{v[T_\star^*(t)]\} \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2},
\]

\[
T(x, 0) = 0 \quad \text{for} \quad x > 0, \\
T(\infty, t) = 0 \quad \text{for} \quad t > 0,
\]

\[
\beta = -\left(\frac{\partial T}{\partial x}\right)_0 + \{v[T_\star^*(t)]\} \theta_v,
\]

where

\[
v^*[T_\star^*(t)] = \frac{\dot{p}_\star^*}{\rho^*} \left[\frac{W^*}{2\pi R^* T_\star^*(t)}\right]^{1/2} e; \tag{7-4.2}
\]

\[
T_\star^*(t) = T^*(0, t); \quad \dot{p}_\star^* \text{ is the equilibrium vapor pressure at } T_\star^*; \quad \rho^* \text{ denotes again the density; } W^* \text{ is the molecular weight of the evaporating material; } R^* \text{ stands for the molar gas constant; and } \epsilon \text{ is a dimensionless evaporation coefficient which is less than or equal to unity. In a manner analogous to that used in Eqs. (7-2.2), we have set } v = v^* c_p R^* \rho^*/\lambda^*.
\]

7-4A CONSTANT REGRESSION RATE. For \( v(T_\star^*) \equiv a = \text{const}, \) \( T_\star^* \neq \text{const}, \) the ablation problem may be solved in analytic form, for example, by using the method of the Laplace transform. The differential equation and boundary conditions become

\[
\frac{\partial T}{\partial t} = a \frac{\partial T}{\partial x} + \frac{\partial^2 T}{\partial x^2}, \\
T(x, 0) = 0, \quad T(\infty, t) = 0, \tag{7-4.3}
\]

\[
\beta = -\frac{\partial T(0, t)}{\partial x} + a \theta_v.
\]

After some manipulation, we obtain the following solution to the specified boundary-value problem:

\[
\frac{T(x, t)}{2(a \theta_v - \beta)} = -\frac{1}{4a} \left(\exp - ax\right) \left[\text{erfc} \left(\frac{x - at}{2 \sqrt{t}}\right)\right] \\
+ \frac{1}{4} \left[(x + at) + \frac{1}{a}\right] \left[\text{erfc} \left(\frac{x + at}{2 \sqrt{t}}\right)\right] \\
- \frac{1}{2} \left(\frac{t}{\pi}\right)^{1/2} \left[\exp - \left(\frac{x + at}{2 \sqrt{t}}\right)^2\right]. \tag{7-4.4}
\]
From Eq. (7-4.4), it is apparent that

$$T(0, \infty) = -\theta_v + \frac{\beta}{a}$$  \hspace{1cm} (7-4.5a)

or, in terms of the physical variables,

$$\frac{T^*(0, \infty) - T_0^*}{T_0^*} = \frac{q_v^* - \dot{m}_v^* \Delta h_{\text{vap}}^*}{\dot{m}_v^* c_v^* T_0^*},$$

which is the physically obvious steady-state expression for the surface temperature. The time-dependent surface temperature may be obtained from the following convenient relation:

$$\frac{T(0, t)}{2(\beta - a\theta_v)} = \frac{1}{4} - \left( \frac{at}{4} + \frac{1}{2a} \right) \left( \text{erfc} \left( \frac{a \sqrt{t}}{2} \right) \right) + \frac{1}{2} \left( \frac{t}{\pi} \right)^{1/2} \exp \left( -\frac{a^2 t}{4} \right).$$  \hspace{1cm} (7-4.5b)

From Eqs. (7-4.5a) and (7-4.5b), we may compute the value of $t_{cr}$ which corresponds to the time required for the surface temperature to equal 0.90 of the steady-state value. Thus

$$\frac{0.90}{2a} = \frac{T(0, t)}{2(\beta - a\theta_v)} \quad \text{where} \quad t = t_{cr};$$

![Fig. 7-4.1. The reduced time $t_{cr}$ required to reach 0.90 of the steady-state surface temperature [i.e., $T(0, t_{cr}) = 0.90 T(0, \infty)$], for various assumed values of the reduced, constant linear regression rate $a$; reproduced from Penner and Sharma.]

---

1 Penner and Sharma.
this "relaxation time" \( t_{cr} \) is plotted as a function of the parameter \( a \) in Fig. 7-4.1. The values for \( t_{cr} \) plotted in Fig. 7-4.1 have no immediate bearing on the time required for reaching the steady-state in problems for which the temperature-dependent regression rate specified in Eq. (7-4.2) applies. In general, \( t_{cr} \) will be much larger than the time required to reach the steady state when the material is heated from low values of the surface temperature until the value of \( T^*_a \) is such that \( v(t) \approx v_{ss} = a \), because, in the latter problem, most of the incident energy is used for heating until \( T^*_a \) approximates almost the steady-state value. This conclusion follows from the fact that ablation is relatively unimportant until the highest allowed temperatures are practically attained. In fact, a crude lower bound (which is correct within about a factor of 2 in most cases) may be obtained for the time required to approximate the steady state when \( v \) is not constant by setting \( a = 0 \) in Eq. (7-4.3), noting that then

\[
T(x, t) = \beta \left[ 2 \left( \frac{x}{\pi} \right)^{1/2} \exp \left( -\frac{x^2}{4t} \right) \right] - x \text{ erfc} \left( \frac{x}{2 \sqrt{t}} \right)
\]

and

\[
T(0, t) = 2\beta \left( \frac{t}{\pi} \right)^{1/2},
\]

and calculating \( t_{cr} \) from the last expression with \( T(0, t) \) replaced by the steady-state value, i.e., a useful lower bound for the time required to reach effectively the steady-state temperature is given by the relation

\[
(t_{cr})^{1/2} \approx T_{ss}(0, t) \frac{\pi^{1/2}}{2\beta}
\]

provided \( v[T^*_a(t)] \) has a strong dependence on \( T^*_a(t) \), which is generally the case. For constant values of \( a \), Fig. 7-4.1 shows that \( t_{cr} \) decreases rapidly as \( a \) is increased. This last conclusion is physically plausible because the total temperature rise, prior to reaching the steady state, decreases as \( a \) is increased.

In Figs.7-4.2 and 7-4.3, we have plotted, respectively, \([T(0, t)/2(\beta - a\theta_v)]\) as a function of \( t \) for various values of \( a \) and \([T(x = a, t)/2(\beta - a\theta_v)]\) as a function of \( t \) for various values of \( a \). Comparison of Figs. 7-4.2 and 7-4.3 shows that the temperature rise occurs much more abruptly at \( x = a \) than at the surface. The values of \( t_{cr} \), which may be estimated from the curves in Fig. 7-4.2, are, of course, consistent with the previous discussion. The rather abrupt rise of temperatures observed in Fig. 7-4.3 is determined by the numerical values of \( \beta \), i.e., by the interplay
Fig. 7-4.2. The reduced surface temperature $T(0, t)/2(\beta - a\theta_c)$, as a function of time, for various assumed values of the reduced, constant linear regression rate $a$; reproduced from Penner and Sharma.¹

Fig. 7-4.3. The reduced temperature $T(x, t)/2(\beta - a\theta_c)$ for $x = a$, as a function of $t$, for various assumed values of the reduced, constant linear regression rate $a$; reproduced from Penner and Sharma.¹
between external radiant heating and heat conduction. For sufficiently small values of \( x = a \), the steady-state values of \( T(0, t) \) and \( T(x = a, t) \) clearly approach each other (cf. the curves in Figs. 7-4.2 and 7-4.3 for \( a = 0.5 \)). Plots of the type shown in Figs. 7-4.2 and 7-4.3 may be used to construct the instantaneous temperature profiles in the ablating slabs.

7-4B Regression rate controlled by a time-dependent rate law.\(^1\) For a regression rate controlled by a surface rate law, Eqs. (7-4.1) and (7-4.2) apply. Representative calculations have been performed on a high-speed computer for Pyrex and iron, using the numerical values listed in Table 7-4.1. The results are plotted in Figs. 7-4.4(a) to 7-4.4(c).

<table>
<thead>
<tr>
<th>Physical property</th>
<th>Pyrex</th>
<th>Iron</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p^* ), dyn/cm(^2)</td>
<td>( 10^8 \exp \left( -\frac{46,400}{T^*} + 14.5 \right) )</td>
<td>( 4.046 \times 10^{10} \exp \left( -\frac{3.717 \times 10^6}{T^*} \right) )</td>
</tr>
<tr>
<td>( W^* ), g/mole</td>
<td>40.24</td>
<td>55.85</td>
</tr>
<tr>
<td>( c_p^* ), cal/g-(^\circ)K</td>
<td>0.25</td>
<td>0.151</td>
</tr>
<tr>
<td>( \lambda^* ), cal/cm-sec-(^\circ)K</td>
<td>( 7 \times 10^{-5} )</td>
<td>( 1.16 \times 10^{-1} )</td>
</tr>
<tr>
<td>( \rho^* ), g/cm(^3)</td>
<td>2.25</td>
<td>6.9</td>
</tr>
<tr>
<td>( \Delta h_{\text{ap}}^* ), cal/g</td>
<td>2,470</td>
<td>1,626</td>
</tr>
<tr>
<td>( v(t) )</td>
<td>( \frac{3.45 \times 10^7}{[1 + T(0, t)]^{1/2}} \left[ \exp -\frac{1.55 \times 10^2}{1 + T(0, t)} \right] )</td>
<td>( \frac{2.99 \times 10^7}{[1 + T(0, t)]^{1/2}} \left[ \exp -\frac{1.24 \times 10^2}{1 + T(0, t)} \right] )</td>
</tr>
</tbody>
</table>

\(^a\) The data for Pyrex are identical with the estimates used by Bethe and Adams.\(^4\)

\(^b\) The data for iron come from various standard sources, such as the International Critical Tables and the Handbook of Chemistry and Physics.

7-4.4a to 7-4.4c. Reference to Figs. 7-4.4a to 7-4.4c shows the following qualitative facts:

(a) The time required to reach the steady state, for a given ablator, increases as the flux density of the external energy source is decreased. This effect is relatively more marked for the material with higher thermal conductivity.
Fig. 7.4.4. (a) The surface temperature $T^*(0, t^*)$ as a function of time $t^*$ for Pyrex and iron, using the numerical estimates listed in Table 7.4.1, for an external energy flux $q_e^* = 210 \text{ cal/cm}^2\text{-sec}$; reproduced from Penner and Sharma.¹

(b) The surface temperature $T^*(0, t^*)$ as a function of time $t^*$ for Pyrex and iron, using the numerical estimates listed in Table 7.4.1, for an external energy flux $q_e^* = 2.1 \text{ cal/cm}^2\text{-sec}$; reproduced from Penner and Sharma.¹

(c) The surface temperature $T^*(0, t^*)$ as a function of time $t^*$ for Pyrex and iron, using the numerical estimates listed in Table 7.4.1, for an external energy flux $q_e^* = 0.21 \text{ cal/cm}^2\text{-sec}$; reproduced from Penner and Sharma.¹
(b) The times required to reach a steady state are appreciably longer for the good conductor (iron) than for the poor conductor (Pyrex).

(c) The temperatures for Pyrex are so high that melting must occur, for all of the energy sources used, long before the steady state is attained. Hence the problem which we have solved must be reformulated and the correct temperature-time history is better approximated by Landau’s solution for melting at constant surface temperature than by the model of time-dependent surface regression. This same statement also applies for iron at the high external flux level of $2.1 \times 10^2$ cal/cm$^2$-sec. On the other hand, for the reduced flux density of 2.1 cal/cm$^2$-sec, iron does not melt and the present formulation of the problem applies, although the steady state is reached only after very long periods of time.

(d) The initial slopes of the $T^*(0, t^*)$ versus $t^*$ curves are well approximated through the theoretical relation obtained by ignoring surface regression altogether, i.e., initially

$$
T(0, t) = 2\beta \left(\frac{t}{\pi}\right)^{1/2}
$$

or

$$
T^*(0, t) = T^*_0 \left[1 + 2\beta \left(\frac{t}{\pi}\right)^{1/2}\right]
$$

where

$$
t = \left[\frac{\lambda^*}{\rho \cdot c_p^* (R_s^*)^2}\right] t^*
$$

as in Eqs. (7-2.2) with $R_s^*$ replaced by $(R_s^*)^{-1}$.

The preceding results emphasize the importance of formulating the ablation problem more properly by allowing for energy absorption through both melting and evaporation.

7-4C Radiant and Convective, Time-Independent and Time-Dependent, Energy Fluxes with a Time-Independent Regression Rate.

For the special case in which radiative-energy transport is absent and the energy absorbed by the phase change is negligibly small, Ostrach and McConnell have previously given an analytical solution. The complete problem leads to substantially equivalent results. The boundary-value problem becomes

$$
\frac{\partial T}{\partial t} = a \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial x^2} ,
$$

$$
T(x, 0) = 0 \quad \text{for} \quad x > 0 ,
$$

$$
T(\infty, t) = 0 \quad \text{for} \quad t > 0 ,
$$

$$
\beta + h[T_{aw} - T(0, t)] = -\left(\frac{\partial T}{\partial x}\right)_{x=0} + a\theta_v ,
$$

(7-4.6)
where \( h[T_{aw} - T(0, t)] \), with \( h \) and \( T_{aw} \) constant, denotes the convective heat transfer at the ablator surface. Proceeding as before, the solution to the problem is found to be

\[
T(x, t) = \left( \frac{hT_{aw} + \beta - a\theta_v}{2} \right) \left[ \frac{1}{h} \left( \text{erfc} \left( \frac{x + at}{2 \sqrt{t}} \right) \right) + \frac{\exp(-ax)}{(h + a)} \left( \text{erfc} \left( \frac{x - at}{2 \sqrt{t}} \right) \right) \right]
\]
\[
- \left( \frac{2h + a}{h(h + a)} \right) \left( \exp[h(x + at) + h^2 t] \right) \left[ \text{erfc} \left( \frac{x + at}{2 \sqrt{t}} + h \sqrt{t} \right) \right].
\]

The surface temperature is evidently given by the relation

\[
T(0, t) = \left( \frac{hT_{aw} + \beta - a\theta_v}{2} \right) \left[ \frac{1}{h} \left( \text{erfc} \left( \frac{a}{2 \sqrt{t}} \right) \right) + \frac{1}{(h + a)} \left( \text{erfc} \left( -\frac{a}{2 \sqrt{t}} \right) \right) \right]
\]
\[
- \left( \frac{2h + a}{h(h + a)} \right) \left( \exp[h(a + h) t] \right) \left[ \text{erfc} \left( \frac{a}{2} + h \sqrt{t} \right) \right].
\]

Throughout the preceding discussion, we have assumed that the radiative heat flux term \( \beta \) and the ambient gas temperature (behind the shock front, i.e., ahead of the nose of a reentry vehicle) \( T_{aw} \) are independent of time. If we are interested in the reentry history of a space vehicle or missile, these two assumptions become clearly inapplicable. It is, therefore, of interest to consider the boundary-value problem specified in Eq. (7-4.6) subject to the constraint that \( \beta \) and \( T_{aw} \) are both functions of the time while \( h \) and \( a\theta_v \) remain constant. The solution to the problem may now be written in the form

\[
T(x, t) = \int_0^t dt' \left[ \beta(t - t') + hT_{aw}(t - t') \right] \Phi'(x, t') - \frac{a\theta_v}{2} \Phi(x, t)
\]

where

\[
\Phi'(x, t') = \left( \frac{1}{(\pi t')^{1/2}} \right) \left[ \exp \left( -\frac{ax}{2} - \frac{a^2 t'}{4} - \frac{x^2}{4 t'} \right) \right]
\]
\[
- \left( h + \frac{a}{2} \right) \left[ \exp[h(x + (a + h) t')] \left[ \text{erfc} \left( \frac{x + at'}{2 \sqrt{t'}} + h \sqrt{t'} \right) \right] \right].
\]

and

\[
\Phi(x, t) = \frac{1}{h} \left( \text{erfc} \left( \frac{x + at}{2 \sqrt{t}} \right) \right) + \frac{1}{h + a} \left[ \exp(-ax) \right] \left( \text{erfc} \left( \frac{x - at}{2 \sqrt{t}} \right) \right)
\]
\[
- \left( \frac{2h + a}{h(h + a)} \right) \left[ \exp[h(x + at) + h^2 t] \right] \left[ \text{erfc} \left( \frac{x + at}{2 \sqrt{t}} + h \sqrt{t} \right) \right].
\]

For time-dependent energy fluxes, corresponding to representative reentry trajectories, the integral in Eq. (7-4.9) must generally be evaluated numerically.
7-5 Surface melting and evaporation with coupled motion between the liquid layer and the external gas flow—the treatment of Bethe and Adams

Following numerical calculations by Sutton and Scala on the motion of a molten layer of Pyrex during atmospheric entry, Bethe and Adams presented the first comprehensive analytical study on the ablation of materials which melt and evaporate. This ingenious analysis, which contains a considerable number of simplifications as compared with the later treatments of Ostrach and Chapman (see Section 7-6 for details), provides an instructive illustration on the coupling between the external flow, the flow and evaporation of a molten surface layer, and heat transfer to the body. In common with the later studies, the treatment of Bethe and Adams involves the implicit assumption that the equilibrium vapor pressure is attained instantaneously at the gas-liquid interface (for a discussion of this problem, see Appendix 7-1).

7-5A Formulation of the Ablation Problem. We use the coordinate system shown in Fig. 7-5.1, in which the ablation velocity has an intrinsically negative sign and \( R_s^* \) represents the vehicle nose radius. In the stagnation region, the steady conservation equations for the liquid layer, neglecting body forces associated with vehicle accelera-
tion as well as dissipation terms and radiative energy transfer, then take the following form:

continuity equation:

$$\frac{\partial (R^* u^*)}{\partial x^*} + \frac{\partial (R^* v^*)}{\partial y^*} = 0;$$  \hspace{1cm} (7-5.1)

\(x\)-component of momentum conservation:

$$\rho_i^* \left( u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} \right) = \frac{\partial}{\partial y^*} \left( \mu_i^* \frac{\partial u^*}{\partial y^*} \right) - \frac{\partial p^*}{\partial x^*};$$  \hspace{1cm} (7-5.2)

the solution of the equation for the \(y\)-component of momentum conservation is

$$\frac{\partial p^*}{\partial y^*} \sim 0;$$  \hspace{1cm} (7-5.3)

energy conservation:

$$u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{\lambda_i^*}{\rho_i^* c_{p,i}^*} \frac{\partial^2 T^*}{\partial y^*} \equiv k_i^* \frac{\partial^2 T^*}{\partial y^*};$$  \hspace{1cm} (7-5.4)

Here we assume that the liquid density \(\rho_i^*\) and the thermal diffusivity \(k_i^*\) are independent of the temperature, whereas the liquid viscosity \(\mu_i^*\) is a strong function of \(T^*\).

The solution to the specified problem is effected by noting that the inertial terms in Eq. (7-5.2) are small compared with the drag term for a thin, molten liquid layer. Hence Eq. (7-5.2) may be integrated with the result

$$u^*(x^*, y^*) = \tau_i^*(x^*) \int_{\infty}^{y^*} \left( \frac{d\eta^*}{\mu_i^*} \right) + \int_{\infty}^{y^*} \frac{\eta^*}{\mu_i^*} d\eta^*, $$  \hspace{1cm} (7-5.5)

where \(\tau_i^*(x^*) = [\mu_i^* (\partial u^*/\partial y^*)]_i\) is the aerodynamic shear at the gas-liquid interface (the interface is identified by the subscript \(i\)). The pressure \(p^*(x^*)\) is specified by the Newtonian flow solution

$$p^* = p_{st}^* \left[ 1 - \left( \frac{x^*}{R_s^*} \right)^2 \right],$$  \hspace{1cm} (7-5.6)

where the subscript \(st\) identifies conditions at the stagnation point. In the stagnation region, \(R_s^* \simeq x^*\), whence integration of the relation for mass conservation given in Eq. (7-5.1) leads to the result

$$v_{t,1}^* - v_{c}^* \simeq - \frac{1}{x^*} \int_{\infty}^{0} \left[ \frac{\partial}{\partial x^*} (x^* u^*) \right] dy^*. $$  \hspace{1cm} (7-5.7)
Reference to Eqs. (7-5.5) and (7-5.7) shows that the velocity components in the liquid phase are coupled to the energy equation through the temperature dependence of $\mu^*_t$.

At the stagnation point, $\partial T^*/\partial x^* = 0$, whence Eq. (7-5.4) may be integrated to yield

$$
\left( \frac{\partial T^*}{\partial y^*} \right) = \left( \frac{\partial T^*}{\partial y^*} \right)_1 \left[ \exp \left( \frac{1}{k_t^*} \int_0^{v^*} v^* dy^* \right) \right],
$$

where $v^*$ is an intrinsically negative quantity for the specified coordinate system. Bethe and Adams⁴ then justify the approximation of replacing $v^*$ by the constant normal ablation velocity $v^*_o$ for both the thin liquid layer and the solid phase, i.e., they replace the preceding relation by the expression

$$
\frac{\partial T^*}{\partial y^*} = \left( \frac{\partial T^*}{\partial y^*} \right)_1 \left[ \exp \left( \frac{v^*_o y^*}{k_t^*} \right) \right],
$$

whence it follows that

$$
T^* = T^*_1 \left[ \exp \left( \frac{v^*_o y^*}{k_t^*} \right) \right] \quad (7-5.8)
$$

if temperatures are measured above the initial, uniform temperature of the reentry vehicle.

The dependence of $\mu^*_t$ on $y^*$ is specified by employing a viscosity-temperature relation of the form

$$
\frac{\mu^*_t}{\mu^*_t, 1} = \left( \frac{T^*}{T^*_1} \right)^{-n} \quad (7-5.9)
$$

where $n$ is assumed to be a constant. From Eqs. (7-5.5), (7-5.8), and (7-5.9) it is found that

$$
u^*(x^*, y^*) = \left\{ \frac{\tau_i^*(x)}{\mu^*_t, 1} \left( \frac{k_t^*}{v^*_o n} \right) + \frac{dp^*}{dx^*} \left( \frac{k_t^*}{\mu^*_t, 1} \right)^2 \frac{1}{\mu^*_t, 1} \left[ \frac{v^*_o \nu^*_o}{k_t^*} - 1 \right] \left[ \exp \left( \frac{y^*_o \nu^*_o}{k_t^*} \right) \right] \right\} \quad (7-5.10)
$$

the use of Eq. (7-5.10) in Eq. (7-5.7) then yields the result

$$\left| \nu^*_{i, 1} \right| = \left| \nu^*_o \right| - 2 \left( \frac{k_t^*}{\nu^*_o \nu^*_o} \right)^2 \frac{1}{\mu^*_t, 1} \left( \frac{d \left[ \tau_i^* \right]}{dx^*} - \frac{2k_t^*}{n} \frac{d^2 \nu^*_o}{dx^*} \right). \quad (7-5.11)
$$

Study of Eq. (7-5.11) shows that specification of $p^*(x^*)$ and $|\tau_i^*(x^*)|$ defines the relation between $|\nu^*_{i, 1}|$ and $|\nu^*_o|$; similarly, it is apparent from Eqs. (7-5.10) and (7-5.8) that $u^*$ and $T^*$ are uniquely related to $|\nu^*_o|$ after these parameters, together with $T^*_1$, are specified. The steady-state
regression velocity \(| v_\infty^* | \) and the interface temperature \( T_1^*(x) \) must now be obtained by enforcing interface boundary conditions that match the external flow to the solution of the conservation equations for the molten liquid layer. The logic of this matching procedure is easily clarified by considering the special case of no evaporation \( (v_{i,1}^* = 0) \).

7-5B Solution of the Problem without Evaporation.\(^4\) In the absence of evaporation, Eq. (7-5.11) is an explicit solution for the steady regression velocity, viz.,

\[
\left| v_\infty^* \right| = - v_\infty^* = + 2 \left( \frac{h_i^*}{nv_\infty^*} \right)^2 \left( \frac{d |\tau_i^*|}{dx^*} \right)^{1/2} - \frac{2h_i^*}{n_{|v_\infty^*|}} \left( \frac{d^2 p^*}{dx^*} \right),
\]

(7-5.12)

where \( p^*(x^*) \) has already been specified in Eq. (7-5.6) for known flight conditions (i.e., for known values of \( p^*_{st} \) and \( R^*_s \)). The shear stress \( (|\tau_i^*|) \) and heat transfer \( (q_{i,0}^*) \) are taken from the stagnation-point heat-transfer solution of Fay et al.\(^{34}\) without mass addition, viz.,

\[
\left| \tau_{i,0}^* \right| \approx q_{i,0}^* \left( x^* \right) R^*_s \left( \frac{2p^*_{st}}{\rho^*_{st}} \right)^{1/2} \frac{1}{(h^*_st - h^*_1)} \text{Pr}^{2/3},
\]

(7-5.13)

and

\[
\frac{q_{i,0}^*}{h^*_st - h^*_1} \approx 0.7 \left( \frac{p^*_{st}l^*_{st}}{\rho^*_{st}} \right)^{1/2} \left( \frac{2p^*_{st}}{\rho^*_{st}} \right)^{1/2} \text{Pr}^{-2/3},
\]

(7-5.14)

where \( \text{Pr} \) denotes the Prandtl number for air at the stagnation point and stagnation-point properties for air pressure, density, and specific enthalpy \( (h^*_st) \) occur in Eqs. (7-5.13) and (7-5.14). Since the specific enthalpy of air at the interface temperature \( T_1^* \), \( h^*_i \), enters in both Eqs. (7-5.13) and (7-5.14), it is apparent that numerical estimates for the velocity components in the liquid phase are weak functions of the interfacial temperature \( T_1^* \). The interface temperature itself is determined by enforcing energy conservation at the boundary, i.e., by writing

\[
q_{i,0}^* = \rho_i^* | v_\infty^* | c_{p,i}^* T_1^*.
\]

(7-5.15)

Equations (7-5.6) and (7-5.12) to (7-5.15) may be combined to yield a single relation from which \( T_1^* \) may be calculated for the viscosity expression given in Eq. (7-5.9). This relation is

\[
\left( \frac{c_{p,i}^* T_1^*}{h^*_st - h^*_1} \right)^3 \left[ 1 + \frac{4}{\text{Pr}^{1/3} n} \left( \frac{c_{p,i}^*}{\lambda_i^*} \left( \frac{\lambda_i^*}{h^*_st - h^*_1} \right)^{1/2} \left( \frac{c_{p,i}^* T_1^*}{h^*_st - h^*_1} \right) \right) \right]
\]

\[
= + \frac{1}{4} \frac{\rho_i^*}{\mu^*_i} \left( \frac{\lambda_i^*}{c_{p,i}^*} \right)^2.
\]

(7-5.16)
Since $\mu_{i,1}^*$ is the only sensitive function of $T_i^*$ appearing in Eq. (7-5.16), it is convenient to effect a solution of Eq. (7-5.16) by assuming a value for $T_i^*$, calculating all other parameters in terms of this estimate except for $\mu_{i,1}^*$, solving for $\mu_{i,1}^*$, using the viscosity-temperature relation to improve on the first calculation of $T_i^*$, etc. For large flight velocities, $h_{st}^* \gg h_t^*$ and Eq. (7-5.16) may be simplified to

\[(c_{p,i}^*T_i^*)^3 \approx c_1 n^2 \rho_{st}^* \mu_{i,1}^* h_{st}^* \]  \hspace{1cm} (7-5.16a)

where $c_1$ is nearly constant for glassy material and a constant Prandtl number has been used. For air, the following approximations are applicable:\(^4\)

\[\rho_{st}^* \sim \rho_{\infty}^* (h_{st}^*)^{0.34} \quad \text{and} \quad \mu_{st}^* \sim (h_{st}^*)^{0.3},\]

where the subscript $\infty$ refers to free-stream conditions; also $\mu_{i,1}^* \sim (T_i^*)^{-n} \sim (c_{p,i}^* T_i^*)^{-n}$. Hence

\[(c_{p,i}^*T_i^*) \sim (\rho_{\infty}^*)^{1/(3+n)} (h_{st}^*)^{3.64/(3+n)} \]  \hspace{1cm} (7-5.16b)

and, therefore, the energy absorbed per unit mass of molten material ($\equiv c_{p,i}^* T_i^*$) is only very weakly dependent on $\rho_{\infty}^*$ (i.e., on flight altitude) and is also not a very sensitive function of flight speed (i.e., of $h_{st}^*$) for a sufficiently temperature-sensitive viscosity law (large $n$).

The method of solution used in the simplified problem of Bethe and Adams\(^4\) shows that the energy equation is coupled to the other conservation relations primarily through the boundary condition for energy conservation, i.e., through the convective heat-transfer term $q_{i,0}$ which is, however, only a weak function of $T_i^*$ [see Eq. (7-5.14)]. In fact, for sufficiently large flight velocities, $h_{st}^* \gg h_t^*$ and $q_{i,0}$ is given independently of $T_i^*$; similarly, $\tau_{i,0}$ becomes now effectively independent of $T_i^*$ while $\rho_{\infty}^*$ is, in any case, not a function of $T_i^*$. Therefore, the ablation velocity $|v_{\infty}^*|$ depends on $T_i^*$ only through the viscosity law [see Eq. (7-5.9)] and, in effect, the solution of the energy conservation equation remains strongly coupled to that of the momentum and mass conservation equations only through Eq. (7-5.9).

It is interesting to note the following conclusions which may be drawn from the preceding relations:

1. The ablation velocity $|v_{\infty}^*|$ is directly proportional to the heat transfer rate (except for an additive term dependent on stagnation pressure $p_{st}^*$) and hence varies with nose radius.
(2) The ablation velocity $|\mathbf{v}_{\text{c}}^*|$ is minimized by low thermal diffusivity $k_f^*$ and high interfacial viscosity $\mu_{\text{f},1}^*$. This last statement defines desirable physical properties for a nonevaporating ablator. For an evaporating ablator, it is apparent that we require high heats of evaporation and low equilibrium vapor pressures for given values of $T_1^*$.

7-5C PARTIAL EVAPORATION OF THE MOLTEN LIQUID. In accord with Reynolds' analogy, the heat transfer and shear stress are assumed to be reduced by the factor $\psi$ when mass addition occurs from the surface. Thus Bethe and Adams write

$$\frac{q_1^*}{q_{1,0}^*} = \frac{\tau_1^*}{\tau_{1,0}^*} = \psi,$$

(7-5.17)

where the expression

$$\psi = 1 - 0.68 \left( \frac{W_A^*}{W_a^*} \right)^{0.26} \left( h_{st} - h_f^* \right) \frac{\rho_1^*}{q_{1,0}^*}$$

(7-5.18)

(with $W_A^*/W_a^*$ representing the ratio of air to ablator molecular weight) was obtained from empirical correlations of experimental measurements by Baron and by Reshotko and Cohen. It is now desirable to relate $\rho_1^* | \mathbf{v}_{\text{c},1}^*$ to the heat-transfer stagnation conditions and the surface temperature of the ablator. For a noncombustible ablator, an effective procedure involves the assumptions of (1) a Lewis number of unity with a binary mixture approximation and (2) instantaneous establishment of the equilibrium vapor pressure at the surface temperature of the ablator. This last assumption may be responsible for the removal of important nonsteady effects during a planetary-entry trajectory.

For a Lewis number ($Le$) of unity, it is well known that the total heat transfer to the surface may be written in the form

$$q_1^* = \psi q_{1,0}^* = -\lambda_g^* \frac{d\theta_g^*}{dY^*},$$

(7-5.19)

where starred properties with the subscript $g$ refer to appropriate variables in the boundary layer adjacent to the molten liquid. The surface boundary condition for the ablating liquid is

$$\rho_i^* | \mathbf{v}_{i,1}^* = \rho_i^* | \mathbf{v}_{i,1}^* | Y_{a,1} + \rho_g^* D_g^* \left( \frac{dY_{a,g}}{dy^*} \right)_1,$$

(7-5.20)

where we have employed a binary mixture approximation and the subscript $a$ identifies the ablating species in the boundary layer. From
Eqs. (7-5.19) and (7-5.20), we obtain the expression for a constant mass-fraction gradient,

\[ \frac{q^*_i}{\rho^*_i|v^*_i,1|} \frac{q^*_i}{\rho^*_i|v^*_i,1|(1 - Y_{a,1})} = - \frac{dh^*_g}{dY_{a,g}} \]

for \( \frac{\lambda^*_g/\rho^*_g c_{p,g} D^*_g}{\rho^*_g c_{p,g} D^*_g} = Le = 1 \). Integration of the preceding relation from the stagnation point (where \( h^*_{g} = h^*_g \) and \( Y_{a,g} = 0 \)) to the interface (where \( h^*_{g} = h^*_1 \) and \( Y_{a,g} = Y_{a,i} \)) leads to the result

\[ \rho^*_i|v^*_i,1| = \frac{\psi q^*_i}{(h^*_g - h^*_1)} \frac{Y_{a,i}}{1 - Y_{a,i}}. \]  

(7-5.21)

Since the pressure is independent of \( y \), it follows that

\[ p^*_s = p^*_i = p^*_s,1 + p^*_A,1 = p^*_s,1 \left(1 + \frac{p^*_s,1}{p^*_A,1}\right), \]

where \( p^*_A,1 \) is the air pressure at the interface. Furthermore,

\[ \frac{p^*_A,1}{p^*_A,1} = \frac{W^*_A}{W^*_A} \frac{1 - Y_{a,1}}{Y_{a,1}}, \]

whence

\[ \left(\frac{p^*_s,1}{p^*_A,1} - 1\right) \frac{W^*_A}{W^*_A} = \frac{1 - Y_{a,1}}{Y_{a,1}}. \]  

(7-5.22)

Equations (7-5.21) and (7-5.22) may be combined to yield the following explicit relation for the surface regression rate:

\[ \rho^*_i|v^*_i,1,1| = \psi q^*_i,0 \left(\frac{W^*_A}{W^*_A} \left(\frac{p^*_s,1}{p^*_A,1} - 1\right)^{-1}. \right. \]  

(7-5.23)

Since \( p^*_A,1 \) is assumed to be the equilibrium vapor pressure at \( T^*_1 \), Eq. (7-5.23) constitutes an expression for \( \rho^*_i|v^*_i,1| \) in terms of externally specified flow conditions and the interface temperature.

With evaporation, the energy-balance equation becomes

\[ \psi q^*_i,0 = p^*_i \left(v^*_i,1,1 \right)^* + |v^*_i,1| \Delta h_{vap}. \]  

(7-5.24)

Combination of Eqs. (7-5.18) and (7-5.23) leads to the result

\[ \rho^*_i|v^*_i,1| = \psi q^*_i,0 \left(\frac{\psi q^*_i,0}{(h^*_g - h^*_1)} \left(\frac{W^*_A}{W^*_A} \left(\frac{p^*_s,1}{p^*_A,1} - 1\right)^{-1}}{1 + 0.68 \left(\frac{W^*_A}{W^*_A}\right)^{0.74}} \right). \]  

(7-5.25)
For the liquid viscosity, Bethe and Adams\textsuperscript{4} use a relation of the form

\[
\frac{\mu_{i,1}^*}{\mu_{i, st}^*} = \left(\frac{p_{i,1}}{p_{i, st}}\right)^{-\alpha}
\]  

(7-5.26)

where \(\mu_{i, st}^*\) is the viscosity of the liquid at the temperature \(T_{a, st}^*\) at which the vapor pressure of the ablator has the value \(p_{a, st}^*\) and \(\alpha\) is a known constant. Equations (7-5.25) and (7-5.26) may now be introduced into Eq. (7-5.11) to yield the following expression for \(|v_{i,1}^*|\):

\[
|v_{i,1}^*| = |v_{i,1}^*| + 2 \left(\frac{\lambda_i^*}{\rho_{i}^{p_{i,1} n}}\right)^2 \frac{1}{\mu_{i, st}^* (v_{\infty}^*)^2} \left[ \frac{d}{t_{c,1}^*} \left(\frac{2\lambda_i^*}{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|} \left(\frac{d^2 p_{\infty}^*}{dx_{c,1}^*}\right)\right) \right] 
\]

\[
\times \left[ 1 - 0.68 \left(\frac{W_a^*}{W_A^*}\right)^{0.74} + \frac{q_{i,0}^*}{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|} \left(\frac{W_A^*}{W_a^*}\right)^{-\alpha} \right], \quad (7-5.27)
\]

where \(|v_{i,1}^*|\) is given in terms of \(|v_{\infty}^*|\) through Eqs. (7-5.18) and (7-5.24).

A second relation involving the interface temperature \(T_i^*\) may also be derived after some manipulation,\textsuperscript{4} viz.,

\[
1 = \left(\frac{q_{i,0}^*}{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|} - c_{p,i} T_i^*\right) \left[ \Delta h_{vap}^* + 0.68 \left(\frac{W_A^*}{W_a^*}\right)^{0.26} (h_{st}^* - h_i^*) \right]^{-1}
\]

\[
+ 4 \left(\frac{c_{p, st}^* \lambda_i^*}{\rho_{p, st}^* \mu_{i, st}^*}\right)^2 \left(\frac{\lambda_i^*}{\rho_{p, st}^* \mu_{i, st}^*}\right) \left[ \frac{q_{i,0}^*}{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|} \left(\frac{1}{h_{st}^* - h_i^*}\right) \right]^3
\]

\[
\times \left\{ \psi + 4 Pr_{st}^{-1/3} \left(\frac{c_{p, st}^* \lambda_i^*}{\rho_{p, st}^* \mu_{i, st}^*}\right) \left[ \frac{q_{i,0}^*}{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|} \left(\frac{1}{h_{st}^* - h_i^*}\right) \right] \left(\frac{p_{st}^*}{p_{i,1}^*}\right)^{-\alpha} \right\}, \quad (7-5.28)
\]

where

\[
\psi = 1 - 0.68 \left(\frac{W_A^*}{W_a^*}\right)^{0.26} \left[ \left(\frac{h_{st}^* - h_i^*}{q_{i,0}^* \rho_{i}^{p_{i,1} n} |v_{\infty}^*|}\right) \left(\frac{\rho_{i}^{p_{i,1} n} |v_{\infty}^*|}{\epsilon_{i,0}^* T_i^*}\right) \right] \left[ \Delta h_{vap}^* + 0.68 \left(\frac{W_A^*}{W_a^*}\right)^{0.26} (h_{st}^* - h_i^*) \right]^{-1}
\]

(7-5.29)

and \(Pr_{st}\) represents again the free-stream Prandtl number at stagnation conditions.

For specified flight conditions and an assumed value of \(q_{i,0}^* / \rho_{i}^{p_{i,1} n} |v_{\infty}^*|\), Eq. (7-5.28) may be readily solved for \(\mu_{i, st}^*\) and, therefore, for \(T_i^*\) since
usable first guesses are easily made for $c_{p,1}T_1^*$ and $h_1^*$, neither of which is a strong function of $T_1^*$. Once $T_1^*$ has been estimated in this manner, Eqs. (7-5.18), (7-5.24), and (7-5.27) may be solved for $|v_{l,1}^*|$ and for $|v_\infty^*|$ which, in turn, provides the necessary information to check on the consistency of the assumed value for $q_{l,0}^*/\rho_1^* |v_\infty^*$ with the specified flight conditions, etc. This type of iterative procedure has been used by Bethe and Adams\(^4\) to calculate $T_1^*$ and the dimensionless parameter $q_{l,0}^*/\rho_1^* |v_\infty^*$ for a reentry vehicle made of Pyrex at various flight speeds and altitudes. Their results are reproduced in Fig. 7-5.2 for the estimated physical properties of Pyrex.

Reference to Fig. 7-5.2 shows that the interface temperature $T_1^*$ and the ratio $q_{l,0}^*/\rho_1^* |v_\infty^*$ are primarily dependent on flight velocity and are relatively insensitive to flight altitude. For the flight-velocity range included in Fig. 7-5.2, less than 2% of the Pyrex glass is evaporated, i.e., Eq. (7-5.16b) holds in good approximation. The onset of evaporation corresponds to the separation of the curves for constant $T_1^*$ from the curves for constant values of $q_{l,0}^*/\rho_1^* |v_\infty^*$. For a particular entry trajectory, $q_{l,0}^*$ is known from earlier heat transfer studies, whence it follows that the data plotted in Fig. 7-5.2 may be used directly to estimate $|v_\infty^*|$, subject to the assumptions made in the analysis. Probably the most important approximations made by Bethe and Adams\(^4\) are those relating to the use of steady-state temperature profiles and evaporation rates (i.e., equilibrium vapor pressures $p_{a,1}^*$). The first of these approximations has been relaxed in the treatments of Ostrach and Chapman, which will now be considered.
7-6 Other treatments of surface melting and the coupled motion of the liquid layer and external gas flow

Detailed theoretical and experimental studies of the coupled motion of the liquid layer and external gas flow for surface melting have been performed in recent years, especially by S. Ostrach and his colleagues and by D. R. Chapman et al. We shall now briefly review these studies.

7-6A The studies of Ostrach et al. (neglecting radiant-energy transport). For a liquid layer that is thin compared with the radius of curvature at the vehicle nose, it is convenient to use the coordinate system shown in Fig. 7-6.1, where the interface between the gas phase and molten liquid is located at the origin $y^* = 0$. Steady-state solutions are used for the external gas flow and for the motion of the liquid layer. However, the temperature in the liquid layer is considered to be unsteady. For an axisymmetric body and an incompressible thin liquid layer, the conservation equations take the following form:

**Mass conservation:**

$$\frac{\partial (u^* R^*)}{\partial x^*} + \frac{\partial (v^* R^*)}{\partial y^*} = 0; \quad (7-6.1)$$

**Momentum conservation:**

$$\frac{\partial p^*}{\partial x^*} + \frac{\partial}{\partial y^*} \left[ \frac{\mu_i}{\partial x^*} \left( \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right) \right] + 2 \frac{\partial}{\partial x^*} \left( \frac{\mu_i}{\partial x^*} \right) = - \frac{\partial p^*}{\partial x^*} + \frac{\partial}{\partial y^*} \left[ \frac{\mu_i}{\partial x^*} \left( \frac{\partial v^*}{\partial x^*} + \frac{\partial u^*}{\partial y^*} \right) \right] + 2 \frac{\partial}{\partial x^*} \left( \frac{\mu_i}{\partial x^*} \right), \quad (7-6.2)$$

*Fig. 7-6.1. Diagram showing the coordinate system used by Ostrach et al.*

Body axis

Gas-liquid interface
7. ABLATION

\[
\rho_i^* \left[ \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} - \frac{dR^*}{dx^*} \alpha^* \right]
\]
\[= - \frac{\partial p^*}{\partial y^*} + 2 \frac{\partial}{\partial y^*} \left( \mu_i^* \frac{\partial v^*}{\partial y^*} \right) + \frac{\partial}{\partial x^*} \left[ \mu_i^* \left( \frac{\partial u^*}{\partial y^*} + \frac{\partial v^*}{\partial x^*} \right) \right], \tag{7-6.3}
\]

where \(\alpha^*\) denotes the acceleration, while \(\rho_i^*[1 - (dR^*/dx^*)^2]^{1/2}\) and \(\rho_i^*(dR^*/dx^*)\alpha^*\) represent, respectively, the \(x^*\)- and \(y^*\)-components of the body forces per unit volume;

energy conservation:

\[
\rho_i^* c_p^* \left( \frac{\partial T^*}{\partial t^*} + u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} \right) = \lambda_i^* \left( \frac{\partial^2 T^*}{\partial x^*} + \frac{\partial^2 T^*}{\partial y^*} \right) + \Phi^*, \tag{7-6.4}
\]

where the dissipation term \(\Phi^*\) is given explicitly by the relation

\[
\Phi^* = \mu_i^* \left[ 2 \left( \frac{\partial u^*}{\partial x^*} \right)^2 + 2 \left( \frac{\partial v^*}{\partial y^*} \right)^2 + 2 \frac{\partial u^*}{\partial y^*} \frac{\partial v^*}{\partial x^*} + \left( \frac{\partial u^*}{\partial x^*} \right)^2 + \left( \frac{\partial v^*}{\partial y^*} \right)^2 \right].
\]

Following Ostrach and other authors, constant values are assumed for the isobaric specific heat \(c_p^*\), the liquid density \(\rho_l^*\), and the liquid thermal conductivity \(\lambda_l^*\). The liquid viscosity \(\mu_l^*\) is expected to increase from a small value at the gas-liquid interface to a very large value at the liquid-solid interface as the local temperature decreases in the liquid phase.

Following Ostrach, we may write the governing equations in dimensionless form by introducing the following dimensionless variables:

\[
x = \frac{x^*}{R_s^*}, \quad y = \frac{R^*}{R_s^*}, \quad y = \frac{y^*}{R_s^*}, \quad T = \frac{T^* - T^*(t = 0)}{T_{st}^*},
\]

\[
p = \frac{p^*}{p_{st}^*}, \quad \mu = \frac{\mu_i^*}{\mu_{i, st}^*}, \quad t = \frac{t^*}{(\rho_i^* c_p^* R_s^* / \lambda^*)^{\delta^2}},
\]

\[
u = \frac{\mu^*}{(p_{st}^* R_s^*/\mu_{i, st}^*)^{\delta^2}}, \quad \psi = \frac{R_s^*}{(p_{st}^* R_s^*/\mu_{i, st}^*)^{\delta^2}},
\]

\[
g = \frac{\alpha^* R_s^*}{p_{st}^*}, \quad \varphi = Re Pr \delta^2 = \left\{ \frac{\rho_i^* R_s^* [(p_{st}^* R_s^*/\mu_{i, st}^*)^{\delta^2}]}{\mu_{i, st}^*} \right\} \left( \frac{c_p^* \mu_{i, st}^*}{\lambda^*} \right)^{\delta^2}.
\]

Here the subscript \(st\) identifies stagnation conditions and \(\delta\) is a dimensionless scale factor which is given by the expression

\[
\delta = \left[ \frac{2\mu_{i, st}^2 (dR^*/dx^*)^2}{(\rho_i^* c_p^* / \lambda^*)^2 n^2 R_{s} R_s^* \mu_{i, st}^*} \right]^{1/9}.
\]
where \( n \) is the exponent in the viscosity-temperature relation

\[
\frac{\mu_i^*}{\mu_{i,\text{st}}^*} = \left( \frac{T^*}{T_{\text{st}}^*} \right)^{-n}
\]

and \( \tau^* \) represents the shear stress. If \( \delta \ll 1 \),

\[
\frac{\rho_i^* R_s^* \left[ \frac{(p_{i,\text{st}}^* R_s^*/\mu_{i,\text{st}}^*) \phi^2}{\mu_{i,\text{st}}^*} \right] \delta^2 \equiv Re \delta^2 \ll 1,
\]

where \( Re \) is a Reynolds number, \((c_p^* \mu_{i,\text{st}}^*/\lambda^*) \equiv Pr \gg 1 \) and \( Pr \) is the Prandtl number for the liquid at the stagnation point, and

\[
\frac{Pr[(p_{i,\text{st}}^* R_s^*/\mu_{i,\text{st}}^*) \delta^2]}{\rho^* T_{\text{st}}^*} \ll 1,
\]

then the governing equations for the liquid phase take the following form:

- **conservation of mass:**
  \[
  \frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial y} (rv) = 0; \quad (7-6.1a)
  \]

- **momentum equations:**
  \[
  \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = \frac{\partial p}{\partial x} - g \left[ 1 - \left( \frac{dr}{dx} \right)^2 \right]^{1/2} = f(x); \quad (7-6.2a)
  \]

  \[
  \frac{\partial p}{\partial y} = \delta g \frac{dr}{dx} \sim 0; \quad (7-6.3a)
  \]

- **energy conservation with negligibly small dissipation:**
  \[
  \frac{\partial T}{\partial t} + \varphi \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial^2 T}{\partial y^2}. \quad (7-6.4a)
  \]

Reference to the preceding expressions shows that the only unsteady term occurs in the energy equation, an assumption which has been discussed by Georgiev et al.,\(^5,6\) as well as by Ostrach and his colleagues.\(^13-15\) In accord with Eq. (7-6.3a), the pressure is assumed to be independent of \( y \) for any given value of \( x \), where it is equated to the estimate at the interface; the \( x \)-dependence of the pressure profile is introduced by assuming \( p(x) \) for a Newtonian fluid. The effect of deceleration is introduced through the dimensionless term \( g \) in the equation for conservation of the \( x \)-component of momentum.
The boundary and initial conditions must now be specified before the problem can be solved. We note that
\[
T(x, y; 0) = u(x, y; 0) = v(x, y; 0) = 0; \quad T(x, \infty; t) = u(x, \infty; t) = 0; \quad v(x, \infty; t) = v_\infty.
\]

From symmetry considerations at the stagnation point,
\[
\left( \frac{\partial T}{\partial x} \right)_{x=0} = \left( \frac{\partial u}{\partial x} \right)_{x=0} = \left( \frac{\partial v}{\partial x} \right)_{x=0} = 0. \quad (7-6.7)
\]

At the liquid-gas interface (identified by the subscript \(i\)), where \(y = 0\) for all \(x\), we enforce not only continuity of the pressure but also of the temperature, shear stress, and energy flux. Therefore,
\[
T^*_g(x, 0^-; t) = T^*(x, 0^+; t) \quad (7-6.8)
\]

where the subscript \(g\) identifies conditions in the gas phase;
\[
\tau^*_g(x, 0^-; t) = \tau^*(x, 0^+; t) = -\left( \rho_i^* \frac{\partial u^*}{\partial y^*} \right)_{y=0} = p_0^* \delta \tau_i; \quad (7-6.9)
\]
\[
q_{\text{conv}}^* = -\left( \lambda_i^* \frac{\partial T^*}{\partial y^*} \right)_{y=0^+} + \rho_i^* v_i^* \Delta l^*, \quad (7-6.10)
\]

where \(\Delta l^*\) is the heat absorbed during fusion per unit mass, \(q_{\text{conv}}^*\) denotes the total convective energy input to the reentry vehicle, and \(v_i^*\) is equivalent to the velocity \(v_i\) used in Section 7-5.

The method of solution now involves first the assumption of a temperature profile at the interface \(T(x, 0; t) = T_i(x; t)\). Next Eq. (7-6.4a) is simplified by assuming that the term in \(u(\partial T/\partial x)\) may be neglected, as for stagnation-point heat transfer. Furthermore, \(v\) is replaced by the steady-state value \(v_\infty\). With these simplifications, the temperature profile is independent of \(x\) except through a linear dependence on the interface temperature \(T_i(x, t)\); furthermore, the \(y\)-dependence of \(T\) is determined through the explicit relation
\[
T = \frac{T_i(x; t)}{2} \times \left[ \exp(qv_\infty y) \right] \left[ \text{erfc} \frac{1}{2} \left( \frac{y}{\sqrt{t} + qv_\infty \sqrt{t}} \right) \right] + \left[ \text{erfc} \frac{1}{2} \left( \frac{y}{\sqrt{t} - qv_\infty \sqrt{t}} \right) \right].
\]

This method of solution evidently also involves the assumption that heat conduction in the \(x\)-direction may be neglected in the liquid phase
when \( T_i(x, t) \) varies with \( x \). For specified values of \( T_i(x; t) \), the shear stress and pressure at the interface are immediately determined from known solutions for the gaseous boundary layer.

The remaining differential equations are then integrated without use of the energy balance condition, which is employed only as a check on the consistency of the solution, i.e., \( T_i(x; t) \) is varied until this equation is satisfied. For known values of \( T_i(x; t) \), solution may be effected, according to Ostrach et al., by proceeding as follows. First Eq. (7-6.2a) is integrated twice with the result

\[
 u = f(x) \int_0^y \frac{y}{\mu} \, dy - \tau_i \int_0^y \frac{dy}{\mu},
\]

(7-6.12)

and, therefore, \( u \) is known after \( \mu \) has been specified as a function of \( y \). The assumed functional form for \( \mu \) is

\[
 \frac{\mu}{\mu_i} = \exp(ay + by^2) = \left( \frac{T}{T_i} \right)^{-n},
\]

(7-6.13)

where the constants \( a \) and \( b \) are determined from the known temperature profile given in Eq. (7-6.11), i.e., from the relations

\[
 a = - \frac{n}{T_i} \left( \frac{dT}{dy} \right)_i,
\]

(7-6.13a)

\[
 b \approx \frac{n}{4t}.
\]

(7-6.13b)

Next the continuity equation is integrated with the results

\[
 v - v_\infty = - \frac{1}{r} \frac{\partial}{\partial x} \left( r \int_0^y u \, dy \right)
\]

(7-6.14)

and

\[
 v_1 - v_\infty = - \frac{1}{r} \frac{\partial}{\partial x} \left( r \int_0^y u \, dy \right).
\]

(7-6.15)

Now the problem has been completely solved, except that the boundary condition for the energy flux has not been used. This condition is enforced by varying the assumed form of \( T_i(x, t) \) until Eq. (7-6.10) is satisfied.

Numerical results have been derived by Ostrach et al. for a specified body shape and representative entry conditions of a Pyrex body. The (normal) ablation velocity at the stagnation point is found to increase continuously with time toward a steady-state value; however, down-
stream of the stagnation point, the ablation velocity first reverses its sign (which corresponds to accumulation of material in a "bump") and is then gradually reduced while the "bump" is swept downstream at still later times. These phenomena are associated with a decrease in shear stress with downstream distance and with the gradual flattening of the temperature profile with time. The velocity structure for the liquid layer in the stagnation region tends toward a steady-state profile, whereas farther downstream the velocity profiles tend to simulate conditions for unsteady heating. Moderate deceleration reduces the normal ablation velocity at the stagnation point, whereas the calculations become inapplicable (presumably because of failure of the boundary-layer approximation as the bumps accumulate because of growth of the liquid layer) for larger decelerations at some distance downstream from the stagnation region. Near the liquid bump, the interface temperature gradients tend to become very small. Reverse flow may be produced when the opposing effect of the deceleration body force overcomes the shear forces at the liquid surface. In practice, it is to be expected that the unusual growth of excessively large bumps, especially on bodies of relatively small size, will be prevented by "pinching" and removal of the accumulated material through phenomena that are not adequately included in the analysis.

The treatment of Ostrach et al. can clearly be improved in a number of important respects, notably by deleting the assumption of a steady regression velocity in the energy equation. A step in this direction has been made by the use of perturbation calculations for short times in terms of the small parameter \( \phi \) without using any approximation for the ablation velocity. Experimental studies on asphalt have shown qualitative agreement with the calculated effect of deceleration on the formation of molten "bumps" of material and with the development of a "melt wave," the detailed behavior of which may be determined by a coupling parameter which relates the heating rate and the melting rate.

7-6B The studies of Chapman et al. (including internal radiant-energy transport for a gray, molten medium). 9–12 Equations (7-6.1a) and (7-6.2a) may be specialized to the stagnation region of a reentry vehicle with the results

\[
\frac{\partial u}{\partial x} + \frac{u}{x} + \frac{\partial v}{\partial y} = 0 \tag{7-6.16}
\]

and

\[
\frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) = \frac{dp}{dx} - gx, \tag{7-6.17}
\]
respectively; in the stagnation region, the energy conservation equation, including the radiant flux term \( \partial F^*(y^*, t^*)/\partial y^* \), becomes

\[
\frac{\partial T}{\partial t} + \varphi \frac{\partial T}{\partial y} = \frac{\partial^2 T}{\partial y^2} + \frac{\partial}{\partial y^*} F(y, t),
\]

(7-6.18)

where

\[
\frac{F^*(y^*, t^*)}{2n^2K^*\sigma^*} = \int_0^{\infty} T^{*4}(\eta^*) \{E_2[\bar{K}^* \mid y^* - \eta^* \}] \, d\eta^*
\]

\[
+ \left(1 - \frac{\epsilon_m}{n^2}\right) \int_0^{\infty} T^{*4}(\eta^*) \{E_2[\bar{K}^* \mid y^* + \eta^* \}] \, d\eta^*
\]

(7-6.19)

and

\[
F(y, t) = \frac{(R_s \delta)^2}{\lambda^* T_{st}^*} F^*(y^*, t^*).
\]

(7-6.20)

Here \( n \) is the index of refraction of the molten liquid, \( K^* \) denotes the (assumed constant) linear absorption coefficient for radiant energy, \( \sigma^* \) is the Stefan-Boltzmann constant, \( \epsilon_m \) is the maximum emissivity for uniform temperature.

The method of solution is similar to that used by Ostrach, after introducing the approximation

\[
\frac{\partial u}{\partial x} \approx \frac{u}{x},
\]

(7-6.21)

which is valid in the stagnation region. The integral of Eq. (7-6.17) becomes

\[
\frac{\partial u}{\partial x} = \frac{u}{x} = \left( \frac{d^2 p}{dx^2} - \tilde{g} \right) \int_\infty^{y} \frac{y \, dy}{\mu} + \frac{d\tau_1}{dx} \int_\infty^{y} \frac{dy}{\mu},
\]

(7-6.22)

which is evidently equivalent to the derivative with respect to \( x \) of Eq. (7-6.12) in the vicinity of the stagnation point. Once \( \partial u/\partial x \) is known, we may use the mass conservation equation as before and also the following expression for calculating \( v \):

\[
v = -2 \int_{y_\infty}^{y} \frac{\partial u}{\partial x} \, dy + v_\infty,
\]

(7-6.23)

where \( v_\infty \) is again the dimensionless ablation rate. Analytic solution of the energy equation is now no longer possible, even for constant values of the regression rate. In general, a computer program is developed to match the solutions of the momentum and mass conservation equations to those of the energy equation subject to a number of boundary constraints, which will now be summarized briefly.
7. Ablation

The condition for energy conservation at the interface of the ablating substance may be conveniently written in the form [compare Eq. (7-6.10)]

\[ q_{\text{conv}}^* = \psi q_0^* = - \left( \lambda_i^* \frac{\partial T^*}{\partial y^*} \right)_{y^*=0^+} + \rho_i^* v_i^* \Delta l^*. \]  

(7-6.10a)

Here

\[ q_0^* = q_{0,c}^* = \frac{1110 (\rho_\infty^*)^{1/2} V_\infty^{*3.15}}{(R_s^*)^{1/2}} \left[ 1 - \left( \frac{h_{g,1}^*}{h_{st}^*} \right) \right] \]  

(7-6.24)

for continuum flow \((q_{0,c}^*\) is in cal/cm\(^2\)-sec, \(\rho_\infty^* = \) free-stream air density in g/cm\(^3\), \(V_\infty^* = \) free-stream or reentry speed in km/sec, \(R_s^* = \) in cm, \(h_{g,1}^* = \) air specific enthalpy at the interface, \(h_{st}^* = \) specific stagnation enthalpy of the air), and

\[ \psi q_0^* = \frac{\psi q_{0,c}^* q_{0,FM}}{(q_{0,FM}^* + q_{0,c}^*)^{1/2}} \]  

(7-6.24a)

in the transition flow regime if \(q_{0,FM}^*\) is the convective heat transfer for free-molecule flow. Also \(\psi\) equals a "heat blockage fraction" associated with vaporization and is given by the relation

\[ \psi = \frac{0.94}{1 + a_1 [(p_{st}^* / p_{v,0}^*)^m - 1]^{-1}} + 0.06, \]  

(7-6.25)

where \(a_1\) is a constant which is determined from various numerical solutions of the boundary-layer equations; \(p_{st}^* / p_{v}^*\) equals the ratio of the stagnation pressure to the vapor pressure of the ablating material and \(p_{v,0}^*\) differs from the normal vapor pressure \(p_{v,0}^*\) but is related to it through the expression

\[ \frac{p_{st}^*}{p_v^*} = \left( \frac{p_{st}^*}{p_{v,0}^*} \right)^m \]  

(7-6.26)

with \(m\) denoting an empirically determined constant the numerical value of which depends on the nature of the ablating material and on the surrounding atmosphere. Combination of all of the preceding relations defines the convective energy input, in the continuum flow regime, through the relation

\[ \frac{0.94}{1 + a_1 [(p_{st}^* / p_{v,0}^*)^m - 1]^{-1}} + 0.06 \left\{ \frac{1110 (\rho_\infty^*)^{1/2} V_\infty^{*3.15}}{(R_s^*)^{1/2}} \left[ 1 - \left( \frac{h_{g,1}^*}{h_{st}^*} \right) \right] \right\} = - \left( \lambda_i^* \frac{\partial T^*}{\partial y^*} \right)_{y^*=0^+} + \rho_i^* v_i^* \Delta l^*. \]  

(7-6.27)
Hence known external flow parameters are related to the interfacial temperature and temperature gradient, except that the interfacial mass velocity per unit area, \( \rho_i^* \psi_i^* \), is also involved; this quantity is determined from the solution of the boundary-layer diffusion equation, for Lewis number \( Le \) and ratio \( M \) of ablator to air molecular weight, by the equation

\[
\rho_i^* \psi_i^* = \psi_0^* \left( \frac{Le}{M} \right) \left[ \left( \frac{\rho_{st}^*}{\rho_v^*} \right) - 1 \right]^{-1} (h_{st}^* - h_{g,1}^*)^{-1}. \tag{7-6.28}
\]

The calculation of \( u(x, y) \) requires not only specification of \( d^2 p/dx^2 \) but also of the dimensionless shear-stress gradient \( d\tau_{1}/dx \). In this connection, we note that mass addition from the ablator surface reduces the wall shear stress from its normal value \( \tau_{1,0}^* \) to the applicable value

\[
\tau_{1}^* = \psi \left( 1 + a_2 \psi \left[ \left( \frac{\rho_{st}^*}{\rho_v^*} \right) - 1 \right]^{-1} \right) \tau_{1,0}^*, \tag{7-6.29}
\]

where \( a_2 \) is also derived from numerical solutions of the boundary-layer equations. It now remains to relate shear stress to heat transfer and external velocity gradient without ablation. The applicable relations are

\[
\frac{d\tau_{1,0}^*}{dx^*} = a_2 \psi_0^* (h_{st}^* - h_{g,1}^*)^{-1} \frac{du_e^*}{dx^*},
\]

where the velocity gradient at the outer edge of the boundary layer is

\[
\frac{du_e^*}{dx^*} = \frac{V^*}{R_g^*} \frac{(2\rho_{st}^*/\rho_{st}^*)^{1/2}}{f}, \tag{7-6.30}
\]

\( f \) is a number close to unity, and \( \rho_{st}^*/\rho_{st}^* \) is the ratio of free-stream to stagnation density.

The boundary conditions given in Eqs. (7-6.27) to (7-6.29) match the convective heat transfer to the energy conservation requirements within the melt layer. Similarly, the use of Eq. (7-6.30) in Eq. (7-6.22) will lead to velocity components in the liquid layer that are not only consistent with the calculated temperature and viscosity profiles but also with the surface shear stresses created by the external gas flow. For tektite ablation, Chapman has employed the following numerical values: \( a_1 = 0.95, a_2 = 0.28, a_3 = 1.025, m = 1.4 \).

Chapman and Larson\(^{10} \) have discussed another important correction for reentry applications—that resulting from wobbling or (slow) turning of the body at nonzero angle of attack.
7-7 Ablation with combustion of gasification products, depolymerization, and liquid-phase reactions

Extensive experimental studies on the ablation mechanisms of various plastic ablators have been published, both in the open literature and in company reports.\textsuperscript{23–33} In general, the details of the chemical processes depend on the nature of the decomposing plastic, on its interactions with other constituents such as the inorganic reinforcement (e.g., glass) used, and on the composition of the ambient atmosphere. Representative simplified models of the reaction, ablation, and boundary-layer zones are sketched in Figs. 7-7.1a, 7-7.1b, and 7-7.2.

The requisite analytical description of ablating materials is, of course, strongly dependent not only on chemical reaction mechanisms and rates but also on the presence or absence of a molten surface layer. We shall indicate briefly the essential features of rather incomplete analyses which have been used for ablators undergoing chemical reactions.

![Diagram of gross features of surface for phenol-formaldehyde resin reinforced with fiberglass](image1)

![Diagram of idealized version of Fig. 7-7.1a for fiberglass-reinforced phenolic resin](image2)

**Fig. 7-7.1.** (a) Schematic diagram of the gross features of the surface for a phenol-formaldehyde resin reinforced with fiberglass; after Beecher and Rosenzweig.\textsuperscript{24,25} (b) The idealized version of Fig. 7-7.1a used in an analytical study of ablation for fiberglass-reinforced phenolic resin; after Rosenzweig and Beecher.\textsuperscript{25}
7-7A ABLATION OF FIBERGLASS-REINFORCED PHENOLIC RESIN. The model shown in Fig. 7-7.1b has been studied by using a treatment that is very similar to the analysis of Bethe and Adams (compare Section 7-5), except that allowance is made for the occurrence of chemical reactions in the melt zone. The chemical processes have been idealized to a one-step, first-order reaction of the type condensed reactants → gaseous products. Rosenzweig and Beecher follow the procedure of Bethe and Adams (see Section 7-5), except that they consider a melt zone, the outer layer of which is described by a continuity equation of the form

$$\frac{\partial (R^* u^*)}{\partial x^*} + \frac{\partial (R^* v^*)}{\partial y^*} = - R^* \frac{k^*}{\delta_{i,R}} \exp \left( - \frac{\Delta E^*}{R T_R^*} \right)$$  

(7-7.1)

where $k^* \exp(-\Delta E^*/RT_R^*)$ measures the rate of mass conversion per unit volume of condensed reactants to gaseous products in the reaction zone of the molten liquid layer (see Fig. 7-7.1b). The gaseous products are assumed to escape instantaneously according to a process that is not described analytically. The reaction zone is assumed to remain isothermal at the temperature $T_R^*$. Nonsteady phenomena are not considered.

The method of solution employed by Rosenzweig and Beecher follows closely the treatment of Bethe and Adams. The unknown parameters, which must be determined theoretically, are $v_{\infty}^*$ as well as the thickness $\delta_{i,R}^*$ and temperature $T_R^*$ of the reaction zone in the molten liquid. The required relations involve an expression that is similar to Eq. (7-5.27), except that account is taken of the fact that the molten layer has been divided into two portions, one of which is identified as the reaction zone. The relation for energy balance at the interface [cf. Eq. (7-5.24)] now contains a heat release term and is written with
The third relation between the unknown parameters \( v^*, \delta_{i,R}^*, \) and \( T_R^* \) is obtained from a study of possible reaction processes. The following chemical changes involving silica and carbon are mentioned:

\[
\begin{align*}
\text{SiO}_2 + 3\text{C} &\rightarrow \text{SiC} + 2\text{CO}, \\
\text{SiO}_2 + 2\text{C} &\rightarrow \text{Si}(l) + 2\text{CO}, \\
\text{SiO}_2 + \text{C} &\rightarrow \text{SiO}(g) + \text{CO}.
\end{align*}
\]

The chemical processes leading to the formation of carbon are not considered in detail; nor is explicit allowance made in the conservation equations of the liquid phase for the escape of gaseous reaction products from the liquid layer. This escape can clearly occur only when the vapor pressure of the reaction products exceeds the stagnation pressure. The process \( \text{SiO}_2 + 3\text{C} \rightarrow \text{SiC} + 2\text{CO} \) is considered to control the reaction rate.

Solution of the specified problem has been carried out on a high-speed computer by using an assumed temperature profile of the form \( T^* = C_1^* + C_2^*y^* + C_3^*(y^*)^2 \) for various reentry trajectories. Representative results for the dependence of ablation rate and \( \delta_{i,R}^* \) on the rate of reaction and heat of reaction are presented graphically.

It is apparent that the treatment of Rosenzweig and Beecher\(^{25}\) is capable of extensive refinement, especially in so far as proper allowance for the detailed chemical processes and for nonsteady phenomena is concerned. In this connection, it should be noted that some experimental information is available on relevant chemical reaction mechanisms and rates.\(^{24,29–33}\)

### 7-7B ABLATION OF CHARRING HEAT SHIELDS

Extensive experimental results are available which have been interpreted in terms of the model shown in Fig. 7-7.2.\(^{23}\) As might be expected, it was found that the observed ablation rate could be correlated in terms of observed surface temperatures. Furthermore, both of these parameters, as well as the heat of ablation, were found to be dependent on the nature of the heating processes (i.e., radiative or convective heating) and on the magnitudes of the heating rates. Observations of this type show that the nature of the charring and reaction processes is closely coupled to the external flow field and that, therefore, a complete theoretical description will require consideration of reaction mechanisms and rates that are themselves dependent on flight conditions. Both blockage effects and reradiation from the char layer were found to be important.\(^{23}\)

Char recession was found to depend not only on surface temperature (it increased with surface temperature) but also on pressure.\(^{23}\) In
hypersonic flight, char ablation may become unimportant since the primary process for removal of the char layer may now involve spallation phenomena. Spallation mechanisms have been studied by Mathieu\textsuperscript{26} on the assumption that the spallation processes depend on the internal pressure drop through the porous char layer; this model leads to the notion of a critical char-layer thickness at which rupture occurs. The practical solution for the improvement of ablator materials clearly leads to the development of char layers with improved structural strength, which has been achieved by using a variety of reinforcement materials (e.g., graphite, silica, asbestos, nylon). When a sufficient amount of reinforcement material is added to provide adequate structural rigidity, melting may again occur before ablation, and the appropriate analytical model will then become similar to those described in Sections 7-5 and 7-6.

The quantitative treatment of Mathieu\textsuperscript{26} is restricted to the analysis of a one-dimensional, porous layer for which the energy conservation, continuity, and momentum equations are supplemented by a single \( n \)-th-order reaction process. One of the spallation criteria involves the use of an empirical relation between critical char thickness and shear stress on a nonablating surface; the other spallation criterion is based on the assumption that failure of the char layer will occur when the normal stress produced by pressure drop in the porous layer exceeds an assumed critical value.\textsuperscript{26} These criteria lead to the idea that periodic removal of the char layer will normally occur during atmospheric entry.

7-7C ABLATION OF GRAPHITE. Ablation of graphite has been treated by several authors, especially by Scala and Gilbert.\textsuperscript{47} The chemical processes are relatively well understood for graphite combustion. The analytical treatments refer to steady-state regression for which a numerical solution has been worked out under representative conditions.\textsuperscript{27} A highly simplified theoretical treatment of graphite ablation forms the contents of Section 7-8.

7-7D CONCLUDING REMARKS. Ablation of reacting plastic materials has been considered by several authors. The steady-state theoretical solutions will probably prove to be inadequate for applications requiring real-time processing of optical or radar observables. Semiempirical approaches of the type used by Sutton,\textsuperscript{29,30} Lundell,\textsuperscript{23} and others, when coupled with flight-test observations, clearly hold the greatest promise for early solution of the practical problem. In the meantime, it appears desirable to develop a rational, highly flexible computer program for effecting solution of the complete nonsteady equations with arbitrary
time-dependent reaction rates. Numerical experiments, coupled with carefully chosen experimental tests, should provide the next important step in the evolution of our quantitative understanding of one of the messiest problems in applied science that the aerospace community has been forced to consider. The discussion in Appendix 7-1 indicates, for example, the type of considerations required for a relatively minor refinement in analysis—that of deleting the assumption that the surface vapor pressure for the ablating species assumes its equilibrium value at the interface.

In addition to an extensive computer program, which must include proper consideration of nonsteady temperature profiles, ablation rates, and reaction rates, it appears appropriate to emphasize the importance of allowing also for such effects as surface forces on a molten liquid layer, which were probably responsible for nonsteady surface flows encountered by Aihara\[21\] in experimental studies on ablation of highly conducting (i.e., iron or copper) conical and blunt bodies exposed to a plasma jet. These nonsteady surface vortex flows would be expected to be "frozen into the surface" for highly conducting materials which are suddenly cooled below the melting point to macroscopic depths, whereas they should be wiped out during the much more gradual cooling of successive, macroscopic layers of poorly conducting materials. A careful discussion of neglected, but possibly important, physical phenomena is clearly a part of a significant quantitative program for the elucidation of ablation processes.

7-8 **Unsteady ablation coupled to a steady, laminar boundary-layer flow**

After commenting briefly on the use of similarity principles in theoretical combustion research, we develop an approximate analytical procedure for deriving relations between ablation rate and surface temperature on the assumption that the external boundary-layer flow is steady, pressure gradients across the boundary layer are negligibly small, and the steady boundary-layer flow is coupled to a steady or unsteady surface ablation rate. We utilize simplified boundary-layer solutions

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*The results described in this Section are taken from recent studies by Penner and Sharma,\[27b\] which were supported, in part, by the Advanced Research Projects Agency under Project Defender (monitored by the U.S. Army Research Office, Durham, under Contract DA-31-124-ARO-D-257) and, in part, by the U.S. Army Research Office, Durham, under Grant No. DA-ARO-(D)-31-124-G747.*
obtained previously for reacting boundary-layer flows; these solutions involve such approximations as Prandtl and Lewis numbers equal to unity, binary diffusion relations, etc.

For a flat plate, we obtain the ablation rate as an explicit function of the distance coordinate and the (time-dependent) wall temperature. A heuristic transformation of the results to a spherical body provides analogous solutions for a spherical reentry vehicle. Comparison between our simplified results for the high-temperature ablation of a graphite sphere and the numerical solutions of Scala and Gilbert\textsuperscript{47} shows acceptable agreement in the diffusion limit.

7-8A Outline of Theoretical Considerations. In this section, we comment briefly on the use of similarity principles in theoretical combustion research and then summarize relevant results for reacting boundary-layer flows.

a. Use of similarity principles in theoretical combustion research. The use of similarity principles in theoretical combustion research is justified because of our inadequate present and anticipated future knowledge of chemical reaction mechanisms and rates\textsuperscript{38} and of the nature of turbulent-flow phenomena with chemical reactions.

All practically useful chemicals undergo chain reactions in producing reaction products. The elementary steps involved in the decomposition or combustion reactions are not well understood, except possibly for some of the hydrogen-halogen mixtures. Specific reaction rates are generally unknown and are only defined within about an order of magnitude for the simplest processes (e.g., recombination of hydrogen atoms in the presence of a third body). An effective, overall rate law for the production of reaction products has sometimes been determined for a limited range of experimental conditions.

The steady-state approximation is often employed for the atomic and free-radical intermediates occurring in combustion processes. The validity of this approximation has been examined in connection with the theoretical calculation of laminar flame velocities\textsuperscript{39–41} in premixed gaseous systems. The steady-state approximation is occasionally a useful device for obtaining first-order estimates for flame-propagation velocities but should probably not be used in estimating concentration profiles for reaction intermediates.

The preceding remarks may be summarized by the statement that combustion processes generally involve unknown reaction steps with unknown rates in complex systems for which no generally useful simplification in analysis is applicable. Furthermore, judging from the
lack of progress that has been made, it is unrealistic to believe that this situation will change materially in the foreseeable future. In fact, agreement between theory and experiment is generally achieved by repetitive iterations which ultimately lead to the selection of particular, non-unique sets of reactions and rate laws that yield "predictions" which are in agreement with very limited experimental data and usually require readjustment as soon as new measurements are performed.

Our past and anticipated future lack of progress in the quantitative understanding of reaction rates and mechanisms has forced us, for all practical purposes, to formulate analyses for "real" combustion processes at such an unsophisticated level that detailed reaction mechanisms and rate laws are not required. The conclusions derived from procedures of this type are then likely to be only of qualitative or, at best, of semi-quantitative significance. Among the schemes that are most useful for avoiding immediate confrontation with the complexities resulting from unknown or inadequate kinetic information are techniques leading to similarity solutions, as introduced by Penner and Williams for one-step reactions into the Western literature and identified as the Shvab-Zel'dovich procedure. This methodology has been generalized to multistep reactions by Penner and Libby (for studies of boundary-layer heat transfer) in a manner that is related to an earlier modification of the Penner-Williams procedure by Lees and also to the work of Denison and Dooley.

b. Boundary-layer flows with chemical reactions. The simplified treatment given for boundary-layer heat transfer by Penner and Libby may be adapted to a discussion of nonsteady ablation with minor modifications. We state first the essential results obtained by Penner and Libby and use the notation employed by these authors.

For \( C = \rho \mu / \rho_e \mu_e = 1 \) (where \( \rho \) and \( \mu \) denote the fluid density and viscosity coefficient, respectively, and the subscript \( e \) identifies external conditions) and negligibly small normal pressure gradients, the momentum conservation equation is readily shown to reduce to the form

\[
\frac{d^2 f}{d \xi^2} + \frac{d f}{d \xi} = 0
\]  

(7-8.1a)

with the boundary conditions

\[
f'(0) = 0, \quad f'(\infty) = 1, \quad f(0) = - (\rho v)_w \left( 2s \right)^{1/2} / \rho_e \mu_e u_e r = f_w = \text{constant},
\]  

(7-8.1b)

where primes denote derivatives with respect to the variable

\[
\eta = \frac{\rho_e u_e r}{(2s)^{1/2}} \int_0^y \left( \frac{\rho}{\rho_e} \right) dy \quad \text{and} \quad s = \int_0^s u_e \rho_e r^{2j} ds.
\]  

(7-8.2)
Also $j = 0$ or $1$ for two-dimensional and axisymmetric flow, respectively. Equation (7-8.1a) is the Blasius equation, the solution of which is well known for $j = 0$ and various values of $f_w$.\textsuperscript{46} The variable $\eta$ is introduced by following Lees\textsuperscript{44} in combining the Levy and Mangier transformations with the Howarth-Dorodnitzyn transformation.\textsuperscript{34a} The function $f$ has been introduced through use of the expressions

$$
\rho vr^j = \frac{\partial}{\partial y} [(2\tilde{s})^{1/2} f(\eta)], \quad \rho vr^j = -\frac{\partial}{\partial \tilde{s}} [(2\tilde{s})^{1/2} f(\eta)].
$$

The velocity components $u$, $v$ are given in terms of $f$ and of its derivatives according to the expressions

$$
u = - (\rho \mu \eta v e^2 \eta^2 / \rho) [(2\tilde{s})^{-1/2} f + (2\tilde{s})^{1/2} (\partial f / \partial \tilde{s})] - (\partial f / \partial \eta) [(2\tilde{s})^{1/2} / \rho] f'.
$$

In terms of the operator $L( )$, which is defined by the relation

$$
L( ) = [(\gamma')^2 + f( \gamma)] (\rho \mu \eta v e^2 \eta^2 / 2\tilde{s}),
$$

we find, for any combination $\alpha_i$ of the dependent variables appearing in the relation

$$
L(\alpha_i) = 0,
$$

that the basic differential equations are of the form

$$
\alpha_i'' + f \alpha_i' = 0
$$

for values of the Prandtl and Lewis numbers equal to unity and constant values of $\rho D = \mu = \lambda / \tilde{s}_\rho$. The solutions of Eq. (7-8.5) obey Crocco-type integral relations, i.e.,

$$
\alpha_i = \alpha_i,w + (\alpha_i,e - \alpha_i,w) f'.
$$

in view of Eqs. (7-8.1a) and (7-8.1b). Here $\alpha_i,e$ and $\alpha_i,w$ are specified constants or derived constraints on the physical variables.

For injection or ablation of a single chemical species into a gas mixture, the heat-transfer boundary condition is readily seen to be of the form\textsuperscript{43}

$$
\left| \left( \lambda \frac{\partial T}{\partial y} \right)_w^+ \right| = m \Delta l, \quad \left| \left( \lambda \frac{\partial T}{\partial y} \right)_w^- \right| = \Delta l
$$

where the subscript $w$ identifies conditions at the wall, $m = (\rho v)_w^+$ is the mass regression rate per unit area, $\Delta l$ denotes the enthalpy change per unit mass at the wall for the injected material, gradients evaluated
on the gas side of the wall are identified by the subscript $w^+$, and gradients on the solid-phase side of the boundary are denoted by the subscript $w^-$. If we assume that the boundary-layer flow adjusts itself to changes in $f_w(t)$ on a time scale that is short compared with the regression rate, then we may couple the solutions for steady boundary-layer flow with the equations for unsteady mass addition and ablation rates. In this case, the temperature gradient at the wall on the gas side, in the transformed coordinates, is given by the simple relation

$$T_{w^+} = (T_{s,e} - T_{w^+} + \bar{Y}_{O,e})f_w^*,$$  \hspace{1cm} (7-8.8)

where $\bar{T}$ denotes a reduced temperature ($= T/T^*$) and $\bar{Y}_{O,e}$ is the reduced ambient mass fraction $= Y_{O,e}/Y_{O}^*$ of the oxidizer.\(^{43}\)

It is now convenient to rewrite Eq. (7-8.7) in the transformed coordinates and to apply the steady-gradient relation given in Eq. (7-8.8) to time-dependent values of $f_w$ and $f_w^*$. In this manner, we find that

$$\left| \left( \frac{\partial T}{\partial y} \right)_{w^+} \right| = \left[ \frac{\lambda_{w^+} \rho_{w^+} u_{e^f}}{(2\delta)^{1/2}} \left( T_{s,e} - T_{w^+} + \bar{Y}_{O,e} \frac{T^*}{Y_{O}^*} \right) f_w^* \right],$$  \hspace{1cm} (7-8.9)

$$| m \Delta l | = \left| \frac{1}{r^f} \left( \rho_{e^f} u_{e^f} \bar{r}^2 \right) \frac{1}{(2\delta)^{1/2}} f_w^* \right| \Delta l,$$  \hspace{1cm} (7-8.10)

and, therefore,

$$\left( T_{s,e} - T_{w^+} + \bar{Y}_{O,e} \frac{T^*}{Y_{O}^*} \right) f_w^*(t) = - \left( \rho_e u_e \bar{r} \right) \frac{\partial T}{\partial y} \bigg|_{w^+} \bigg| \frac{(2\delta)^{1/2}}{\lambda_{w^+} \rho_{w^+} u_{e^f}} \left( \frac{\partial T}{\partial y} \right)_{w^+},$$  \hspace{1cm} (7-8.11)

where the rate of heat conduction into the wall is, of course, also time-dependent. For two-dimensional flow ($j = 0$) with effectively constant external flow variables, Eq. (7-8.11) reduces to the simpler form

$$a_1 f_w^* + a_2 f_w = a_3 \left( \frac{\partial T}{\partial y} \right)_{w^+},$$  \hspace{1cm} (7-8.11a)

where the time-dependent parameters $a_1$, $a_2$, and $a_3$ are given by the expressions

$$a_1 = T_{s,e} - T_{w^+} + \bar{Y}_{O,e} \frac{T^*}{Y_{O}^*},$$  

$$a_2 = \left( \rho_{e^f} \bar{r} \right) \frac{\lambda_{w^+} \rho_{w^+}}{(2\delta)^{1/2}} \Delta l,$$  

$$a_3 = \left( 2\rho_{e^f} \bar{r} \right)^{1/2} \lambda_{w^+} \rho_{w^+} (u_e)^{1/2},$$  \hspace{1cm} (7-8.12)
for any given value of $T_w(t) = T_w^+(t) = T_w^-(t)$ which defines $f_w(t)$, $\lambda_w^*(t)$, and $\rho_w^+(t)$.

Much more complicated ablation and boundary-flow models than we are considering have appeared in the recent literature (see, for example, Refs. 27, 44a, 47–49).

7-8B ABLATION RATES FOR A FLAT PLATE. The energy balance at the surface of an ablator for a boundary-layer flow determines a unique relation between surface temperature and blowing rate. Thus the mass-addition rate is implicitly given by the use of the boundary-layer equations and it is, therefore, not permissible to specify a priori the applicable functional relation between $(\rho_w^+v_w^*)$ and $T_w^+$.

For a flat plate with negligibly small temperature gradients in the $y^-$-direction, Eq. (7-8.11a) becomes

$$f_w^+(-f_w) \simeq a_2/a_1,$$

(7-8.13)

![Fig. 7-8.1. The ratio of $f_w^+(-f_w)$ as a function of $(-f_w)$ for the Blasius equation; this plot has been constructed from the data of Emmons and Leigh. Reproduced from Penner and Sharma.](image)
where \( f_w''(\tilde{f}_w) \) is a known function of \( (\tilde{f}_w) \) according to the solutions obtained by Emmons and Leigh.\(^{46}\) The functional relation
\[
f_w''(\tilde{f}_w) = \varphi(\tilde{f}_w)
\]
is indicated in Fig. 7-8.1 for the Blasius equation.

a. Flows with \( T_{s,e} \gg T_w, \ T_{s,e} \gg Y_{O_e}(T^*/Y_O^*) \) and comparison with the results of Scala and Gilbert.\(^{47}\) For sufficiently large free-stream Mach numbers, \( T_{s,e} \gg T_w, \ T_{s,e} \gg Y_{O,e}(T^*/Y_O^*) \); hence \( a_1 \sim T_{s,e} \). Therefore,
\[
\frac{f_w^*}{(-f_w)} = \frac{a_2}{a_1} \sim \frac{\rho_{e} \mu_{e}}{\lambda_{w} + \rho_{w}^{+} \lambda_{w}^{+}} \frac{\Delta l}{T_{s,e}} - \frac{\rho_{e} \mu_{e}}{\rho_{w}^{+} \mu_{w}^{+} \lambda_{w}^{+}} \frac{\Delta l}{T_{s,e}} \sim \frac{4}{15} \frac{\Delta l}{T_{s,e}} \frac{\bar{W}_{w}^{+}}{R} \tag{7-8.14}
\]
where we have again used the approximation \( C = \rho_{e} \mu_{e} / \rho_{w}^{+} \mu_{w}^{+} = 1 \), as well as the kinetic theory relation for a monatomic gas
\[
\frac{\mu_{w}^{+}}{\lambda_{w}^{+}} \sim \frac{4}{15} \frac{\bar{W}_{w}^{+}}{R}
\]
with \( R \) representing the molar gas constant. Here \( \bar{W}_{w}^{+} \) is a mean molecular weight for the gases at the wall. Combination of Eq. (7-8.14) with the Blasius solution leads to the conclusion that a plateau value exists for the ablation rate which is independent of the value of the wall temperature for sufficiently large flight velocities (i.e., large \( T_{s,e} \)). For the sublimation of graphite to form \( C(g) \) at the surface, \( \Delta I \sim 14 \text{ kcal/g} \) for the ablation conditions, whereas \( \bar{W}_{w}^{+} \sim 20 \text{ g/mole} \); hence Eq. (7-8.14) becomes
\[
\frac{f_w^*}{(-f_w)} \sim 3.8 \times 10^4 \frac{1}{T_{s,e}} \tag{7-8.14a}
\]
with \( T_{s,e} \) expressed in degrees Kelvin. Comparison with the data plotted in Fig. 7-8.1 leads to the conclusion that the blowing rate \( (-f_w) \) reaches the plateau values shown in Fig. 7-8.2 for various assumed values of \( T_{s,e} \).

Our theoretical procedure and the relations which we have derived for the calculation of ablation rates are valid only in case of complete removal of the free-stream reactants before these can reach the ablator surface. Because of this constraint, only a limited comparison is justified with the more detailed numerical solutions of graphite ablation by Scala and Gilbert.\(^{47}\)

Scala and Gilbert refer to a low-temperature regime in which direct surface attack by oxygen occurs as "reaction-rate-controlled" and
"transition regimes" for which the following empirical expressions are used for the mass ablation rate:

\[ \dot{m}_R \simeq K_0 \exp(-44,000/RT_w) (p_{O_2,w})^{1/2}. \]

Here \( T_w \) is the wall temperature, \( p_{O_2,w} \) denotes the partial pressure of oxygen at the wall, and \( K_0 \) is an effective collision frequency, which is determined from a comparison with experimentally measured results for "slow and fast" ablators.

At the opposite extreme of temperature, the mass-ablation rate is said to be determined by the "sublimation regime," which Scala and Gilbert define, for all practical purposes, as the ablation regime in which chemical equilibrium is maintained everywhere and which is dominated by direct surface attack of nitrogen on carbon, producing CN. Our similarity solution is clearly inapplicable when this type of direct surface attack occurs.
For intermediate temperatures, Scala and Gilbert identify a "diffusion-controlled regime" involving direct surface attack, with mass ablation rate $\dot{m}_d$. Our similarity solution actually constitutes a generalization of a "diffusion flame" (i.e., a process not involving direct surface attack) since, in this case, the boundary conditions on reactant mass fractions at the wall and in the free stream are satisfied. Nevertheless, it is of interest to make a comparison between our results and those of Scala and Gilbert for $\dot{m}_d$. In particular, we shall now show that we have the same functional form for $\dot{m}_d$, as well as acceptable numerical agreement in an absolute sense.

Scala and Gilbert\textsuperscript{47} define a diffusion limit for the mass addition rate of graphite when the free-stream velocity is $u_\infty = 20,000$ ft/sec at an altitude of 100,000 ft; this diffusion limit is independent of the wall temperature and is determined\textsuperscript{47} by the equation

$$\dot{m}_d = 6.35 \times 10^{-3} (p_e/R_B)^{1/2} \text{ lb/ft}^2\text{-sec},$$

where $p_e$ is the ambient pressure in atmospheres and $R_B$ equals the body radius in feet.

According to our analysis and Eq. (7-8.1), the mass injection rate is

$$\left(\rho_w v_w^+\right) \sim \frac{\rho_e \mu \mu_e u_e}{(2s)^{1/2}} (-f_w) = \sqrt{\frac{\rho_e \mu \mu_e u_e}{2s}} (-f_w)$$

for external flow variables independent of $s$. For the specified flight conditions, it is appropriate to use in our flat-plate analysis for the external flow the values corresponding to the bow shock at the stagnation point, viz., $T_{s,e} = 6968^\circ\text{K}\textsuperscript{50}$ so that $(-f_w) = 0.0765$ according to Fig. 7-8.2. Furthermore, $T_o = 6930^\circ\text{K}$, $p_e = 5.6$ atm, and $\rho_e = 1.285 \times 10^{-2} \text{ lb/ft}^3\textsuperscript{50}$ so that $\bar{W}_e = (\rho_e RT_e/p_e) = 20.9 \text{ lb/mole}$ and $u_e = (p_o u_o/p_e) = 1.667 \times 10^3$ ft/sec.

If we now identify $s$ with the radius of the blunt body $R_B$ multiplied by an appropriate constant factor (see Appendix 7-2 for the origin of this result), then we obtain a heuristic transformation to the conditions of Scala and Gilbert\textsuperscript{47} and the expression

$$\left(\rho_w v_w^+\right) = \left(\frac{\bar{W}_e \mu \mu_e u_e}{2RT_e}\right)^{1/2} \left(\frac{\rho_e}{s}\right)^{1/2} (-f_w) = 4.4 \times 10^{-3} (\rho_e/R_B)^{1/2} \text{ lb/ft}^2\text{-sec}$$

\textit{(7-8.14b)}

for $\mu_e = 9.26 \times 10^{-5} \text{ lb/ft}\text{-sec}$. Our final relation is seen to have the same functional form and to be of the same order of magnitude as the diffusion-limit result of Scala and Gilbert. However, it should be noted that the present analysis is not restricted, in any way, to a particular reaction model but rather remains applicable for all reaction models as
long as the constant-property assumptions which we have made constitute reasonable approximations and the surface- and free-stream boundary conditions are satisfied.

b. Flows with arbitrary values of $T_{s,e}$. For flows with arbitrary values of $T_{s,e}$, Eq. (7-8.14) is replaced by the expression

$$f_w^\nu (-f_w) \approx \frac{4}{15} \frac{A I}{T_{s,e}} \frac{W_{w*}}{R} \frac{1}{[1 - (T_w/T_{s,e}) + (Y_{O,e}T^*/Y_{O}T_{s,e})]}.$$  (7-8.15)

Since the value of $T^*$ generally depends on the assumed reaction model, it follows immediately from Eq. (7-8.15) that the rate of ablation is also dependent on the detailed nature of the reaction processes. Problems of this type have been considered by various authors\textsuperscript{47-49} using partially verified assumptions about reaction models and rates.

Direct surface attack by oxygen atoms. For direct surface attack by oxygen atoms, the heat release is determined by the surface process

$$C(s) + O(g) \rightarrow CO(g).$$

Hence, for a fixed wall temperature $T_{w+}$, the applicable value of $T^*$ approaches the limiting value which corresponds to the ratio of the molar heat of surface reaction to the molar heat capacity of CO, i.e., to the limiting process in which no energy absorption occurs by inert diluents. The corresponding value of $T^*$, and hence of $T_{w+}$ is far too high to be admissible for any surviving reentry heat shield.

c. Flat plates with temperature gradients in the y-direction. With nonnegligible temperature gradients in the heat-shield material, it follows from Eq. (7-8.11) that

$$f_w^\nu (-f_w) = (a_2/a_1) + [a_2/a_1(-f_w)] \lambda(\partial T/\partial y)_{w-}. \quad (7-8.16)$$

Solution of the ablation problem now requires solution of the unsteady heat-conduction problem into the solid body.

Appendix 7-1

The mass fraction of ablator at the interface, where evaporation occurs, for a boundary-layer flow\textsuperscript{1}

Implicit in the derivations of the expression for the mass evaporation rate of ablator by Bethe and Adams\textsuperscript{4} is the idea that the surface vapor pressure for the ablating species approximates closely the equilibrium
value at the interfacial temperature. This assumption is clearly valid if the evaporation and condensation rates are very rapid in comparison with the mass removal rates by diffusion. The conditions under which this constraint is satisfied have been studied previously by Bauer and Zlotnick.\textsuperscript{51} We shall reconsider this problem for a boundary-layer flow and obtain results in a form suitable for making numerical estimates.

The mass conservation condition for the ablating species at the interface is well known and may be written in the form

\[
\rho_i^* v_{i,1}^* = - \frac{\rho_g^* D_g^*}{1 - Y_{a,1}^*} \left( \frac{dY_{a,g}^*}{dy^*} \right)_1,
\]

where \(Y_{a,i}^*\) and \(Y_{a,g}^*\) represent, respectively, the mass fraction of evaporating material at the interface and in the gas phase. In terms of the evaporation frequency \(v_{\text{evap}}^*\) and condensation velocity \(v_{\text{cond}}^*\), it is apparent that

\[
\rho_i^* v_{i,1}^* = (v_{\text{evap}}^* N_{a,s}^* - v_{\text{cond}}^* m_a^*) m_a^* \text{, (A-7.2)}
\]

where \(N_{a,s}^*\) is the number of surface atoms per unit area, \(n_{a,1}^*\) is the number of ablator molecules per unit volume in the gas phase at the interface, and \(m_a^*\) is the mass of the gas-phase ablation atom or molecule. Since \(\rho_i^* v_{i,1}^*\) must vanish at equilibrium, detailed balancing shows that

\[
v_{\text{cond}}^* = \frac{v_{\text{evap}}^* N_{a,s}^*}{n_{a,1}\text{eq}} \text{, (A-7.3)}
\]

where we assume a planar surface for the evaporating material and use the notion that the relaxation times for establishing evaporation frequencies and condensation velocities at any given temperature are very short in comparison with other characteristic times of the problem; the subsidiary subscript eq identifies equilibrium conditions. Combination of Eqs. (A-7.1) to (A-7.3) leads to the result

\[
\rho_i^* v_{i,1}^* = v_{\text{evap}}^* N_{a,s}^* m_a^* \left[ 1 - \left( \frac{Y_{a,1}}{Y_{a,1}\text{eq}} \right) \right] = - \frac{\rho_g^* D_g^*}{1 - Y_{a,1}^*} \left( \frac{dY_{a,g}^*}{dy^*} \right)_1 \text{. (A-7.4)}
\]

Equation (A-7.4) may be solved for the interface mass fraction \(Y_{a,i}^*\) if we assume that the ablator mass fraction decays to zero as \(|y_i^*|\) increases from zero to \(|y_{a,i}^*|\) in the stagnation region; this type of approximation may also be used for a boundary layer flow with \(Y_{a,g}^*\) decaying from \(Y_{a,i}^*\) to zero in a distance that is equal to the boundary layer thickness \(\delta^*\).
In terms of the dimensionless parameter
\[ K \equiv \frac{\rho_g^* D_g^*}{\nu_{evap} N_a^* m_a^* \delta^*}, \]  
(A-7.5)
we obtain the quadratic expression
\[ \frac{Y_{a,1}}{Y_{a,1;eq}} - Y_{a,1} \left(1 + \frac{1}{Y_{a,1;eq}} + K\right) + 1 = 0. \]  
(A-7.6)

Equation (A-7.6) leads to the result
\[ \frac{Y_{a,1}}{Y_{a,1;eq}} = \frac{1}{2} \left(1 + \frac{1}{Y_{a,1;eq}} + K\right) + \left(1 - \frac{1}{Y_{a,1;eq}}\right) \left(1 + \frac{2K[1 + (1/Y_{a,1;eq})] + K^2}{1 - (1/Y_{a,1;eq})^2}\right)^{1/2} \]
\[ = 1 - K \frac{Y_{a,1;eq}}{1 - Y_{a,1;eq}} + \mathcal{O}(K^2). \]  
(A-7.7)

Reference to Eq. (A-7.7) shows that the approximation \( Y_{a,1} \approx Y_{a,1;eq} \) may be used in the limit as \( K \) goes to zero, provided \( Y_{a,1;eq} \) does not approach unity. In general, we conclude that the use of local equilibrium values at the interface requires verification by first making independent estimates of the appropriate value of \( K \). We shall now calculate typical values of \( K \) for evaporating liquids and solids.

A general procedure for estimating the evaporation flux rate involves the use of the Knudsen equation in the form
\[ \nu_{evap} N_a^* m_a^* = \epsilon_{evap} \rho_{a,1}^*[m_a^*/(2\pi k^* T_1^*)]^{1/2}, \]  
(A-7.8)
where \( \rho_{a,1}^* \) is the saturated vapor pressure of ablator at the interface temperature \( T_1^* \); \( \epsilon_{evap}(\leq 1) \) is an empirically determined, dimensionless evaporation coefficient; \( k^* \) represents the Boltzmann constant. There are many empirical equations that have been used to represent the vapor pressure as a function of temperature. The following expression is a useful representation:
\[ \rho_{a,1}^* = \rho_0^* \exp(-\tau_s^*/T_1^*), \]  
(A-7.9)
where \( \tau_s^* = \Delta H_{evap}^*/R^* \) is the ratio of the molar heat of evaporation \( (\Delta H_{evap}^*) \) to the molar gas constant \( R^* \), and the integration constant \( \rho_0^* \) has the value 1 atm when \( T_3^* \) equals the normal boiling point. The temperature dependence of \( \rho_g^* D_g^* \), which has been treated as a constant.
equal to its interface value throughout the present discussion, may be expressed, approximately, as

$$\rho_{g}^{*} D_{g}^{*} = (\rho_{g}^{*} D_{g}^{*})_0 \left( \frac{T_{1}^{*}}{T_{0}^{*}} \right).$$  \hspace{1cm} (A-7.10)

The preceding relations may now be combined to exhibit explicitly the temperature dependence of $K$. Thus

$$K = K_0 \left( \frac{T_{1}^{*}}{T_{0}^{*}} \right)^{3/2} \exp \left( \frac{\tau_{g}^{*}}{T_{1}^{*}} \right),$$  \hspace{1cm} (A-7.11)

where

$$K_0 = \left( \frac{2 \pi k^{*} T_{0}^{*}}{m_{a}^{*}} \right)^{1/2} \frac{(\rho_{g}^{*} D_{g}^{*})_0}{p_0^{*}} \frac{1}{\delta^{*} \epsilon_{evap}}.$$  \hspace{1cm} (A-7.12)

Reference to Eq. (A-7.11) shows that, for values of $\tau_{g}^{*}$ which are large, $K$ increases very rapidly as the temperature is reduced. Thus, for relatively small values of the temperature, $K$ may become so large as to vitiate the assumption that the equilibrium vapor pressure is approximated at the ablator interface.

The formulation of the boundary condition for a substance which dissociates on evaporation or undergoes chemical reactions has also been considered in simplified form.\(^{52}\)

Appendix 7-2

Heuristic transformation of flat-plate results to the stagnation region of a sphere\(^{*}\)

We first determine the velocity of the flow external to the boundary layer in the stagnation region of a spherical body using the Newtonian-flow approximation. From Newtonian-flow theory, the ratio of the static pressure ($p_e$) to the stagnation pressure ($p_{s,e}$) is (see Fig. A-7.1)

$$\frac{p_e}{p_{s,e}} = \cos^2 \varphi \sim 1 - \left( \frac{s}{R_b} \right)^2,$$  \hspace{1cm} (A-7.13)

where the last equality holds good in the neighborhood of the stagnation point. Now,

$$p_{s,e} = p_e + \frac{1}{2} \rho_e u_e^2,$$

\(^{*}\) We are indebted to Professor P. A. Libby for suggesting the methodology described in this appendix.
APPENDIX 7-2 FLATE-PLATE RESULTS AND STAGNATION REGION OF SPHERE

Fig. A-7.1. Schematic diagram of the stagnation region for a blunt reentry vehicle, defining the coordinates used in the text. Reproduced from Penner and Sharma.\textsuperscript{37b}

so that

$$u_e = \left[2(p_{s,e} - p_e)/p_e \right]^{1/2}$$

$$= (2p_{s,e}[1 - (p_e/p_{s,e})]/p_e)^{1/2}$$

$$= (2p_{s,e}/p_e)^{1/2}(s/R_B)$$

$$= \xi s$$

where $$\xi = (2p_{s,e}/p_e)^{1/2}/R_B.$$ \hspace{1cm} (A-7.14)

Furthermore, in the immediate neighborhood of the stagnation point, we note that

$$r \approx s, \quad \rho_e \approx \rho_{s,e}, \quad \mu_e \approx \mu_{s,e}.$$ 

In the following discussion, the subscripts e, c identify conditions behind a normal shock, which we have used as the constant free-stream conditions for a flat-plate boundary-layer flow. Referring to Eq. (A-7.14) and noting that \( r^f = 1 \) for the flat plate (fp) and \( r^s = r \) for a spherical body (sp), we find that

$$\delta_{fp} = \rho_{e,c}u_{e,c} \delta, \quad \eta_{fp} = (u_{e,c}/2\rho_{e,c} \mu_{e,c})^{1/2} \int_0^y \rho \, dy,$$ \hspace{1cm} (A-7.15)

and

$$\delta_{sp} \approx \frac{1}{2} \xi \rho_{s,c} \mu_{s,c} \delta, \quad \eta_{sp} = (2\xi/\rho_{s,c} \mu_{s,c})^{1/2} \int_0^y \rho \, dy.$$ \hspace{1cm} (A-7.16)

for the flat-plate and stagnation-point regions of a blunt body of radius \( R_B \), respectively.
Since the conservation equations for the boundary-layer flows over a blunt body and a flat plate in their respective coordinates [see Eqs. (A-7.16) and (A-7.15)] are equivalent in the "cold-wall" approximation, we obtain the following transformation (by equating $\eta_{fp}$ to $\eta_{sp}$) relating $s$ and $R_B$:

$$s \sim \frac{u_{e,c}}{4\xi} \sim \frac{u_{e,c}}{4\sqrt{2}} \left(\frac{\rho_{s,e}}{\rho_{s,e}}\right)^{1/2} R_B,$$

(A-7.17)

where we have used the relation $\mu_{s,e}\rho_{s,e} = \mu_{e,c}\rho_{e,c}$. For the specified flight conditions, Eq. (A-7.17) becomes

$$s = 5.36 \times 10^{-2} R_B.$$

This numerical value has been used in the main text in the derivation of Eq. (7-8.14b).

REFERENCES


